N° d'ordre : 3591

THÈSE

PRÉSENTÉE À

L'UNIVERSITÉ BORDEAUX I

ÉCOLE DOCTORALE DE MATHÉMATIQUES
ET D'INFORMATIQUE

Par **Louis Esperet**

POUR OBTENIR LE GRADE DE

DOCTEUR

SPÉCIALITÉ : INFORMATIQUE

Distance-two colorings of graphs

Soutenue le : 30/05/2008

Après avis des rapporteurs :

Frédéric Havet	Chargé de Recherche (HDR)
Stéphan Thomassé	Professeur

Devant la commission d'examen composée de :

Adrian Bondy	Professeur	Examinateur
Frédéric Havet	Chargé de Recherche (HDR)	Rapporteur
André Raspaud ...	Professeur	Examinateur
Martin Škoviera ..	Professeur	Examinateur
Éric Sopena	Professeur	Président
Stéphan Thomassé	Professeur	Rapporteur

– 2008 –

Colorations à distance deux dans les graphes

Résumé : Dans cette thèse, on s'intéresse en particulier à la coloration du carré des graphes planaires (deux sommets à distance au plus deux ont des couleurs distinctes) et à la coloration cyclique des graphes planaires (deux sommets incidents à la même face ont des couleurs distinctes). On montre un résultat général qui implique que deux conjectures importantes sur ces colorations (Wegner 1977 et Borodin 1984) sont vraies asymptotiquement.

On s'intéresse également à d'autres colorations à distance deux, qui ont des liens (plus ou moins vagues) avec l'allocation de fréquences dans les réseaux radios, la théorie des jeux, la sociologie, et l'écologie.

Mots clés :

théorie des graphes

coloration de graphes

graphes planaires

Discipline: Informatique

LaBRI,
Université Bordeaux 1,
351, cours de la libération
33405 Talence Cedex (FRANCE)

Distance-two colorings of graphs

Abstract: In this thesis, we study the coloring of the square of planar graphs (two vertices at distance at most two receive distinct colors) and the cyclic coloring of plane graphs (two vertices incident to the same face receive distinct colors). We show a general result implying that two important conjectures on these colorings (Wegner 1977 and Borodin 1984) hold asymptotically.

We also study other types of distance-two colorings, (more or less) related to frequency assignment in radio networks, game theory, sociology, and ecology.

Keywords:

graph theory

graph coloring

planar graphs

Discipline: Computer-Science

LaBRI,
Université Bordeaux 1,
351, cours de la libération
33405 Talence Cedex (FRANCE)

Remerciements

Merci à toutes les personnes que j'ai pu rencontrer pendant ma thèse, à Bordeaux ainsi que durant mes pérégrinations.

Un énorme merci à André pour m'avoir supporté et aidé pendant ces trois années.

Merci à Pascal, Daniel, Alex, Mickaël, Arnaud, et Jérémie qui ont fait de ces années au LaBRI une expérience à la fois enrichissante sur le plan de la recherche et sur le plan humain.

Many thanks to Louigi Addario-Berry, Omid Amini, Jan van den Heuvel, Ross Kang, Colin McDiarmid, and Xuding Zhu for sharing their knowledge with me. Many thanks also to all the people in Montreal, Oxford, Nice, and Kaohsiung who helped me to have a wonderful time over there.

Merci à Frédéric Havet et Stéphan Thomassé d'avoir accepté d'être rapporteurs de cette thèse, et merci à Adrian Bondy, Martin Škoviera, et Éric Sopena d'avoir bien voulu faire partie du jury de ma thèse.

Merci à mes parents et à mon frère.

Merci à Céline.

Introduction

Un des points de départ de cette thèse est le *problème d'allocation de fréquences* dans les réseaux. Dans un réseau radio, on cherche à assigner des fréquences aux antennes de manière à éviter les interférences. Pour cela deux antennes très proches l'une de l'autre doivent émettre sur des fréquences très éloignées, tandis que deux antennes relativement proches doivent simplement émettre sur des fréquences suffisamment éloignées.

Ce problème peut être modélisé par le $L(p,q)$-étiquetage des graphes, introduit par Griggs et Yeh [GY92]. Un $L(p,q)$-étiquetage d'un graphe G est une assignation d'entiers aux sommets de G telle que deux sommets adjacents reçoivent des entiers distants d'au moins p, tandis que deux sommets à distance deux dans G reçoivent des entiers distants d'au moins q. On suppose en général que $p \geq q$, étant donné que deux antennes très proches sont plus sujettes aux interférences que deux antennes relativement proches.

Le *nombre* $\lambda_{p,q}$ de G, noté $\lambda_{p,q}(G)$, est le plus petit entier t tel qu'il existe un $L(p,q)$-étiquetage de G utilisant des étiquettes de $\{1,2,\ldots,t\}$. On remarque qu'un $L(0,1)$-étiquetage d'un graphe G est équivalent à une coloration propre de G, on a donc $\lambda_{1,0}(G) = \chi(G)$. Si l'on définit le *carré* G^2 d'un graphe $G = (V,E)$ comme le graphe ayant pour ensemble de sommets V et dans lequel deux sommets sont adjacents s'ils sont à distance au plus deux dans G, on observe qu'un $L(1,1)$-étiquetage de G est exactement une coloration propre de G^2; on a donc $\lambda_{1,1}(G) = \chi(G^2)$.

En général, il est NP-difficile de déterminer le nombre $\lambda_{p,q}$ d'un graphe [GMW94]. Toutefois, il est possible d'obtenir des bornes intéressantes en se restreignant à des classes de graphes spécifiques. Dans le **Chapitre 2** on donnera des détails sur le $L(p,q)$-étiquetages des graphes planaires, dans le **Chapitre 3**, on utilisera des résultats existants sur le $L(p,q)$-étiquetage des graphes planaires de maille bornée, et enfin dans le **Chapitre 5**, on étudiera le $L(p,q)$-étiquetage des graphes d'incidence. Pour plus de détails sur le $L(p,q)$-étiquetage, le lecteur est invité à consulter [Cal06].

Dans le cas des graphes de degré maximum Δ, il est facile de voir qu'en appliquant un algorithme glouton on peut obtenir la borne $\lambda_{2,1}(G) \leq \Delta^2 + 2\Delta + 1$. Griggs et Yeh ont proposé la conjecture suivante:

Conjecture 1 [GY92] *Pour tout graphe G de degré maximum $\Delta \geq 2$, on a $\lambda_{2,1}(G) \leq \Delta^2 + 1$.*

Cette borne est optimale étant donné que pour $\Delta = 2, 3, 7$ il existe des graphes de diamètre deux et de degré maximum Δ ayant Δ^2+1 sommets. Cette conjecture a été récemment prouvée pour Δ assez grand par Havet *et al.* [HRS08] en utilisant des techniques de preuves probabilistes.

Étant donné que les antennes dans les réseaux radios sont généralement réparties sur la surface de la terre, un intérêt particulier a été accordé cette dernière décennie au $L(p,q)$-étiquetage des graphes planaires. Dans le cas où $p = q = 1$, il est connu depuis une trentaine d'années qu'il existe des graphes planaires G_Δ de degré maximum Δ tels que $\lambda_{1,1}(G_\Delta) = \chi(G_\Delta^2) = \lfloor \frac{3}{2}\Delta \rfloor + 1$. Wegner [Weg77] a conjecturé que cette valeur est optimale.

Conjecture 2 [Weg77] *Pour tout graphe planaire G de degré maximum $\Delta \geq 8$ on a $\chi(G^2) \leq \lfloor \frac{3}{2}\Delta \rfloor + 1$.*

La première borne supérieure sur le nombre chromatique du carré des graphes planaires en terme de Δ, $\chi(G^2) \leq 8\,\Delta - 22$, était implicite dans un manuscrit de Jonas [Jon93]. Cette borne a été ensuite améliorée par Wong [Won96], qui a montré $\chi(G^2) \leq 3\,\Delta + 5$ puis par Van den Heuvel et McGuinness [HM03], qui ont prouvé $\chi(G^2) \leq 2\,\Delta + 25$. De meilleures bornes ont ensuite été obtenues pour des valeurs suffisamment grandes de Δ. Agnarsson et Halldórsson [AH00] ont montré $\chi(G^2) \leq \lceil \frac{9}{5}\Delta \rceil + 1$ lorsque $\Delta \geq 750$, et la même borne lorsque $\Delta \geq 47$ a ensuite été montrée par Borodin *et al.* [BBG+01]. Molloy et Salavatipour [MS05] ont prouvé que $\chi(G^2) \leq \lceil \frac{5}{3}\Delta \rceil + 78$, et ont montré que la constante 78 pouvait être réduite lorsque Δ était suffisamment grand.

Récemment, Havet *et al.* ont montré le théorème suivant :

Théorème 3 [HHM+07] *Pour tout p fixé et pour tout graphe planaire G de degré maximum Δ, on a $\lambda_{p,1}(G) \leq \left(\frac{3}{2} + o(1)\right)\Delta$.*

En prenant $p = 1$, cela implique que le carré de tout graphe planaire de degré maximum Δ admet une coloration propre avec au plus $\left(\frac{3}{2}+o(1)\right)\Delta$ couleurs, ce qui améliore le résultat de Molloy et Salavatipour [MS05]. Notre but dans le **Chapitre 2** est d'étendre leur approche à une famille plus large de colorations à distance deux.

Une *coloration cyclique* d'un graphe planaire G (dont le dessin dans le plan est fixé) est une coloration des sommets de G telle que toute paire de sommets incidents à la même face reçoive des couleurs différentes. Le nombre minimum de couleurs dans une coloration cyclique de G est appelé le *nombre chromatique cyclique* de G, noté $\chi^*(G)$. Si on note $\Delta^*(G)$ la taille (nombre de sommets) de la plus grande face de G, il est clair que

$\chi^*(G) \geq \Delta^*(G)$ pour tout graphe planaire G. Ore and Plummer [OP69], qui ont introduit la notion de coloration cyclique, ont également montré que pour tout graphe planaire G, on a $\chi^*(G) \leq 2\,\Delta^*(G)$. Borodin [Bor84] (voir également Jensen et Toft [JT95, page 37]) a proposé la conjecture suivante :

Conjecture 4 [Bor84] *Pour tout graphe planaire G, on a*

$$\chi^*(G) \leq \left\lfloor \tfrac{3}{2}\,\Delta^*(G) \right\rfloor.$$

Il a donné des exemples montrant que cette borne était atteinte et a prouvé la conjecture pour $\Delta^* = 4$. Pour des valeurs générales de Δ^*, la borne originale $\chi^*(G) \leq 2\,\Delta^*(G)$ d'Ore et Plummer [OP69] a été améliorée par Borodin *et al.* [BSZ99], qui ont montré $\chi^*(G) \leq \left\lfloor \tfrac{9}{5}\,\Delta^*(G) \right\rfloor$. La meilleure borne connue dans le cas général est due à Sanders et Zhao [SZ01] : $\chi^*(G) \leq \left\lceil \tfrac{5}{3}\,\Delta^*(G) \right\rceil$.

En étudiant ces colorations, il apparaît non seulement que les conjectures de Wegner et de Borodin ont une ressemblance frappante, mais aussi que les techniques utilisées pour obtenir des bornes sur la coloration du carré et sur la coloration cyclique sont similaires. Pourtant, aucun lien direct permettant de relier les deux colorations n'a été trouvé jusqu'à présent.

Dans le **Chapitre 2**, on introduit une notion qui unifie la coloration du carré et la coloration cyclique des graphes planaires, et on utilise des idées de [HHM$^+$07] pour prouver un résultat général [AEH08] impliquant que :

- tout graphe planaire G admet une coloration cyclique avec au plus $\left(\tfrac{3}{2} + o(1)\right) \Delta^*(G)$ couleurs ;
- tout graphe planaire G admet une coloration de son carré avec au plus $\left(\tfrac{3}{2} + o(1)\right) \Delta(G)$ couleurs.

Notre preuve est légèrement plus directe que la preuve de [HHM$^+$07], et améliore le résultat de Sanders et Zhao [SZ01]. De plus, notre résultat améliore également la meilleure borne connue sur la taille d'une clique maximale dans le carré d'un graphe planaire. Comme dans [HHM$^+$07], on réduit le problème à un problème de coloration par listes des arêtes d'un multigraphe, et on utilise ensuite le fait que l'indice chromatique par listes est proche de l'indice chromatique fractionnaire.

On a vu dans ce qui précède que le $L(p, q)$-étiquetage peut être considéré comme une généralisation de la coloration du carré. Il existe une autre généralisation qui permet d'établir des liens entre la coloration du carré et la coloration cyclique des graphes planaires. Une *coloration p-frugale* d'un graphe G est une coloration propre des sommets de G telle qu'aucune couleur n'apparaît plus de p fois dans le voisinage d'un sommet. Le *nombre chromatique p-frugal* de G, noté $\chi_p(G)$, est le nombre minimum de couleurs dans une coloration p-frugale de G.

Cette coloration a été introduite par Hind, Molloy et Reed [HMR97] dans le but de montrer des résultats sur la coloration totale des graphes. Une *coloration totale* d'un graphe G est une coloration des sommets et des arêtes de G telle que (i) toute paire de sommets adjacents reçoive des couleurs distinctes, (ii) toute paire d'arêtes incidentes reçoive des couleurs distinctes, et (iii) la couleur d'une arête est distincte des couleurs de ses extrémités. Le nombre minimum de couleurs dans une coloration totale de G est appelé le *nombre chromatique total* de G, noté $\chi^T(G)$. À la fin des années 60, Behzad [Beh65] et Vizing [Viz68] ont proposé de manière indépendante la conjecture suivante :

Conjecture 5 (Conjecture de la Coloration Totale)
Pour tout graphe G de degré maximum Δ, $\chi^T(G) \leq \Delta + 2$.

Hind, Molloy et Reed [HMR97] ont prouvé que tout graphe de degré maximum Δ suffisamment grand admet une coloration $(\log^8\Delta)$-frugale avec au plus $\Delta + 1$ couleurs, et ont utilisé ce résultat pour en déduire que tout graphe de degré maximum Δ suffisamment grand admet une coloration totale avec $\Delta + \log^{10}\Delta$ couleurs [HMR99].

Une coloration p-frugale peut aussi être vue comme une coloration propre dans laquelle toute paire de classes de couleurs induit une graphe (biparti) de degré maximum au plus p. Le cas $p = 1$ étant équivalent à la coloration du carré, il est intéressant de voir de quelle manière la conjecture de Wegner se généralise à la coloration p-frugale des graphes planaires. Dans le **Chapitre 3** on propose la conjecture suivante :

Conjecture 6 [AEH07] *Pour tout entier $p \geq 1$ et tout graphe planaire G de degré maximum $\Delta \geq \max\{2p, 8\}$ on a*

$$\chi_p(G) \leq \begin{cases} \left\lfloor \frac{\Delta-1}{p} \right\rfloor + 2, & \textit{si } p \textit{ est pair ;} \\ \left\lfloor \frac{3\Delta-2}{3p-1} \right\rfloor + 2, & \textit{si } p \textit{ est impair.} \end{cases}$$

On prouve également des résultats sur les graphes planaires, les graphes planaires de maille bornée, et les graphes planaire-extérieurs [AEH07]. Pour cela, on montre qu'il existe des connections reliant la coloration frugale, le $L(p,q)$-étiquetage, et la coloration cyclique des graphes.

Une *arête coloration p-frugale* d'un multigraphe G est une coloration (potentiellement impropre) des arêtes de G telle qu'aucune couleur n'apparaît plus de p fois parmi les arêtes incidentes à un sommet. Le nombre minimum de couleurs dans une arête coloration p-frugale de G est appelé l'*indice chromatique p-frugal* de G, noté $\chi'_p(G)$. On peut observer qu'une arête coloration 1-frugale correspond exactement à une coloration propre des arêtes, on a donc $\chi'(G) = \chi'_1(G)$ pour tout graphe G.

Hilton *et al.* [HSS01] ont prouvé que lorsque p est pair, tout graphe vérifie $\chi'_p(G) = \left\lceil \frac{1}{p}\,\Delta(G) \right\rceil$. Dans le **Chapitre 3** on montre que quand p est impair, tout multigraphe G vérifie $\chi'_p(G) \leq \left\lceil \frac{3\,\Delta(G)}{3\,p-1} \right\rceil$.

Lorsque $p = 2$, une coloration p-frugale des sommets d'un graphe G est une coloration propre telle que l'union de toute paire de classes de couleurs est un graphe de degré maximum au plus deux (une union disjointe de chaînes et de cycles). De manière surprenante, il y a peu de différences si l'on autorise seulement l'union de toute paire de classes de couleurs à être une forêt de chaînes. Une *coloration linéaire* d'un graphe G est définie comme une coloration propre des sommets de G telle que l'union de toute paire de classes de couleurs est une forêt de chaînes (une forêt de degré maximum au plus deux).

Cette coloration, équivalente à une coloration acyclique et 2-frugale, a été introduite par Yuster [Yus98], qui a prouvé que tout graphe de degré maximum Δ admet une coloration linéaire avec $O(\Delta^{\frac{3}{2}})$ couleurs (la même borne avait été montrée dans le cas de la coloration 2-frugale par Hind *et al.* [HMR97]). Dans le **Chapitre 4**, on étudie plusieurs classes de graphes, comme les graphes de degré borné, les graphes planaires, les graphes planaires de degré moyen maximum borné, et les graphes planaire-extérieurs [EMR08], et on obtient (la plupart du temps) des résultats assez proches des résultats obtenus pour la coloration 2-frugale dans le **Chapitre 3**. On étudie également la complexité de la coloration linéaire : on montre que déterminer si un graphe planaire biparti de degré maximum trois admet une coloration linéaire avec au plus trois couleurs est un problème NP-complet.

Pour tout graphe G, soit $G^\star$ le graphe d'incidence de G, c'est-à-dire le graphe obtenu à partir de G en remplaçant chaque arête par une chaîne de longueur (nombre d'arêtes) deux. On peut remarquer que les colorations à distance deux dans les graphes d'incidence ont une signification particulière : pour tout graphe G, la coloration du carré de $G^\star$ est par exemple équivalente à une coloration totale de G.

Un $L(p,1)$-étiquetage de $G^\star$ correspond à une assignation d'entiers aux sommets de G telle que (i) toute paire de sommets adjacents reçoive des entiers distincts, (ii) toute paire d'arêtes incidentes reçoive des entiers distincts, et (iii) les entiers assignés à une arête et à ses extrémités sont distants d'au moins p. Cet étiquetage est appelé un *$(p,1)$-étiquetage total* de G, et le plus petit entier t tel qu'il existe un $(p,1)$-étiquetage total de G utilisant des étiquettes de $\{1, \dots, t\}$ est appelé le *nombre* $(p, 1)$-*total* $\lambda_p^T(G)$ de G. Cette notion a été introduite par Havet et Yu [HY08], et correspond exactement à la coloration totale quand $p = 1$. Havet et Yu ont proposé une conjecture qui généralise la Conjecture de la Coloration Totale :

Conjecture 7 [HY08] *Si G est un graphe de degré maximum Δ, on a $\lambda_p^T(G) \leq \Delta + 2p$.*

Dans le **Chapitre 5** on étudie le nombre $(p, 1)$-total des graphes clairsemés et on montre que pour tout $0 < \varepsilon < \frac{1}{2}$, et pour tout entier p, il existe une constante $C_{p,\varepsilon}$ telle que tout graphe $\varepsilon\Delta$-clairsemé G de degré maximum Δ vérifie $\lambda_p^T(G) \leq \Delta + C_{p,\varepsilon}$ [EMR06]. Cela implique notamment que les graphes aléatoires du modèle Erdös-Rényi satisfont cette propriété avec une probabilité tendant vers 1 lorsque Δ tend vers l'infini.

Nous avons également étudié les colorations à distance deux sous l'angle d'un jeu à deux joueurs. Alice et Bob colorient chacun leur tour et de manière propre le carré d'un graphe (à chaque étape, toute paire de sommets à distance au plus deux doit avoir des couleurs distinctes). Si le jeu s'arrête avant que tous les sommets ne soient coloriés, Bob est le vainqueur et sinon c'est Alice qui gagne. Dans le **Chapitre 6**, on étudie des stratégies gagnantes pour Alice dans les arbres, les graphes planaires-extérieurs, les 2-arbres partiels, et les graphes planaires [EZ08].

On peut remarquer qu'une stratégie gagnante dans un graphe G ne l'est pas nécessairement dans un sous-graphe H de G. De plus, avoir une stratégie gagnante avec k couleurs ne garantit pas qu'il existe une stratégie gagnante avec $k+1$ couleurs. Pour ces raisons, l'étude de ce jeu à deux joueurs nécessite d'utiliser des techniques de preuve profondément différentes des techniques utilisées dans les chapitres précédents.

Dans le chapitre final, on montre comment utiliser des techniques de coloration à distance deux pour obtenir des informations sur la structure des graphes. La *boxicité* d'un graphe $G = (V, E)$ est le plus petit entier k pour lequel il existe k graphes d'intervalle $G_i = (V, E_i)$, $1 \leq i \leq k$, tels que $E = E_1 \cap \ldots \cap E_k$. Les graphes de boxicité au plus d sont exactement les graphes d'intersection de boîtes en dimension d. La boxicité des graphes a été introduite par Roberts [Rob69] et a de nombreuses applications dans les réseaux sociaux et dans les réseaux écologiques. Le cas $d = 2$ correspond également à un problème de gestion de parc automobile.

Dans le **Chapitre 7**, on utilise une coloration à distance deux spécifique pour montrer que les graphes de degré maximum Δ ont une boxicité au plus $\Delta^2 + 2$ [Esp08].

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En annexe, nous ajoutons à ce mémoire des articles sur la coloration orientée des graphes planaire-2-extérieurs [EO07a], la densité des graphes de cordes de maille au moins cinq [EO07b], les graphes universel-induits [ELO07], la coloration acyclique impropre des graphes de degré maximum borné [AEK+07], et la coloration adaptable des graphes planaires [EMZ08].

Introduction

One of the main motivations of the work presented in this manuscript is *the channel assignment problem*: in a radio or mobile phone network, we need to assign radio frequency bands to transmitters (every station is assigned an integer, which corresponds to a specific channel). In order to minimize interference, the separation between the channels assigned to two stations that are very close must be sufficiently large. Additionally, two stations that are close (but not very close) must also receive channels that are sufficiently far apart.

This problem may be modelled by *$L(p,q)$-labellings* of graphs, first introduced by Griggs and Yeh [GY92]. An $L(p,q)$-labelling of a graph G is an assignment of integers to the vertices of G in such way that any two adjacent vertices receive integers that differ by at least p, and any two vertices at distance two receive integers that differ by at least q. We often assume that $p \geq q$, since very close stations are more subject to interference than close stations.

The *$\lambda_{p,q}$-number* of G, denoted by $\lambda_{p,q}(G)$, is the smallest t such that there exists an $L(p,q)$-labelling of G using labels from $\{1, 2, \ldots, t\}$. Observe that an $L(0,1)$-labelling of a graph G is equivalent to a proper coloring of G, so $\lambda_{1,0}(G) = \chi(G)$. Define the *square* G^2 of a graph $G = (V,E)$ as the graph with vertex set V in which two vertices are adjacent if they are at distance at most two in G, then an $L(1,1)$-labelling of a graph G is exactly a proper coloring of G^2, thus $\lambda_{1,1}(G) = \chi(G^2)$.

In general, it is NP-hard to determine the $\lambda_{p,q}$-number of a graph [GMW94]. However, general bounds can be given for specific classes of graphs. In **Chapter 2** we will give details about $L(p,q)$-labellings of planar graphs, in **Chapter 3**, we will use existing results on $L(p,q)$-labellings of planar graphs with bounded girth, and in **Chapter 5**, we will study $L(p,q)$-labellings of incidence graphs. For a survey on $L(p,q)$-labellings of graphs, the reader is referred to [Cal06].

For a graph G with maximum degree Δ, it is easy to see that a greedy algorithm gives the bound $\lambda_{2,1}(G) \leq \Delta^2 + 2\Delta + 1$. Griggs and Yeh conjectured the following:

Conjecture 1 [GY92] *For every graph G with maximum degree $\Delta \geq 2$, we have $\lambda_{2,1}(G) \leq \Delta^2 + 1$.*

This bound would be tight since for $\Delta = 2, 3, 7$ there exist graphs with diameter two, maximum degree Δ, and order $\Delta^2 + 1$. This conjecture was recently proved for large enough Δ by Havet *et al.* [HRS08] using probabilistic techniques.

Since transmitters are in general spread over the surface of the earth, a particular interest has been shown over the last decade for $L(p,q)$-labellings of planar graphs. For the case $p = q = 1$, its is known for more that thirty years that there exist planar graphs G_Δ with maximum degree Δ such that $\chi(G_\Delta^2) = \lfloor \frac{3}{2}\Delta \rfloor + 1$. Wegner [Weg77] conjectured that this is optimal:

Conjecture 2 [Weg77] *For any planar graph G of maximum degree $\Delta \geq 8$ we have $\chi(G^2) \leq \left\lfloor \frac{3}{2}\,\Delta \right\rfloor + 1$.*

The first upper bound on $\chi(G^2)$ for planar graphs in terms of Δ, $\chi(G^2) \leq 8\,\Delta - 22$, was implicit in the work of Jonas [Jon93]. This bound was later improved by Wong [Won96] to $\chi(G^2) \leq 3\,\Delta + 5$ and then by Van den Heuvel and McGuinness [HM03] to $\chi(G^2) \leq 2\,\Delta + 25$. Better bounds were then obtained for large values of Δ. It was shown that $\chi(G^2) \leq \lceil \frac{9}{5}\,\Delta \rceil + 1$ for $\Delta \geq 750$ by Agnarsson and Halldórsson [AH00], and the same bound for $\Delta \geq 47$ by Borodin *et al.* [BBG$^+$01]. Molloy and Salavatipour [MS05] proved that $\chi(G^2) \leq \lceil \frac{5}{3}\,\Delta \rceil + 78$, and showed that the constant 78 could be reduced for sufficiently large Δ. For example, it was improved to 24 when $\Delta \geq 241$.

Recently, Havet *et al.* proved the following:

Theorem 3 [HHM$^+$07] *For any fixed p, and any planar graph G of maximum degree Δ, we have $\lambda_{p,1}(G) \leq \left(\frac{3}{2} + o(1)\right)\Delta$.*

If we take $p = 1$, this theorem implies that the square of any planar graph with maximum degree Δ can be colored with $\left(\frac{3}{2} + o(1)\right)\Delta$ colors, which improves the result of Molloy and Salavatipour [MS05]. Our aim in **Chapter 2** is to extend their approach to a wider family of distance-two colorings.

A *cyclic coloring* of a plane graph G (a planar graph with a prescribed embedding) is a vertex coloring of G such that any two vertices incident to the same face have distinct colors. The minimum number of colors required in a cyclic coloring of a plane graph G is called the *cyclic chromatic number* $\chi^*(G)$. Denote by $\Delta^*(G)$ the size (number of vertices in its boundary) of a largest face of G. It is clear that $\chi^*(G) \geq \Delta^*(G)$ for any plane graph G. Ore and Plummer [OP69], who introduced the concept of cyclic coloring, also proved that for any plane graph G, we have $\chi^*(G) \leq 2\,\Delta^*(G)$. Borodin [Bor84] (see also Jensen and Toft [JT95, page 37]) conjectured the following:

Conjecture 4 [Bor84] *For a plane graph G of maximum face degree Δ^* we have $\chi^*(G) \leq \left\lfloor \frac{3}{2}\,\Delta^* \right\rfloor$.*

He gave examples showing that this would be best possible and also proved Conjecture 4 for $\Delta^* = 4$. For general values of Δ^*, the original bound $\chi^*(G) \leq 2\,\Delta^*$ of Ore and Plummer [OP69] was improved by Borodin *et al.* [BSZ99] to $\chi^*(G) \leq \left\lfloor \frac{9}{5}\,\Delta^* \right\rfloor$. The best known upper bound in the general case is due to Sanders and Zhao [SZ01]: $\chi^*(G) \leq \left\lceil \frac{5}{3}\,\Delta^* \right\rceil$.

The main point is that not only Wegner's and Borodin's conjectures look the same, but the proof techniques used in order to obtain bounds on the chromatic number of the square and the cyclic chromatic number are very similar. However, it seems that no one ever found a direct connection between these two colorings.

In **Chapter 2**, we introduce a notion that unifies colorings of the square and cyclic colorings of plane graphs, and then use ideas from [HHM+07] to prove a general result [AEH08] implying that

- every planar graph G admits a cyclic coloring with at most $\left(\frac{3}{2} + o(1)\right)$ $\Delta^*(G)$ colors;
- every planar graph G admits a coloring of its square with at most $\left(\frac{3}{2} + o(1)\right)\Delta(G)$ colors.

Our proof is slightly more direct than the proof of [HHM+07], and improves the result of Sanders and Zhao [SZ01]. Besides, our result also improves the best known bound on the size of a largest clique in the square of a planar graph. As in [HHM+07], we reduce the problem to a list edge coloring problem, and then use the fact that the list chromatic index is close from the fractional chromatic index.

Another way to relate cyclic coloring and coloring of the square of plane graphs is through frugal coloring. A *p-frugal coloring* of a graph G is a proper coloring of the vertices of G such that no color appears more than p times in the neighborhood of a vertex. The p-frugal chromatic number of G, denoted $\chi_p(G)$, is the smallest number of colors in a p-frugal coloring of G.

This coloring was introduced by Hind, Molloy and Reed [HMR97] in order to obtain bounds on the total coloring of graphs. A *total coloring* of a graph G is a coloring of the vertices and edges of G so that (i) any two adjacent vertices have distinct colors, (ii) any two incident edges have distinct colors, and (iii) the color of any edge is distinct from the colors of its ends. The minimum number of colors in a total coloring of G is called the *total chromatic number* of G, denoted $\chi^T(G)$. In the late sixties, Behzad [Beh65] and Vizing [Viz68] independently proposed the following conjecture:

Conjecture 5 (The Total Coloring Conjecture) *For any graph G with maximum degree Δ, $\chi^T(G) \leq \Delta + 2$.*

Hind, Molloy and Reed [HMR97] proved that any graph with large enough maximum degree Δ has a $(\log^8 \Delta)$-frugal coloring using at most $\Delta + 1$ colors, and used this result to prove that any graph with large enough maximum degree Δ has a total coloring with $\Delta + \log^{10} \Delta$ colors [HMR99].

A p-frugal coloring can also be seen as a proper coloring such that any two color classes induce a (bipartite) graph with maximum degree p. The case $p = 1$ is equivalent to a coloring of the square of G, so it is interesting to see how Wegner's conjecture can be generalized to frugal coloring of planar graphs. In **Chapter 3** we propose the following conjecture:

Conjecture 6 [AEH07] *For any integer $p \geq 1$ and planar graph G with maximum degree $\Delta \geq \max\{2p, 8\}$ we have*

$$\chi_p(G) \leq \begin{cases} \left\lfloor \frac{\Delta - 1}{p} \right\rfloor + 2, & \text{if } p \text{ is even;} \\ \left\lfloor \frac{3\Delta - 2}{3p - 1} \right\rfloor + 2, & \text{if } p \text{ is odd.} \end{cases}$$

We also prove results on planar graphs, planar graphs with given girth, and outerplanar graphs [AEH07]. To show these results, we relate frugal coloring with $L(p, q)$-labelling of graphs and cyclic coloring of plane graphs.

A *p-frugal edge coloring* of a multigraph G is a (possibly improper) coloring of the edges of G such that no color appears more than p times on the edges incident with a vertex. The least number of colors in a p-frugal edge coloring of G, the *p-frugal chromatic index* of G, is denoted by $\chi'_p(G)$. Remark that for $p = 1$ we have $\chi'_1(G) = \chi'(G)$, the usual chromatic index of G.

Hilton *et al.* [HSS01] proved that for even p, any multigraph G satisfies $\chi'_p(G) = \left\lceil \frac{1}{p} \Delta(G) \right\rceil$. In **Chapter 3** we prove that for odd p, any multigraph G satisfies $\chi'_p(G) \leq \left\lceil \frac{3\Delta(G)}{3p-1} \right\rceil$, which is optimal.

When $p = 2$, a p-frugal coloring of the vertices of a graph G is such that the union of any two color classes is a graph with maximum degree two (a union of paths and cycles). Surprisingly, there are only few differences if we only allow the union of any two color classes to be a union of paths: define a *linear coloring* of a graph G as a proper coloring of the vertices of G such that the subgraph induced by any two color classes is a forest of paths (a forest with maximum degree at most two), then a linear coloring is exactly an acyclic and 2-frugal coloring.

This coloring was introduced by Yuster [Yus98], who proved that any graph with maximum degree Δ has a linear coloring with $O(\Delta^{\frac{3}{2}})$ colors (the same bound was proven by Hind *et al.* for 2-frugal coloring in [HMR97]). In **Chapter 4**, we study several classes of graphs, such as planar graphs, planar graphs with bounded maximum average degree, outerplanar graphs [EMR08], and we obtain bounds which are (most of the time) close from the bounds obtained for the 2-frugal chromatic number in [AEH07]. We also study complexity aspects of linear coloring: we show that deciding whether a bipartite planar graph with maximum degree three admits a linear coloring with three colors is an NP-complete problem.

For a graph G, let $G^\star$ be the incidence graph of G, that is the graph obtained from G by inserting one vertex along each edge. Observe that an $L(p,1)$-labelling of $G^\star$ corresponds to an assignment of integers to the vertices and edges of G such that two adjacent vertices have distinct integers, any two incident edges have distinct integers, and the difference between the integer assigned to an edge and the integers assigned to its ends is at least p. This coloring is called a $(p,1)$-*total labelling* of G, and the smallest t such that there exists a $(p,1)$-total labelling of G using labels from $\{1,2,\ldots,t\}$ is the $(p,1)$-*total number* $\lambda_p^T(G)$ of the graph G. This coloring was introduced by Havet and Yu [HY08], and corresponds exactly to the notion of total coloring when $p=1$. Havet and Yu proposed the following conjecture, which generalizes the total coloring conjecture:

Conjecture 7 [HY08] *Let G be a graph with maximum degree Δ, then $\lambda_p^T(G) \leq \Delta + 2p$.*

In **Chapter 5** we study the $(p,1)$-total number of sparse graphs and prove that for any $0 < \varepsilon < \frac{1}{2}$, and for any integer p, there exists a constant $C_{p,\varepsilon}$ such that every $\varepsilon\Delta$-sparse graph G with maximum degree Δ satisfies $\lambda_p^T(G) \leq \Delta + C_{p,\varepsilon}$ [EMR06]. This implies that Erdös-Rényi random graphs satisfy this property asymptotically almost surely.

Consider a two player game in which Alice and Bob alternatively color the square of a graph G properly (that is, at any step, any two vertices at distance at most two in G have distinct colors). If the game stops before all the vertices are colored, Bob wins and otherwise Alice wins. In **Chapter 6**, we study winning strategies for Alice in trees, outerplanar graphs, partial 2-trees, and planar graphs [EZ08].

Observe that if we have a winning strategy for a graph G, we cannot necessarily use it to obtain a winning strategy in a subgraph H of G. Furthermore, having a winning strategy with k colors for a graph G does not mean that we have a strategy with $k+1$ colors for G. As a consequence, we have to use completely different techniques than the one used in the previous chapters.

In the final chapter, we show how to use distance-two colorings to obtain specific information on the structure of graphs. The *boxicity* of a graph $G = (V, E)$ is the smallest integer k for which there exist k interval graphs $G_i = (V, E_i)$, $1 \leq i \leq k$, such that $E = E_1 \cap \ldots \cap E_k$. Graphs with boxicity at most d are exactly the intersection graphs of (axis-parallel) d-dimensional boxes. Boxicity of graphs has been introduced by Roberts [Rob69] and has several applications in social networks and ecology. The case $d = 2$ also corresponds to a fleet maintenance problem.

In **Chapter 7**, we use a specific distance-two coloring to prove that graphs with maximum degree Δ have boxicity at most $\Delta^2 + 2$ [Esp08].

*****

In appendix, we add articles about oriented coloring of 2-outerplanar graphs [EO07a], the density of circle graphs with girth at least five [EO07b], induced-universal graphs [ELO07], acyclic improper colorings [AEK+07], and adapted coloring of planar graphs [EMZ08].

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Chapter 1

Preliminaries

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1.1 Graph theory

Most of the terminology and notation we use in this thesis is standard and can be found in any text book on graph theory (such as [BM76] or [Die05]). For the French terminology, please refer to [Ber69].

1.1.1 Basic definitions

A *graph* is a pair $G = (V(G), E(G))$ of sets, such that $E(G) \subseteq \{\{x, y\}, x, y \in V(G)\}$. The elements of $V(G)$ are called the *vertices* of G, whereas the elements of $E(G)$ are called the *edges* of G. We usually write xy or yx instead of $\{x, y\}$ when considering an edge. If $e = xy$ is an edge of a

$G = (V(G), E(G))$

graph G, the vertices x and y are said to be *incident with* or *to* the edge e. The two vertices incident to an edge e are called the *end points*, or *end vertices* of e. Two vertices x and y are *adjacent* or *neighbors* in a graph G if xy is an edge of G. Two edges $e \neq f$ are said to be *incident* if they have a common end vertex.

The number of vertices of a graph G is called the *order* of G. Most of the graphs we consider in this thesis are *finite* (they have finite order), and *simple* : for any edge xy, $x \neq y$ (we say that there are no loops) and for any two vertices x and y, there is at most one edge xy (we say that there are no multiple edges). Such requirements correspond exactly to the definition of graphs given above. In Chapters 2 and 3, however, we will study *multigraphs* (graphs with multiple edges). The only difference is that in this case, $E(G)$ is a multiset (instead of a set).

A subset U of vertices of a graph G is called a *stable* or *independent set* if any two vertices of U are non adjacent in G. If any two vertices of U are adjacent in G, the set U is called a *clique* of G.

1.1.2 Relations between graphs

We say that $\varphi : V(G) \to V(H)$ is a *homomorphism* between G and H, if for every edge xy of G, $\varphi(x)\varphi(y)$ is an edge of H. The existence of a
$G \to H$ homomorphism between G and H is denoted by $G \to H$.

Two graphs G and H are said to be *isomorphic* if there exists a bijective homomorphism between G and H. Usually, we do not make any distinction between isomorphic graphs. In other words, when considering a graph G, we implicitly consider the equivalence class for the relation *being isomorphic to* containing the graph G.

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. If $V \subseteq V'$ and
$G \subseteq G'$ $E \subseteq E'$ we say that G is a *subgraph* of G', denoted by $G \subseteq G'$. If $G \subseteq G'$ and G contains all the edges $xy \in E'$ with $x, y \in V$, we say that G is the subgraph of G' *induced* by V, or more simply that G is an *induced*
$G[X]$ *subgraph* of G', and we denote this by $G = G'[V]$. If $G \subseteq G'$ and $V = V'$, we say that G is a *spanning subgraph* of G'.

We now define basic operations on graphs. Let G be a graph and U
$G - U$ be a subset of vertices of G. We denote by $G - U$ the graph obtained from G by removing all the vertices from U as well as the edges incident to any vertex of U. Observe that $G - U$ is the subgraph of G induced by $V(G)\backslash U$. If U is a single vertex u, we write $G - u$ instead of $G - \{u\}$. Let
$G - F$ F be a subset of edges of G, we denote by $G - F$ (or $G - f$ if $F = \{f\}$) the graph obtained from G by removing all the edges from F. We call

these two operations the *deletion* of vertices and edges from G.

Let $e = xy$ be an edge of a graph G. We denote by G/e the graph obtained from G by deleting the vertices x and y and adding a vertex z adjacent to all the neighbors of x or y in G. This operation is called the *contraction* of the edge e. G/e

If a graph G can be obtained from a subgraph of H by a sequence of edge contractions, we call G a *minor* of H, denoted by $G \preceq H$. $G \preceq H$

1.1.3 Degree and neighborhood

Let G be a non-empty graph and x be a vertex of G. The set of vertices adjacent to x in G is called the *neighborhood* of x, denoted $N_G(x)$ or $N(x)$ when the graph G is clear from the context. The number of neighbors of the vertex x in G is called the *degree* of x in G, denoted $d_G(x)$ or $d(x)$ when G is clear from the context. $N(x)$ $d(x)$

We call *k-vertex* (resp. *$^{\le}k$-vertex*, *$^{\ge}k$-vertex*) a vertex of degree k (resp. at most k, at least k). If for some k, all the vertices of G are k-vertices, then G is said to be *k-regular*, or *regular*. A 3-regular graph is also called a *cubic* graph.

The value $\delta(G) = \min\{d(x), x \in V(G)\}$ is called the *minimum degree* of G and the value $\Delta(G) = \max\{d(x), x \in V(G)\}$ is called the *maximum degree* of G. Let n and m be the order and the number of edges of G. The value $\mathrm{ad}(G) = \sum_{v \in V(G)} d(v)/n = 2m/n$ is called the *average degree* of G. The *maximum average degree* of G, denoted by $\mathrm{mad}(G)$, is the maximum of $\mathrm{ad}(H)$ over all subgraphs H of G. $\delta(G)$ $\Delta(G)$ $\mathrm{ad}(G)$ $\mathrm{mad}(G)$

If for some integer k, any subgraph H of G is such that $\delta(H) \le k$, then G is said to be *k-degenerate*. Observe that every graph G is $\lfloor \mathrm{mad}(G) \rfloor$-degenerate, and every k-degenerate graph has maximum average degree at most $2k$.

1.1.4 Distance

A *path* P is a graph with vertex set $V = \{x_0, x_1, \ldots, x_k\}$ and edge set $E = \{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k\}$, where all the x_i are distinct vertices and $k \ge 0$ is an integer. We often write $P = x_0x_1 \ldots x_k$ to denote such a path, and say that P is path *between x_0 and x_k* (resp. *between x_k and x_0*), or *from x_0 to x_k* (resp. *from x_k to x_0*). The number of edges in a path is called the *length* of the path. A path of length k is denoted by P_k. P_k

The graph obtained from a path $P = x_0x_1 \ldots x_{k-1}$ by adding an edge between x_0 and x_{k-1} is called a *cycle* of length k, denoted by C_k. We also C_k

call *k-cycle* (resp. *${}^{\leq}k$-cycle*, *${}^{\geq}k$-cycle*) a cycle of length k (resp. at most
$g(G)$ k, at least k). The *girth* $g(G)$ of a graph G is the length of a shortest cycle contained by G. If G does not contain any cycle, we set $g(G)$ to be infinite. An edge joining two non-consecutive vertices of a cycle is called a *chord*. An *induced cycle* in a graph G is a chordless cycle of G (that is, a cycle which is an induced subgraph of G).

$d(x, y)$ The *distance* $d_G(x, y)$ or $d(x, y)$ of two vertices x and y in G is the length of a shortest path between x and y in G (if such a path does not exist, we set $d(x, y)$ to be infinite). Given a graph G, the *square of* G,
G^2 denoted G^2, is the graph having the same vertex set as G, with an edge between any two different vertices that have distance at most two in G (see Figure 1.1).

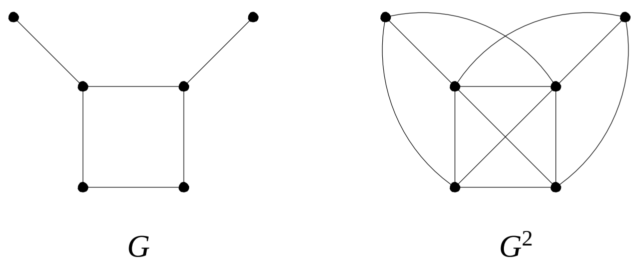

Figure 1.1: The square of G.

1.1.5 Connectivity

Let G be a non-empty graph. If for any two vertices x and y of G, there is a path in G between x and y, then G is said to be *connected*. A maximal connected subgraph of G is called a *component* of G. If a vertex x of G is such that $G - x$ has more components than G, then x is said to be a *cut-vertex* of G. If an edge e of G is such that $G - e$ has more components than G, then e is said to be a *bridge* of G.

A graph G is said to be *k-connected* if for some integer $k \geq 1$, G has at least $k + 1$ vertices and the graph $G - X$ is connected for any set X of at most $k - 1$ vertices of G.

1.1.6 Trees and bipartite graphs

A graph without cycles is called a *forest*, and a connected forest is called a *tree*. A vertex of degree 1 in a tree is called a *leaf*. Observe that a path $P = x_0x_1 \ldots x_k$ is a tree with exactly two leaves: x_0 and x_k. Sometimes we distinguish one vertex of a tree, and call it *the root*. In this case, we say that we consider a *rooted tree*.

A graph G is *bipartite* if its set of vertices can be partitioned into two sets V and V', such that every edge of G has one end point in V and the other one in V'. Observe that forests are bipartite. A bipartite graph is said to be a *complete bipartite* graph if it contains all possible edges between the two sets V and V' of the bipartition. The complete bipartite graph with m vertices in the first set and n vertices in the second set is denoted by $K_{m,n}$. $K_{m,n}$

1.1.7 Some classes of graphs

In this subsection, we define some classes of graphs that will be studied throughout this thesis.

The graph with n vertices and all possible edges is called the *complete graph* of order n, denoted by K_n. K_n

A *plane graph* is a graph drawn in the plane in such a way that there is no crossing of edges. A *planar graph* is a graph that admits a drawing in the plane with this property. An *outerplanar graph* is a planar graph that can be drawn in the plane without crossing of edges, in such a way that every vertex lies on the outer face.

A graph is *chordal* if it contains no cycle of length at least four as an induced subgraph. A *clique* of G is a set a pairwise adjacent vertices of G. For any integer $k \geq 1$, a *k-tree* is a chordal graph in which every (inclusion-) maximal clique as order exactly $k+1$. A *partial k-tree* is a subgraph of a k-tree. For example, the class of partial 2-trees is exactly the class of graphs which do not contain the complete graph K_4 as a minor.

The *treewidth* of a graph G, denoted by $\mathrm{tw}(G)$, is the smallest integer k such that G is a partial k-tree. $\mathrm{tw}(G)$

1.2 Graph coloring

For some integer $k \geq 1$, a *(proper) k-coloring* of the vertices of G is a map $c: V(G) \rightarrow \{1, \ldots, k\}$ such that for every edge xy of G, $c(x) \neq c(y)$. The elements from $\{1, \ldots, k\}$ are called *colors*, and the set of all vertices colored with a specific color is called a *color class*. Observe that a proper coloring of a graph is a partition of its set of vertices into color classes, each of which is an independent set. If a graph admits a k-coloring, it is said to be *k-colorable*. The smallest k such that a graph G is k-colorable is called the *chromatic number* of G, denoted by $\chi(G)$. $\chi(G)$

A *list assignment* $L : V(G) \to 2^{\mathbb{N}}$ on the vertices of a graph is a map which assigns to each vertex v of the graph a list $L(v)$ of prescribed integers. If for some integer t, every list has size at least t, then L is called a *t-list assignment.*

Let L be a list assignment on the vertices of a graph G. A coloring c of the vertices of G such that for any vertex v, $c(v) \in L(v)$ is called an *L-coloring* of G. If such a coloring exists, then G is said to be

$ch(G)$ *L-colorable.* The *list chromatic number* or *choice number* $ch(G)$ is the minimum value t, so that for every t-list assignment L on the vertices of G, the graph G is L-colorable.

The concept of choosability was introduced by Vizing [Viz76], and Erdös, Rubin, and Taylor [ERT79]. This generalization of the notion of coloring has been applied to various problems, especially to the field of coloring under constraints ((a, b)-choosability [Tuz97], k-improper l-choosability [EH99, Skr99], acyclic choosability [BFK$^+$02]).

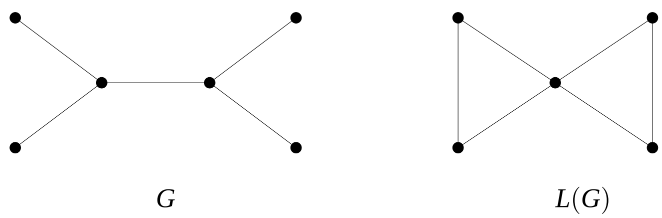

Figure 1.2: The line graph of G.

$L(G)$ For any graph $G = (V, E)$, we define the *line graph* $L(G)$ of G to be the graph with vertex set E, where two vertices $u, v \in E$ are adjacent in $L(G)$ if and only if the corresponding edges are incident in G (see Figure 1.2 for an example).

The smallest integer k, such that the edges of a graph G can be colored with k colors in such a way that any two incident edges have

$\chi'(G)$ distinct colors, is called the *chromatic index* of G, denoted by $\chi'(G)$. Such a coloring is called a *(proper) edge coloring* of G. Note that $\chi'(G) =$

$ch'(G)$ $\chi(L(G))$. We also define the *list chromatic index* $ch'(G)$ of G as the choice number of the line graph of G.

1.3 Probabilistic tools

In this section, we recall some notions of discrete probabilities, as well as some useful probabilistic tools, as they appear in [MR02].

We only consider experiments which have a finite number of possible outcomes. For example when tossing a coin, there are only two possible outcomes: head and tail. The set of all possible outcomes of an experiment is called the *sample space*, denoted by Ω. A *finite probability space* $(\Omega, \mathbf{Pr})$ consists of a sample space Ω and a *probability function* $\mathbf{Pr} : \Omega \to [0,1]$ (where $[0,1]$ denotes the closed real interval between 0 and 1) such that: Ω $\mathbf{Pr}$

$$\sum_{x\in\Omega} \mathbf{Pr}(x) = 1.$$

When considering a probability function verifying $\mathbf{Pr}(x) = 1/|\Omega|$ for every $x \in \Omega$, we say that the distribution is *uniform*.

We extend $\mathbf{Pr}$ to 2^Ω (the set of *events*) by setting for every $A \subseteq \Omega$:

$$\mathbf{Pr}(A) = \sum_{x\in A} \mathbf{Pr}(x)$$

If we denote by $\bar{A}$ the event that A does not occur, then we have :

1. $\mathbf{Pr}(\bar{A}) = 1 - \mathbf{Pr}(A)$,
2. $\mathbf{Pr}(A \cup B) = \mathbf{Pr}(A) + \mathbf{Pr}(B) - \mathbf{Pr}(A \cap B)$,
3. $\mathbf{Pr}\left(\cup_{i=1}^n A_i\right) \le \sum_{i=1}^n \mathbf{Pr}(A_i)$.

The *conditional probability of A given B*, denoted by $\mathbf{Pr}(A|B)$, is defined as the ratio between $\mathbf{Pr}(A \cap B)$ and $\mathbf{Pr}(B)$. Two events A and B are said to be *independent* if $\mathbf{Pr}(A|B) = \mathbf{Pr}(A)$, or equivalently if $\mathbf{Pr}(A \cap B) = \mathbf{Pr}(A)\mathbf{Pr}(B)$. A set of events $\mathcal{E}$ is *mutually independent* if for any subset $\{A_0, \ldots, A_n\}$ of $\mathcal{E}$, we have $\mathbf{Pr}(A|B)$

$$\mathbf{Pr}\left(A_0 | \cap_{i=1}^n A_i\right) = \mathbf{Pr}(A_0).$$

Note that a set of events which is *pairwise independent* (every two events are independent) is not necessarily mutually independent. We also say that an event A is mutually independent from a set of events $\mathcal{E}$ if for any subset $\{B_1, \ldots, B_n\}$ of $\mathcal{E}$, we have

$$\mathbf{Pr}\left(A | \cap_{i=1}^n B_i\right) = \mathbf{Pr}(A).$$

A *random variable* defined on a probability space $(\Omega, \mathbf{Pr})$ is a function $X : \Omega \to \mathbb{R}$. The *expected value*, or *expectation* of a random variable X is $\mathbf{E}$

$$\mathbf{E}(X) = \sum_{x\in\Omega} \mathbf{Pr}(x)X(x).$$

A major property of expectation is its linearity : $\mathbf{E}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \mathbf{E}(X_i)$. The *conditional expectation of X given B*, denoted by $\mathbf{E}(X|B)$ is equal $\mathbf{E}(X|B)$

to $\sum_{x \in \Omega_X} \mathbf{Pr}\,(X = x|B)$, where Ω_X denotes the range of X. Note that linearity of expectation extends to conditional expectation :

$$\text{if } X = \sum_{i=1}^{n} X_i, \text{ then } \mathbf{E}(X|B) = \sum_{i=1}^{n} \mathbf{E}(X_i|B).$$

The next results characterize the concentration of random variables with specific properties, in other words they give bounds on the probability that the value taken by a random variable is close from its expectation.

Lemma 1.1 (Simple Concentration Bound) *Let X be a random variable determined by n independent trials $T_1, \ldots, T_n$ and satisfying:*

1. *Changing the outcome of any one trial can affect X by at most c.*

Then,

$$\mathbf{Pr}(|X - \mathbf{E}(X)| > t) \le 2e^{-\frac{t^2}{2c^2 n}}$$

Lemma 1.2 (Talagrand's Inequality) *Let X be a non-negative random variable, not identically 0, which is determined by n independent trials $T_1, \ldots, T_n$, and satisfying the following for some $c, r > 0$:*

1. *Changing the outcome of any one trial can affect X by at most c.*

2. *For any s, if $X \ge s$ then there is a set of at most rs trials whose outcomes certify that $X \ge s$.*

Then for any $0 \le t \le \mathbf{E}(X)$,

$$\mathbf{Pr}\left(|X - \mathbf{E}(X)| > t + 60c\sqrt{r\mathbf{E}(X)}\right) \le 4e^{-\frac{t^2}{8c^2 r \mathbf{E}(X)}}$$

Lemma 1.3 (McDiarmid's Inequality) *Let X be a non-negative random variable, not identically 0, which is determined by n independent trials $T_1, \ldots, T_n$ and m independent permutations $\Pi_1, \ldots, \Pi_m$ and satisfying the following for some $c, r > 0$:*

1. *Changing the outcome of any trial can affect X by at most c.*

2. *Interchanging two elements in any one permutation can affect X by at most c.*

3. *For any s, if $X \ge s$ then there is a set of at most rs choices whose outcomes certify that $X \ge s$.*

Then for any $0 \le t \le \mathbf{E}(X)$,

$$\mathbf{Pr}\left(|X - \mathbf{E}(X)| > t + 60c\sqrt{r\mathbf{E}(X)}\right) \le 4e^{-\frac{t^2}{8c^2 r \mathbf{E}(X)}}$$

We denote by $\mathrm{BIN}(n, p)$ the variable which is the sum of n variables each of which is 1 with probability p and 0 with probability $1 - p$. The expectation of $\mathrm{BIN}(n, p)$ is known to be np, so the next result gives a bound on the concentration of $\mathrm{BIN}(n, p)$.

Lemma 1.4 (Chernoff Bound) *For any $0 \leq t \leq np$:*

$$\mathbf{Pr}\left(|BIN(n,p) - np| > t\right) < 2e^{-t^2/3np}.$$

It is easy to see that if $\{A_1, \ldots A_n\}$ is a mutually independent set of events with $\mathbf{Pr}(A_i) < 1$ for every i, then with positive probability, none of the events occur. The last result shows that under certain assumption, the same is true even if the events are not mutually independent.

Lemma 1.5 (Lovász Local Lemma) *Consider a set $\mathcal{E}$ of (typically bad) events such that for each $A \in \mathcal{E}$*

1. $\mathbf{Pr}(A) \leq p < 1$, *and*
2. *A is mutually independent of a set of all but at most d of the other events.*

If $4pd \leq 1$ then with positive probability, none of the events in $\mathcal{E}$ occur.

Chapter 2

Coloring of the square and cyclic coloring

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In this chapter, we prove a general result on the structure of planar graphs, which implies that

- the vertices of any planar graph with maximum degree Δ can be colored with $\left(\frac{3}{2}+o(1)\right)\Delta$ colors, in such a way that any two vertices at distance at most two apart have distinct colors;
- the faces of any plane graph with maximum degree Δ can be colored with $\left(\frac{3}{2}+o(1)\right)\Delta$ colors, in such a way that any two faces sharing a vertex have distinct colors.

2.1 Introduction

The Four Color Theorem can be stated as follows: the faces of any plane graph can be colored with four colors, such that any two faces sharing an edge have distinct colors. In [OP69], Ore and Plummer considered the same problem, but requiring that any two faces sharing a vertex have distinct colors.

To study this problem, it is convenient to consider the corresponding vertex coloring problem: a *cyclic coloring* of a plane graph G is a vertex coloring of G such that any two vertices incident to the same face have distinct colors. The minimum number of colors in a cyclic coloring of G is called the *cyclic chromatic number* $\chi^*(G)$.

$\chi^*(G)$

A list version of this coloring can also be considered: the least integer t such that for any t-list assignment L, there exists a cyclic coloring c of G satisfying $c(v) \in L(v)$ for every vertex v of G is called the *cyclic choice number* of G, denoted by $ch^*(G)$.

$ch^*(G)$

Let us denote by G^* the dual graph of G, that is the plane graph in which the vertices are the faces of G, and such that two vertices are adjacent in G^* if and only if the corresponding faces share an edge. Clearly, a cyclic coloring of G^* is a coloring of the faces of G in which any two faces sharing a vertex have distinct colors. If we denote the size of the largest face of any plane graph H by $\Delta^*(H)$, we clearly have $\Delta^*(G^*) = \Delta(G)$. Ore and Plummer [OP69] proved that any plane graph G has a cyclic coloring with at most $2\Delta^*(G)$ colors, which implies that the faces of any plane graph with maximum degree Δ can be colored with 2Δ colors in such a way that any two faces sharing a vertex have distinct colors.

$\Delta^*(H)$

From now on, we forget about the original face coloring problem, and concentrate on cyclic coloring of plane graphs. Borodin [Bor84] (see also Jensen and Toft [JT95, page 37]) conjectured the following:

Conjecture 2.1 [Bor84] *Any plane graph G has a cyclic coloring with $\left\lfloor \frac{3}{2}\Delta^*(G) \right\rfloor$ colors.*

Additionally, he proved this conjecture for $\Delta^* = 4$. The best known upper bound in the general case is due to Sanders and Zhao [SZ01], who proved that any plane graph G has a cyclic coloring with $\left\lceil \frac{5}{3}\Delta^*(G) \right\rceil$ colors. Observe that Borodin's conjecture is optimal: in the graph depicted in Figure 2.1(a), every pair of vertices is incident to the same face, and must receive distinct colors in any cyclic coloring. There are $3k+1$ vertices, and every face has size $2k+1$, hence at least $\left\lfloor \frac{3}{2}\Delta^* \right\rfloor$ colors are necessary.

In this chapter, we relate cyclic coloring with another vertex coloring of graphs. Recall that the *square* G^2 of a graph G is the graph with vertex set $V(G)$, with an edge between any two different vertices that have distance at most two in G. The *chromatic number* of G^2, denoted $\chi(G^2)$, is the least number of colors needed in a proper coloring of G^2: that is, such that any two adjacent vertices of G^2 have distinct colors (or equivalently, such that any two vertices at distance at most two in G have distinct colors). A conjecture by Wegner [Weg77] about the chromatic number of planar graphs has been the starting point of several articles, the most recent of which proves an asymptotic version of the conjecture [HHM+07].

Conjecture 2.2 [Weg77] *For a planar graph G of maximum degree $\Delta \geq 8$ we have $\chi(G^2) \leq \left\lfloor \frac{3}{2}\Delta \right\rfloor + 1$.*

Observe that Wegner's conjecture is also optimal. In the graph depicted in Figure 2.1(b), all the vertices except z are pairwise at distance at most two. Hence the graph needs at least $3k + 1 = \left\lfloor \frac{3}{2}\Delta \right\rfloor + 1$ colors, since $\Delta = 2k$.

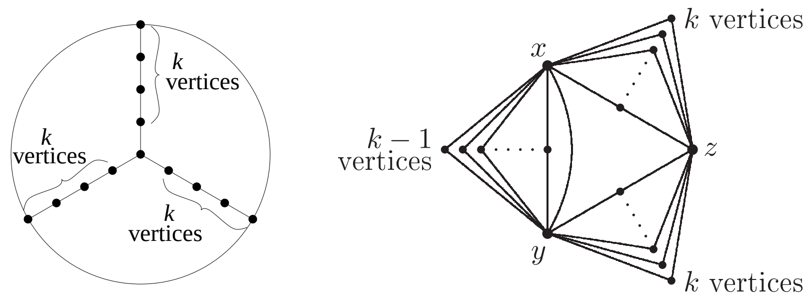

Figure 2.1: (a) A graph showing that Borodin's conjecture is optimal (b) A graph showing that Wegners's conjecture is optimal.

An $L(p, q)$-labelling of a graph G is an assignment of integers to the vertices of G in such way that any two adjacent vertices receive integers that differ by at least p, and any two vertices at distance two receive integers that differ by at least q. The $\lambda_{p,q}$-*number* of G, denoted by $\lambda_{p,q}(G)$, is the smallest integer t such that there exists an $L(p, q)$-labelling of G using labels from $\{1, 2, \ldots, t\}$. $\lambda_{p,q}(G)$

Of course we can also consider the list version of $L(p, q)$-labellings. Given a graph G, the *list* $\lambda_{p,q}$-*number*, denoted $\lambda^l_{p,q}(G)$, is the smallest integer t such that, for every t-list assignment L on the vertices of G, there exists an $L(p, q)$-labelling f such that $f(v) \in L(v)$ for every vertex v. $\lambda^l_{p,q}(G)$

Havet *et al.* recently proved the following result, which implies that Wegner's conjecture holds asymptotically:

Theorem 2.3 [HHM+07] *For any fixed p, and any planar graph G with maximum degree* Δ*, we have* $\lambda^l_{p,1}(G) \leq \left(\frac{3}{2}+o(1)\right)\Delta$.

Although Wegner's and Borodin's conjectures seem to be tightly related, nobody has ever been able to bring to light a direct connection between them. Most of the results approaching these conjectures use the same ideas, but at this point (as far as we know), no one proved a general theorem implying a result on the coloring of the square and a result an the cyclic coloring of plane graphs.

This is exactly our approach in this chapter: we define a coloring that generalizes both the coloring of the square and the cyclic coloring of plane graphs, and we prove a result on this coloring which implies asymptotic versions of both conjectures.

Let A and B be two subsets of the vertex set V. (Note that we do not require A and B to be disjoint.) An (A,B)*-coloring* of G is an assignment of colors to the vertices in B so that:

- vertices of B that are adjacent must receive different colors, and
- vertices of B that have a common neighbor from A must receive different colors.

When each vertex $v \in B$ has its own list $L(v)$ of colors from which its color must be chosen, we talk about a *list* (A,B)*-coloring*.

We denote by $\chi(G;A,B)$ the minimum number of colors required for an (A,B)-coloring to exist. Its list variant is denoted by $ch(G;A,B)$, and is defined as the minimum integer t so that for every t-list assignment $L(v)$ to the vertices $v \in B$, there exists a proper (A,B)-coloring of G in which the vertices in B are assigned colors from their own lists. Notice that we trivially have $\chi(G) = \chi(G;\varnothing,V)$ and $\chi(G^2) = \chi(G;V,V)$; and the same relations hold for the list variant.

For a vertex $v \in V$, let $N_B(v) = N(v) \cap B$, and $d_B(v) = |N_B(v)|$ (so $d_G(v) = d_V(v)$). If we set $\Delta(G;A,B) = \max\{\, d_B(v) \mid v \in A \,\}$, then it is clear that we always need at least $\Delta(G;A,B)$ colors in a proper (A,B)-coloring.

Our main result in this chapter is the following:

Theorem 2.4 [AEH08] *Let G be a planar graph and* $A,B \subseteq V$*. Then* $ch(G;A,B) \leq (1+o(1))\,\frac{3}{2}\,\Delta(G;A,B)$.

In other words, for all $\varepsilon > 0$*, there exists* D_ε*, so that for all* $D \geq D_\varepsilon$ *we have: If G is a planar graph, with* $A,B \subseteq V$ *so that* $\Delta(G;A,B) \leq D$*, and L is a list assignment so that each vertex v in B gets a list* $L(v)$ *of at* *least* $\left(\frac{3}{2}+\varepsilon\right) D$ *colors, then there exists an* (A,B)*-coloring of* G *in which the vertices in* B *are assigned colors from their own lists.*

A trivial lower bound for the (list) chromatic number of a graph G is the *clique number* $\omega(G)$, the maximal size of a clique in G. For (A,B)-colorings, where $A,B \subseteq V$, we can define the following related concept. An (A,B)-*clique* is a subset $C \subseteq B$ so that every two different vertices in C are adjacent or have a common neighbor in A. Denote by $\omega(G;A,B)$ the maximal size of an (A,B)-clique in G. Then we trivially have $ch(G;A,B) \geq \omega(G;A,B)$, and so Theorem 2.4 means that for a planar graph G we have $\omega(G;A,B) \leq (1+o(1))\,\frac{3}{2}\,\Delta(G;A,B)$. $\omega(G)$

But in fact, the structural result we use to prove Theorem 2.4 fairly easily gives a better estimate.

Theorem 2.5 [AEH08] *Let* G *be a planar graph and* $A,B \subseteq V$*. Then* $\omega(G;A,B) \leq \frac{3}{2}\,\Delta(G;A,B) + O(1)$*.*

We now discuss two special consequences of these results. These special versions of Theorems 2.4 and 2.5 also show that the term $\frac{3}{2}\,\beta$ in these results is best possible.

Since $ch(G^2) = ch(G;V,V)$, as immediate corollaries of Theorems 2.4 and 2.5 we obtain.

Corollary 2.6 *The square of every planar graph* G *of maximum degree* Δ *has list chromatic number at most* $(1+o(1))\,\frac{3}{2}\,\Delta$*.*

Corollary 2.7 *The square of every planar graph* G *of maximum degree* Δ *has clique number at most* $\frac{3}{2}\,\Delta + O(1)$*.*

In order to show that our Theorem 2.4 provides an asymptotically best possible upper bound for the cyclic chromatic number of plane graphs G, we need some extra notation. For each face f of G, add a vertex x_f and call X_F the set of vertices that were added to G. For any face f of G, and any vertex v incident with f, add an edge between v and x_f. We denote by G^F the graph obtained from G by this construction, so $V(G^F) = V(G) \cup X_F$. Observe that a (list) $(X_F, V(G))$-coloring of G^F is exactly a cyclic (list) coloring of G and that $\Delta(G^F; X_F, V(G)) = \Delta^*(G)$. We get the following corollary of Theorem 2.4.

Corollary 2.8 *Every plane graph* G *of maximum face degree* Δ^* *has cyclic list chromatic number at most* $(1+o(1))\,\frac{3}{2}\,\Delta^*$*.*

For a plane graph G, the *cyclic clique number* $\omega^*(G)$ is the maximal size of a set $C \subseteq V$ so that every two vertices in C have some face they are both incident with. Note that the plane graph depicted in Figure 2.1(a) satisfies $\omega^*(G) = 3\,k = \left\lfloor \frac{3}{2}\,\Delta^* \right\rfloor$. This shows that the following corollary of Theorem 2.5 is best possible, up to the constant term. $\omega^*(G)$

Corollary 2.9 *Every plane graph G of maximum face degree Δ^* has cyclic clique number at most $\frac{3}{2}\Delta^* + O(1)$.*

To prove Theorems 2.4 and 2.5 we can as well assume that A contains all vertices of degree at most $\Delta(G; A, B)$. To simplify things, define $B^\beta = \{\, v \in V \mid d_B(v) \leq \beta \,\}$. So to prove Theorems 2.4 and 2.5 it is enough to prove the following theorems.

Theorem 2.10 *For all real $\varepsilon > 0$, there exists a β_ε so that the following holds for all $\beta \geq \beta_\varepsilon$. Let G be a planar graph, with $B \subseteq V$ a set of vertices, and suppose every vertex $v \in B$ has a list $L(v)$ of at least $\left(\frac{3}{2} + \varepsilon\right)\beta$ colors. Then a list (B^β, B)-coloring of G with those colors exist.*

Theorem 2.11 *There exist constants γ_1, β_1 so that the following holds for all $\beta \geq \beta_1$. Let G be a planar graph, with $B \subseteq V$ a set of vertices. Then every (B^β, B)-clique in G has size at most $\frac{3}{2}\beta + \gamma_1$.*

The main steps in the proof of Theorem 2.10 can be found in Section 2.2. The proof relies on two technical lemmas; the proofs of those can be found in Section 2.3. After that we use one of those lemmas to provide the relatively short proof of Theorem 2.11 in Section 2.4. In Section 2.5 we discuss some of the aspects of our work, give details about the main differences with the proof of [HHM+07], and discuss open problems related to (list) (A, B)-coloring of graphs.

2.2 Proof of Theorem 2.10

We use the terminology and notation from the previous section. Throughout this section we assume that $G = (V, E)$ is a plane graph with $B \subseteq V$, and β is a positive integer. Recall the notation $U^\beta = \{\, v \in V \mid d_U(v) \leq \beta \,\}$ for a subset $U \subseteq V$. Note that this means that V^β is the set of all vertices of degree at most β

Our goal is to show that for all $\varepsilon > 0$, if we take β large enough, then for every assignment $L(v)$ of at least $\left(\frac{3}{2} + \varepsilon\right)\beta$ colors to the vertices $v \in B$, there is a list (B^β, B)-coloring of G where each vertex in B receives a color from its own list. In other words, we want an assignment $c(v)$ for each $v \in B$ so that :

- for all $v \in B$ we have $c(v) \in L(v)$;
- for all $u, v \in B$ with $uv \in E$ we have $c(u) \neq c(v)$; and
- for all $u, v \in B$ with a common neighbor in B^β (i.e., with a common neighbor of degree at most β) we have $c(u) \neq c(v)$.

2.2.1 The First Steps

A *β-neighbor* of v is a vertex $u \neq v$, so that u and v are adjacent, or u and v have a common neighbor in B^β. Denote the set of β-neighbors of v by $N^\beta(v)$, and its number by $d^\beta(v)$. Note that we have

$$d^\beta(v) \;\leq\; d(v) + \sum_{u \in N(v) \cap B^\beta} (d(u) - 1).$$

For $P, Q \subseteq V$, the set of edges between P and Q is denoted by $E(P, Q)$, and the number of edges between P and Q by $e(P, Q)$ (edges with both ends in $P \cap Q$ are counted twice).

An important tool in our proof of Theorem 2.10 is the following structural result.

Lemma 2.12 *There exist constants γ, γ', so that for all $\beta \geq \gamma'$ and plane graphs $G = (V, E)$ we have that G contains one of the following:*

(S1) *a vertex with degree zero or one;*

(S2) *a face f and two vertices u, v on the boundary of f with $d(u) + d(v) \leq \beta$ and $d^\beta(u) \leq \frac{3}{2}\beta$;*

(S3) *two disjoint nonempty sets $X, Y \subseteq V^\beta$ with the following properties:*

(i) *Every vertex $y \in Y$ has degree at most four. Moreover, y is adjacent to exactly two vertices of X and the other neighbors of y have degree at most four as well.*

For $y \in Y$, let X^y be the set of its two neighbors in X. And for $W \subseteq X$, let Y^W be the set of vertices $y \in Y$ with $X^y \subseteq W$ (that is, the set of vertices of Y having their two neighbors from X in W).

(ii) *For all pairs of vertices $y, z \in Y$, if y and z are adjacent or have a common neighbor $w \notin X$, then $X^y = X^z$.*

(iii) *For all nonempty subsets $W \subseteq X$, we have the following inequality:*

$$e(W, V \setminus W) \;\leq\; e(W, Y) + e(W, Y \setminus Y^W) + \gamma\,|W|.$$

The proof of Lemma 2.12 can be found in Subsection 2.3.1. In the proof we obtain $\gamma = 132$ and $\gamma' = 1060$, values that are probably far from best possible. The important point, to our mind, is that these are constant.

We continue with a description how to apply the lemma to prove Theorem 2.10, assuming that $\beta \geq \gamma'$. We use induction on the number of vertices of G. By Lemma 2.12, G contains one of (S1), (S2) or (S3).

(S1) If G contains a vertex v of degree at most one, we consider the graph G_1 obtained from G by removing v. If $v \notin B$, then a list (B^β, B)-coloring of G_1 is also a list (B^β, B)-coloring of G. Otherwise set $B_1 = B \setminus \{v\}$. Now find a list (B_1^β, B_1)-coloring of G_1, and give an appropriate color to v at the end. This is always possible since v is in conflict with at most β other vertices, and we have $\left(\frac{3}{2} + \varepsilon\right) \beta \geq \beta + 1$ colors available for v.

(S2) Let f be a face with two vertices u, v on its boundary such that $d(u) + d(v) \leq \beta$ and $d^\beta(u) \leq \frac{3}{2}\beta$. In this case we construct a new planar graph G_2 by identifying u and v into a new vertex w. Set $V_2 = (V \setminus \{u, v\}) \cup \{w\}$, and notice that G_2 has strictly fewer vertices than G, and w has degree at most $d_G(u) + d_G(v) \leq \beta$ in G_2. In other words, $w \in V_2^\beta$. If $v \notin B$, then set $B_2 = B$. Otherwise, set $B_2 = (B \setminus \{u, v\}) \cup \{w\}$ and give w a list of colors $L(w)$ with $L(w) = L(v)$.

By induction there exists a list (B_2^β, B_2)-coloring of G_2. We define a coloring of G as follows: every vertex different from u and v keeps its color from the coloring of G_2. If $v \in B$, then we color v with the color given to w in G_2. And if $u \in B$, then we use the assumption, $d_G^\beta(u) \leq \frac{3}{2}\beta$, and hence there exists a color for u different from the color of all the vertices in conflict with u. We color u with one of these colors. It is easy to verify that this defines a list (B^β, B)-coloring of G.

(S3) This is the only non-trivial case. In the remaining of this subsection we describe how to reduce this case to a list edge-coloring problem. In the next subsection, we then describe how Kahn's approach to prove that the list edge-chromatic number is asymptotically equal to the fractional edge-chromatic number can be used to conclude the proof of Theorem 2.10.

Let X and Y be the two disjoint sets as in (S3). This means that every vertex in X has degree at most β. Also recall that by (S3)(i), every vertex $y \in Y$ has degree at most four. Moreover, y is adjacent to exactly two vertices of X and the other neighbors of y have degree at most four as well. As in (S3), let X^y be the set of the two neighbors of y in X.

Suppose there is a vertex $y \in Y$ with $y \notin B$. If $N(y) = X^y$, then contract y to one of its two neighbors in X^y. If y has a neighbor u outside X^y, then contract the edge uy. Call the resulting graph G_3. It is easy to check that a list (B^β, B)-coloring of G_3, which exists by induction, also is a proper list (B^β, B)-coloring of G.

So from now on we assume that all vertices in Y are contained in B.

Let Y_0 be the set of vertices from Y with no neighbor outside $X \cup Y$. Consider the graph $G[V \setminus Y_0]$ induced on the set of vertices outside Y_0. For every vertex $y \in Y \setminus Y_0$ with a unique neighbor u outside $X \cup Y$, or with exactly two neighbors u and v outside $X \cup Y$, contract the edge yu into a new vertex u^*. The graph obtained is denoted by G_0. And let B_0

be the union of $B \setminus Y_0$ and all new vertices u^* that originated from an edge yu with $u \in B$.

By the construction of G_0, it is easy to verify the following statement.

Claim 2.13 *For all $u \in V(G_0)$ we have $(N^\beta_G(u) \setminus Y) \subseteq N^\beta_{G_0}(u)$.*

For each vertex u^* of B_0 corresponding to the contraction of an edge uy ($y \in Y \setminus Y_0$) in G, set $L_0(u^*) = L(u)$ and for all other vertices v of B_0 set $L_0(v) = L(v)$. By the induction hypothesis, the graph G_0 admits a list (B_0^β, B_0)-coloring c_0 with respect to the list assignment L_0.

We now transform this coloring into a list (B^β, B)-coloring of G with respect to the original list assignment L. For each vertex $u \in B \setminus Y$, if an edge incident to u has been contracted in the construction of G_0 to form a new vertex u^*, set $c(u) = c_0(u^*)$. Otherwise set $c(u) = c_0(u)$. Using Claim 2.13, this is a good partial (B^β, B)-coloring of all the vertices of $B \setminus Y$. The difficult part of the proof is to show that c can be extended to Y.

By assumption, at the beginning every vertex in Y has a list of at least $\left(\frac{3}{2} + \varepsilon\right)\beta$ available colors. For each vertex y in Y, let us remove from $L(y)$ the colors which are forbidden for y according to the partial (B^β, B)-coloring c of G. In the worst case, these forbidden colors are exactly the colors of the vertices of $V \setminus Y$ at distance at most two from y.

Let us define the multigraph H as follows: H has vertex set X. And for each vertex $y \in Y$ we add an edge e_y between the two neighbors of y in X (in other words, between the two vertices from X^y). We associate a list $L(e_y)$ to e_y in H by taking the list of y obtained after removing the set of forbidden colors for y from the original list $L(y)$. Finally, for every edge e in $G[X]$, we add the same edge e to H and associate a list $L(e)$ of at least $\left(\frac{3}{2} + \varepsilon\right)\beta$ colors to such an edge. (The colors within these lists are irrelevant for what follows, we just have to make sure that the lists of these specific edges of H are large enough.)

We now prove the following lemma.

Lemma 2.14 *A list edge-coloring for H, with the list assignment L defined as above, provides an extension of c to a list (B^β, B)-coloring of G by giving to each vertex $y \in Y$ the color of the edge e_y in H.*

Proof. This follows from property (S3)(ii) in Lemma 2.12: for every two vertices $y, z \in Y$, if y and z are adjacent or have a common neighbor $w \notin X$, then $X^y = X^z$. This proves that the two vertices adjacent in Y or with a common neighbor not in X define parallel edges in H and so will have different colors. If two vertices y_1 and y_2 of Y have a common neighbor in X, e_{y_1} and e_{y_2} will be adjacent in H and so will get different colors. Since we have already removed from the list of vertices in Y the set of forbidden colors (defined by the colors of the vertices in $V \setminus Y$), there will be no conflict between the colors of a vertex from Y and a vertex from $V \setminus Y$. We conclude that the edge coloring of H will provide an extension of c to a list (B^β, B)-coloring of G. ∎

The following lemma provides a lower bound on the size of $L(e)$ for the edges e in H.

Lemma 2.15 *Let $e = uv$ be an edge in H. Then we have*

$$|L(e)| \;\geq\; \left(\tfrac{3}{2} + \varepsilon\right)\beta - (d_G(u) - d_H(u)) - (d_G(v) - d_H(v)) - 10.$$

Proof. If e originated because there was already an edge in $G[X]$, then by construction we have $|L(e)| \geq \left(\tfrac{3}{2}+\varepsilon\right)\beta$. On the other hand, suppose that $e = e_y$, i.e., e originated because of a vertex $y \in Y$ in G with $X^y = \{u, v\}$. Let Z be the set of vertices adjacent in G to y in $V \setminus X$. Then by (S3), $|Z| \leq 2$ and $|N_G(Z) \setminus Y| \leq 6$. The colors that are forbidden for y are the colors of $\{u, v\}$, plus the colors of vertices in $(N_G(u) \cup N_G(v)) \setminus Y$, plus the colors of vertices in $(Z \cup N_G(Z)) \setminus Y$. The number of vertices in these three sets add up to $(d_G(u) - d_H(u)) + (d_G(v) - d_H(v)) + 10$. The lemma follows. ∎

In the remainder of this subsection, we apply Lemma 2.12 to obtain information on the density of subgraphs in H, which we will need in the next subsection. As in Lemma 2.12, for all non-empty subsets $W \subseteq X$, we define Y^W as the set of vertices $y \in Y$ with $X^y \subseteq W$ (that is, the set of vertices of Y having their two neighbors from X in W). By (S3)(iii) we have:

$$e_G(W, V \setminus W) \;\leq\; e_G(W, Y) + e_G(W, Y \setminus Y^W) + \gamma\,|W|.$$

This inequality has the following interpretation in H.

Lemma 2.16 *For all non-empty subsets $W \subseteq X(= V(H))$ we have*

$$\sum_{w \in W} (d_G(w) - d_H(w)) \;\leq\; e_H(W, X \setminus W) + \gamma\,|W|.$$

Proof. We partition $E_G(W, V \setminus W)$ into three parts E_1, E_2 and E_3 as follows: For E_1 we take the set of edges from W to $V \setminus (X \cup Y)$, i.e., $|E_1| = e_G(W, V \setminus (X \cup Y)) = \sum_{w \in W} (d_G(w) - d_H(w))$. The set E_2 contains the edges from W to Y, $|E_2| = e_G(W, Y)$, and E_3 is the set of edges from W to $X \setminus W$ in G. By (S3)(iii) (see also the inequality for $e_G(W, V \setminus W)$ above), we have

$$|E_1| + |E_2| + |E_3| \;\leq\; e_G(W, Y) + e_G(W, Y \setminus Y^W) + \gamma\,|W|.$$

Note that $e_G(W, Y \setminus Y^W) = e_H(W, X \setminus W) - e_G(W, X \setminus W)$ and $|E_2| = e_G(W, Y)$. This results in the following stronger inequality, which in turn implies the lemma:

$$|E_1| + |E_3| \leq e_H(W, X \setminus W) - |E_3| + \gamma\,|W|,$$

and so

$$\sum_{w \in W} (d_G(w) - d_H(w)) \leq e_H(W, X \setminus W) - 2\,|E_3| + \gamma\,|W|.$$

∎

At this point, our aim will be to apply Kahn's approach to the multigraph H with the list assignment L, to prove the existence of a proper list edge-coloring for H. This is described in the next subsection.

We summarize the properties we assume are satisfied by the multigraph H and the list assignment L of the edges of H. For these conditions we just consider $d_G(v)$ as an integer with certain properties, assigned to each vertex of H.

(H1) For all vertices v in H we have $d_H(v) \leq d_G(v) \leq \beta$.

(H2) For all edges $e = uv$ in H: $|L(e)| \geq \left(\frac{3}{2} + \varepsilon\right)\beta - (d_G(u) - d_H(u)) - (d_G(v) - d_H(v)) - 10$.

(H3) For all non-empty subsets $W \subseteq V(H)$: $\sum_{w \in W} (d_G(w) - d_H(w)) \leq e_H(W, X \setminus W) + \gamma\,|W|$, for some constant γ.

2.2.2 The Matching Polytope and Edge-Colorings

We briefly describe the matching polytope of a multigraph. More about this subject can be found in [Sch03, Chapter 25].

Let H be a multigraph with m edges. Let $\mathcal{M}(H)$ be the set of all matchings of H, including the empty matching. For each $M \in \mathcal{M}(H)$, let us define the m-dimensional *characteristic vector* $\mathbf{1}_M$ as follows: $\mathbf{1}_M = (x_e)_{e \in E(H)}$, where $x_e = 1$ for an edge $e \in M$, and $x_e = 0$ otherwise. The *matching polytope* of H, denoted by $\mathcal{MP}(H)$, is the polytope defined by taking the convex hull of all the vectors $\mathbf{1}_M$ for $M \in \mathcal{M}(H)$.

Edmonds [Edm65] gave the following characterisation of the matching polytope:

Theorem 2.17 [Edm65] *A vector $\vec{x} = (x_e)$ is in $\mathcal{MP}(H)$ if and only if $x_e \geq 0$ for all x_e and the following two types of inequalities are satisfied:*

- *For all vertices $v \in V(H)$: $\sum_{e:\ v \text{ incident to } e} x_e \leq 1$;*
- *for all subsets $W \subseteq V(H)$ with $|W| \geq 3$ and $|W|$ odd: $\sum_{e \in E(W)} x_e \leq \frac{1}{2}(|W| - 1)$.*

The significance of the matching polytope and its relation with list edge-coloring is indicated by the following important result. Recall the notation $\lambda\,\mathcal{MP}(H) = \{\,\lambda x \mid x \in \mathcal{MP}(H)\,\}$, for a real number λ.

Theorem 2.18 [Kah00] *For all real numbers δ, μ, $0 < \delta < 1$ and $\mu > 0$, there exists a $\Delta_{\delta,\mu}$ so that for all $\Delta \geq \Delta_{\delta,\mu}$ the following holds. If H is a multigraph and L is a list assignment of colors to the edges of H so that*

- *H has maximum degree at most Δ;*
- *for all edges $e \in E(H)$: $|L(e)| \geq \mu\,\Delta$;*
- *the vector $\vec{x} = (x_e)$ with $x_e = \dfrac{1}{|L(e)|}$ for all $e \in E(H)$ is an element of $(1-\delta)\,\mathcal{MP}(H)$.*

Then there exists a proper edge-coloring of H where each edge gets a color from its own list.

The theorem above is actually not explicitly stated this way in [Kah00], but can be obtained from the appropriate parts of that paper. For further details, the reader is referred to [AEH08].

The next lemma shows how to use Theorem 2.18 to complete the induction.

Lemma 2.19 *Let γ be a real number. Then there exists $K_\gamma > 0$, so that for all $K \geq K_\gamma$ the following holds. Let H be a multigraph, so that for each vertex v an integer $D(v)$ is assigned and for each edge e a positive real number b_e is given. Suppose that the following three conditions are satisfied:*

(H1') For all vertices v in H: $d(v) \leq D(v) \leq \beta$.

(H2') For all edges $e = uv$ in H: $b_e \geq \left(\tfrac{3}{2}\beta + K\right) - (D(u) - d(u)) - (D(v) - d(v))$.

(H3') For all non-empty subsets $W \subseteq V(H)$: $\sum\limits_{w \in W} (D(w) - d(w)) \leq e_H(W, V(H) \setminus W) + \gamma\,|W|$.

Then for all edges $e \in E(H)$ we have $b_e \geq \tfrac{1}{2}\beta$. And the vector $\vec{x} = (x_e)$ defined by $x_e = \dfrac{1}{b_e}$ for $e \in E(H)$ is an element of $\mathcal{MP}(H)$.

The proof of Lemma 2.19 will be given in Subsection 2.3.2. This lemma guarantees that for all $\varepsilon > 0$, there exists a β_ε, so that for all $\beta \geq \beta_\varepsilon$ Theorem 2.18 can be applied to a multigraph H with an edge list assignment L satisfying properties (H1) – (H3) stated at the end of the previous subsection.

To see this, take $0 < \delta_\varepsilon = \frac{\varepsilon}{3+2\,\varepsilon} < 1$. In order to be able to apply Theorem 2.18, we want to prove the existence of $\beta_{\varepsilon,\gamma}$ such that for any $\beta \geq \beta_{\varepsilon,\gamma}$ the vector $\vec{x} = (x_e),\ \ x_e = \frac{1}{|L(e)|}$, is in $(1-\delta_\varepsilon)\mathcal{MP}(H)$. Let K_γ be the number given by Lemma 2.19. By condition (H2) we have

$$\begin{aligned}(1-\delta_\varepsilon)|L(e)| &\geq (1-\delta_\varepsilon)\Big(\big(\tfrac{3}{2}+\varepsilon\big)\,\beta - (D(u)-d(u)) - (D(v)-d(v)) - 10\Big)\\ &\geq (1-\delta_\varepsilon)\big(\tfrac{3}{2}+\varepsilon\big)\,\beta - (D(u)-d(u)) - (D(v)-d(v)) - 10\\ &= \big(\tfrac{3}{2}\,\beta + \tfrac{\varepsilon}{2}\,\beta\big) - (D(u)-d(u)) - (D(v)-d(v)) - 10.\end{aligned}$$

Let $\beta_{\varepsilon,\gamma} = \frac{2(K_\gamma+10)}{\varepsilon}$. For $\beta \geq \beta_{\varepsilon,\gamma}$, we have

$$(1-\delta_\varepsilon)|L(e)| \geq \big(\tfrac{3}{2}\,\beta + K_\gamma\big) - (D(u)-d(u)) - (D(v)-d(v)).$$

So by Lemma 2.19, for $b_e = (1-\delta_\varepsilon)|L(e)|$, the vector $\vec{x}' = (x'_e),\ \ x'_e = \frac{1}{(1-\delta_\varepsilon)}x_e$ is in $\mathcal{MP}(H)$. We infer that $\vec{x} \in (1-\delta_\varepsilon)\mathcal{MP}(H)$ and the lemma follows.

Now assume $\beta \geq \max\{\,\gamma',\ \beta_{\varepsilon,\gamma},\ \Delta_{\delta_\varepsilon,1/2}\,\}$ (where γ, γ' are determined by Lemma 2.12, $\beta_{\varepsilon,\gamma}$ and δ_ε are related to K_γ from Lemma 2.19 as explained above, and $\Delta_{\delta_\varepsilon,1/2}$ is according to Theorem 2.18). Then using Lemma 2.19, we can now apply Theorem 2.18 which implies that the multigraph H defined in Subsection 2.2.1 has a list edge-coloring corresponding to the list assignment L. Lemma 2.14 then implies that the coloring c can be extended to a list (B^β, B)-coloring of the original graph G. This concludes the induction and also completes the proof of Theorem 2.10. ∎

2.3 Proofs of the Main Lemmas

We use the terminology and notation from the previous sections.

2.3.1 Proof of Lemma 2.12

In what follows, we take $\gamma = 132$ and $\gamma' = 1060$. So take $\beta \geq 1060$ and let G be a plane graph. We need some further notation and terminology.

The set of faces of G is denoted by F. For a face f, a *boundary walk of f* is a walk consisting of vertices and edges as they are encountered when walking along the whole boundary of f, starting at some vertex. The *degree of a face f*, denoted $d(f)$, is the number of edges on the boundary walk of f. Note that this means that if f is incident with a bridge (cut edge) of G, that bridge will be counted twice in $d(f)$. The *size of a face f* is the number of vertices on its boundary. We always have that the size of f is at most $d(f)$, with strict inequality if and only if the face has a cut vertex on its boundary.

We start by proving that we can assume that G is a 2-connected triangulation of the plane. First suppose that G is not connected. Then we can take two vertices u, v from different components so that adding the edge uv to G gives a simple plane graph G'.

Next, consider the case that G is connected but contains a face f of degree more than three. If this face contains a vertex v that is a cut vertex, then the vertices u and w that come before and after v on a boundary walk of f are different and not adjacent. Form the simple plane graph G' by adding the edge uw to G. If f contains no cut vertex, then it has four vertices u_1, u_2, u_3, u_4 that are consecutive on a boundary walk. And since G is planar, at least one of the pairs u_1, u_3 and u_2, u_4 are not adjacent. Form the simple plane graph G' by adding an edge between one of these non-adjacent pairs.

Suppose G' contains one of the structures (S1) – (S3) in the lemma. We claim that then also G contains one of these structures. This is obvious if G' contains (S1) or (S2). So suppose G' has sets X, Y according to (S3), and let uv be the edge that was added to G to give G'.

It is easy to check that exactly the same pair X, Y works for G as well in the following cases: if $\{u, v\} \cap (X \cup Y) = \varnothing$, or if $u, v \in X$, or if $u, v \in Y$, or if $u \in Y$ and $v \in V \setminus (X \cup Y)$. If $u \in X$ and $v \in V \setminus (X \cup Y)$, then going from G' to G for $W \subseteq X$ with $x \in W$, we loose one on the left hand side of the inequality in (iii). Hence the pair X, Y also works for G. If $u \in X$ and $v \in Y$, then in G either v has degree at most one, and then G contains structure (S1), or v is adjacent to one vertex $x \in X$ and at most two more vertices of degree at most four. But then v has a neighbor w with $d(v) + d(w) \le 7 \le \beta$. Moreover, since $x \in X \subseteq V^\beta$, we have $d^\beta(v) \le 8 + \beta \le \frac{3}{2}\,\beta$. Hence in this case G contains structure (S2). Finally, the possibilities $v \in Y$ and $u \in V \setminus (X \cup Y)$, or $v \in X$ and $u \in V \setminus (X \cup Y)$, or $v \in X$ and $u \in Y$, can be done by symmetry with the cases above.

So, by adding edges we can transform G to a connected graph G^* in which each face has degree three (which implies that G^* is indeed 2-connected) and so that if G^* satisfies the lemma, then so does G. Hence we might as well assume the following:

(a) *The graph G is 2-connected and all its faces have degree three.*

Now suppose that G does not contain any of the structures (S1) or (S2). In order to prove Lemma 2.12, we only need to prove that G contains structure (S3). We can observe that:

(b) *All vertices have degree at least three.* (Since G does not contain (S1), degrees must be at least two. And we cannot have a vertex of degree two, since otherwise, for each face to have degree three, we have a multiple edge as well.)

(c) *For all pairs of adjacent vertices u, v we have $d(u) + d(v) > \beta$ or $d^\beta(u) > \frac{3}{2}\,\beta$* (otherwise we have structure (S2)).

Let $\mathcal{B} \subseteq V$, the *big* vertices, be the vertices of degree at least 133; the other vertices are called *small*. Define $\mathcal{B}^\beta = \mathcal{B} \cap V^\beta$ (the big vertices with degree at most β) and $\mathcal{B}^{>\beta} = \mathcal{B} \setminus \mathcal{B}^\beta$.

(d) *If a vertex u of degree three has a small neighbor, then its other two neighbors are in $\mathcal{B}^\beta$.*

This follows since if u has a small neighbor v, then $d(u) + d(v) \leq \beta$. But then, by observation (c), we must have $d^\beta(u) > \frac{3}{2}\beta$, which is only possible if both its other neighbors are in $\mathcal{B}^\beta$ (note that a neighbor from $\mathcal{B}^{>\beta}$ adds at most one to $d^\beta(u)$).

In the same way we can prove:

(e) *If a vertex of degree four has a small neighbor, then it also has at least two neighbors from $\mathcal{B}^\beta$.*

(f) *A vertex u of degree five has at least two big neighbors* (otherwise we have $d^\beta(u) \leq 5 + 4 \cdot (132 - 1) + (\beta - 1) \leq \frac{3}{2}\beta$, since $\beta \geq 1060$).

We continue our analysis using the classical technique of *discharging*. Give each vertex $v \in V$ an initial charge $\mu(v) = \frac{2}{3}d(v) - 4$. Using the fact that every face has degree three, Euler's Formula $|V| - |E| + |F| = 2$ can be rewritten as $\sum_{x \in V} \mu(x) = -8$.

We next redistribute initial charges according to the following rules:

(R1) Each vertex of degree three that is adjacent to three big vertices receives a charge $2/3$ from each of its neighbors.

(R2) Each vertex of degree three that is adjacent to two big vertices receives a charge 1 from each of its big neighbors.

(R3) Each vertex of degree four that is adjacent to four big vertices receives a charge $1/3$ from each of its big neighbors.

(R4) Each vertex of degree four that is adjacent to three big vertices receives a charge $4/9$ from each of its big neighbors.

(R5) Each vertex of degree four that is adjacent to two big vertices receives a charge $2/3$ from each of its big neighbors.

(R6) Each vertex of degree five receives a charge $1/3$ from each of its big neighbors.

Denote the resulting charge of an element $v \in V$ after applying rules (R1) – (R6) by $\mu'(v)$. Since the global charge has been preserved, we have $\sum_{v \in V} \mu'(v) = -8$. We will show that for most $v \in V$, $\mu'(v)$ is non-negative.

Combining observations (d) – (f) with rules (R1) – (R6) and our knowledge that $\mu(v) = \frac{2}{3}d(v) - 4$, we find that $\mu'(v) = 0$ if $d(v) = 3, 4$, while $\mu'(v) \geq 0$ if $d(v) = 5$.

For a small vertex v with $d(v) \geq 6$, we have $\mu'(v) = \mu(v) = \frac{2}{3}d(v) - 4 \geq 0$.

So we are left to consider vertices $v \in \mathcal{B}$. The plane embedding of G imposes a clockwise order on the neighbors of v. If u is a neighbor of v, then by u^- (resp. u^+) we indicate the neighbor of v that comes before (resp. after) u in that order. Similarly, we denote by u^{--} (resp. u^{++}) the neighbor of v that comes before u^- (resp. after u^+) in the same order.

Let us take a vertex $v \in \mathcal{B}^{>\beta}$. We distinguish 5 different types of neighbors of v:

$$\begin{aligned}
N_3(v) &= \{\, u \in N(v) \mid d(u) = 3 \text{ and all neighbors of } u \text{ are big}\,\};\\
N_{4a}(v) &= \{\, u \in N(v) \mid d(u) = 4 \text{ and all neighbors of } u \text{ are big}\,\};\\
N_{4b}(v) &= \{\, u \in N(v) \mid d(u) = 4 \text{ and } u \text{ has exactly one small neighbor}\,\};\\
N_5(v) &= \{\, u \in N(v) \mid d(u) = 5\,\};\\
N_6(v) &= \{\, u \in N(v) \mid d(u) \geq 6\,\}.
\end{aligned}$$

Notice that each neighbor of v is in one of these sets. (For a neighbor of degree three, this follows from observation (d). And for a neighbor u of degree four, it follows from observation (e) that, since $v \in \mathcal{B}^{>\beta}$, if u has a small neighbor, then the remaining two neighbors are in $\mathcal{B}^{\beta}$.)

Moreover, by observation (d) we must have that if $u \in N_3(v)$, then $u^-, u^+ \in N_6(v)$. Similarly, if $u \in N_{4a}(v)$, then we also have $u^-, u^+ \in N_6(v)$. While if $u \in N_{4b}(v)$, then at least one of u^-, u^+ is in $N_6(v)$. Set $n_3 = |N_3(v)|$, $n_{4a} = |N_{4a}(v)|$, $n_{4b} = |N_{4b}(v)|$, $n_5 = |N_5(v)|$, and $n_6 = |N_6(v)|$. From the previous observation, we deduce

$$n_6 \;\geq\; n_3 + n_{4a} + \frac{1}{2}\, n_{4b}.$$

We also have, using $\mu(v) = \frac{2}{3}\, d(v) - 4$ and applying rules (R1), (R3), (R4) and (R6), that

$$\mu'(v) \;=\; \tfrac{2}{3}\, d(v) - 4 - \tfrac{2}{3}\, n_3 - \tfrac{1}{3}\, n_{4a} - \tfrac{4}{9}\, n_{4b} - \tfrac{1}{3}\, n_5.$$

Combining this with $d(v) = n_3 + n_{4a} + n_{4b} + n_5 + n_6$ and $\frac{1}{3}\, n_6 \geq \frac{1}{3}\, n_3 + \frac{1}{3}\, n_{4a} + \frac{1}{6}\, n_{4b}$, we find

$$\begin{aligned}
\mu'(v) &= \tfrac{2}{3}\, n_6 + \tfrac{1}{3}\, n_{4a} + \tfrac{2}{9}\, n_{4b} + \tfrac{1}{3}\, n_5 - 4\\
&\geq \tfrac{1}{3}\, n_6 + \tfrac{1}{3}\, n_3 + \tfrac{2}{3}\, n_{4a} + \tfrac{7}{18}\, n_{4b} + \tfrac{1}{3}\, n_5 - 4\\
&\geq \tfrac{1}{3}\, (n_6 + n_3 + n_{4a} + n_{4b} + n_5) - 4\\
&\geq \tfrac{1}{3}\, d(v) - 4 \;\geq\; 0.
\end{aligned}$$

So we found that for all $v \notin \mathcal{B}^{\beta}$ we have $\mu'(v) \geq 0$, and hence we must have

$$\sum_{v \in \mathcal{B}^{\beta}} \mu'(v) \;\leq\; -8 < 0.$$

To derive the relevant consequence of that formula, we must make a detailed analysis of the neighbors of a vertex $v \in \mathcal{B}^{\beta}$. We again distinguish different types of neighbors of v:

$$\begin{aligned}
M_1(v) &= \{\, u \in N(v) \mid \{u^-, u^{--}, u^+, u^{++}\} \cap \mathcal{B}^\beta \neq \varnothing \,\}; \\
M_{4a}(v) &= \{\, u \in N(v) \setminus M_1(v) \mid d(u) = 4 \text{ and } u^- \text{ or } u^+ \text{ have degree at least five} \,\}; \\
M_{4b}(v) &= \{\, u \in N(v) \setminus M_1(v) \mid d(u) = d(u^-) = d(u^+) = 4 \,\}; \\
M_5(v) &= \{\, u \in N(v) \setminus M_1(v) \mid d(u) = 5 \,\}; \\
M_6(v) &= \{\, u \in N(v) \setminus M_1(v) \mid d(u) \geq 6 \,\}.
\end{aligned}$$

First observe that if $u \in N(v) \setminus M_1(v)$ is a small vertex, then u^- and u^+ both have degree at least four: Assume that u^- has degree three, then by observation (d) the neighbor w of u^- distinct from v and u is in $\mathcal{B}^\beta$. By observation (a), $w = u^{--}$, which contradicts the fact that $u \notin M_1(v)$. If u^+ has degree three, we find that $u^{++} \in \mathcal{B}^\beta$, which again contradicts $u \notin M_1(v)$.

As a consequence, every neighbor of v is in exactly one set. Our aim in the following, in order to prove Lemma 2.12, is to show that most neighbors of v are in $M_{4b}(v)$.

We now evaluate the charge that a vertex $v \in \mathcal{B}^\beta$ has given to its neighbors. If $u \in M_1(v)$, then v gave at most $1+1+1 = 3$ to $\{u^-, u, u^+\}$; if $u \in M_{4a}(v)$, then v gave at most $1/3 + 2/3 + 2/3 = 5/3$ to $\{u^-, u, u^+\}$; if $u \in M_{4b}(v)$, then v gave at most $2/3 + 2/3 + 2/3 = 2$ to $\{u^-, u, u^+\}$; if $u \in M_5(v)$, then v gave at most $1/3 + 2/3 + 2/3 = 5/3$ to $\{u^-, u, u^+\}$; and, finally, if $u \in M_6(v)$, then v gave at most $2/3 + 0 + 2/3 = 4/3$ to $\{u^-, u, u^+\}$. Setting $m_1 = |M_1(v)|$, $m_{4a} = |M_{4a}(v)|$, $m_{4b} = |M_{4b}(v)|$, $m_5 = |M_5(v)|$, and $m_6 = |M_6(v)|$, we can conclude that v gave at most

$$\begin{aligned}
\tfrac{1}{3}\left(3\, m_1 + \tfrac{5}{3}\, m_{4a} + 2\, m_{4b} + \tfrac{5}{3}\, m_5 + \tfrac{4}{3}\, m_6\right) &\leq m_1 + \tfrac{2}{3}\, m_{4b} + \tfrac{5}{9}\,(m_{4a} + m_5 + m_6) \\
&\leq \tfrac{5}{9}\, d(v) + \tfrac{4}{9}\, m_1 + \tfrac{1}{9}\, m_{4b}
\end{aligned}$$

to its neighborhood. This means that the remaining charge $\mu'(v)$ of a vertex $v \in \mathcal{B}^\beta$ must satisfy

$$\mu'(v) \geq \left(\tfrac{2}{3}\, d(v) - 4\right) - \left(\tfrac{5}{9}\, d(v) + \tfrac{4}{9}\, m_1 + \tfrac{1}{9}\, m_{4b}\right) = \tfrac{1}{9}\,(d(v) - m_{4b}) - \tfrac{4}{9}\, m_1 - 4.$$

By definition, $M_1(v)$ is at most four times the number of neighbors of v in $\mathcal{B}^\beta$. Since the subgraph of G induced by $\mathcal{B}^\beta$ is planar, it has at most $3\,|\mathcal{B}^\beta| - 6$ edges, and so

$$\sum_{v \in \mathcal{B}^\beta} |M_1(v)| < 24\,|\mathcal{B}^\beta|.$$

Combining the last two inequalities gives

$$0 > \sum_{v \in \mathcal{B}^\beta} \mu'(v) \geq \left(\sum_{v \in \mathcal{B}^\beta} \tfrac{1}{9}\,(d(v) - |M_{4b}(v)|)\right) - \tfrac{4}{9} \cdot 24\,|\mathcal{B}^\beta| - 4\,|\mathcal{B}^\beta|,$$

which can be written as

$$\sum_{v\in\mathcal{B}^\beta}(d(v)-|M_{4b}(v)|)\ <\ 132\,|\mathcal{B}^\beta|.$$

We can assume $\mathcal{B}^\beta\neq\varnothing$, otherwise G contains structure (S1) or (S2). Define $X_0=\mathcal{B}^\beta$ and $Y_0=\bigcup_{v\in\mathcal{B}^\beta}M_{4b}(v)$. Note that the previous inequality can be written $e(X_0,V\setminus Y_0)<132\,|X_0|$. Also observe that the pair (X_0,Y_0) satisfies the conditions (i) and (ii) for X and Y in part (S3) of Lemma 2.12 :

(i) For all vertices $u\in M_{4b}(v)$, u^- and u^+ have degree four in G, and the fourth neighbor of u is in $\mathcal{B}^\beta=X_0$ by observation (e).

(ii) By observation (a), all pairs of adjacent vertices $u,v\in Y_0$, satisfy $X_0^u=X_0^v$. If $u,v\in Y_0$ share a neighbor $w\notin X_0$, then w has degree at most four and its possible neighbors distinct from u and v are in X_0^u. Again by observation (a), we must have $X_0^u=X_0^v$.

So we are done if the pair (X_0,Y_0) also satisfies condition (iii) (with $X=X_0$ and $Y=Y_0$). Suppose this is not the case. So there must exist a set $Z_1\subseteq X_0$ with

$$e(Z_1,V\setminus Z_1)\ >\ e(Z_1,Y_0)+e(Z_1,Y_0\setminus Y_0^{Z_1})+132\,|Z_1|.$$

Define $X_1=X_0\setminus Z_1$ and $Y_1=Y_0^{X_1}$. Again, by construction, (X_1,Y_1) satisfies conditions (i) and (ii) of (S3). If it does not satisfy condition (iii) we iterate the process (see Figure 2.2) and eventually obtain a pair (X_k,Y_k) satisfying conditions (i), (ii) and (iii) of (S3). We only need to check that $X_k\neq\varnothing$ and $Y_k\neq\varnothing$.

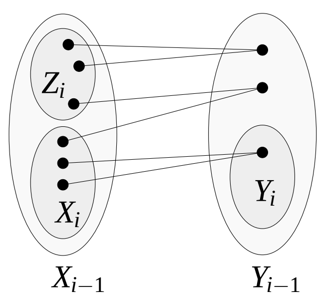

Figure 2.2: $X_i=X_{i-1}\setminus Z_i$ and $Y_i=Y_{i-1}^{X_i}$.

Let $1\le i\le k$. Since $X_i=X_{i-1}\setminus Z_i$, we have

$$\begin{aligned}
e(X_i,V\setminus Y_i) &= e(X_{i-1},V\setminus Y_i)-e(Z_i,V\setminus Y_i)\\
&= e(X_{i-1},V\setminus Y_{i-1})+e(X_{i-1},Y_{i-1}\setminus Y_i)\\
&\qquad\qquad -e(Z_i,V\setminus Y_{i-1})-e(Z_i,Y_{i-1}\setminus Y_i)\\
&= e(X_{i-1},V\setminus Y_{i-1})-e(Z_i,V\setminus Y_{i-1})+e(X_i,Y_{i-1}\setminus Y_i).
\end{aligned}$$

Since $Y_i = Y_{i-1}^{X_i}$, every neighbor $u \in Y_{i-1} \setminus Y_i$ of a vertex from X_i has exactly one neighbor in Z_i (see Figure 2.2). Hence, $e(X_i, Y_{i-1} \setminus Y_i) = e(Z_i, Y_{i-1} \setminus Y_{i-1}^{Z_i})$. So we have

$$\begin{aligned} e(X_{i-1}, V \setminus Y_{i-1}) &= e(X_i, V \setminus Y_i) + e(Z_i, V \setminus Y_{i-1}) - e(Z_i, Y_{i-1} \setminus Y_{i-1}^{Z_i}) \\ &= e(X_i, V \setminus Y_i) + e(Z_i, V) - e(Z_i, Y_{i-1}) - e(Z_i, Y_{i-1} \setminus Y_{i-1}^{Z_i}). \end{aligned}$$

By the definition of Z_i, we have $e(Z_i, V) \geq e(Z_i, V \setminus Z_i) > e(Z_i, Y_{i-1}) + e(Z_i, Y_{i-1} \setminus Y_{i-1}^{Z_i}) + 132\,|Z_i|$. Hence we obtain

$$\begin{aligned} e(X_{i-1}, V \setminus Y_{i-1}) &\geq e(X_i, V \setminus Y_i) - e(Z_i, Y_{i-1}) - e(Z_i, Y_{i-1} \setminus Y_{i-1}^{Z_i}) + e(Z_i, Y_{i-1}) \\ &\qquad\qquad + e(Z_i, Y_{i-1} \setminus Y_{i-1}^{Z_i}) + 132\,|Z_i| \\ &\geq e(X_i, V \setminus Y_i) + 132\,|Z_i|. \end{aligned}$$

Setting $Z^* = \bigcup_{1 \leq i \leq k} Z_i$, we have $e(X_k, V \setminus Y_k) \leq e(X_0, V \setminus Y_0) - 132\,|Z^*|$. As a consequence,

$$|Z^*| \leq \frac{e(X_0, V \setminus Y_0) - e(X_k, V \setminus Y_k)}{132} \leq \frac{e(X_0, V \setminus Y_0)}{132} < \frac{132\,|X_0|}{132} = |X_0|.$$

Since $X_k = X_0 \setminus Z^*$, this implies $|X_k| > 0$, which leads to $X_k \neq \varnothing$.

Finally, let $v \in X_k$. Taking $W = \{v\}$ in the inequality (iii) of (S3) (which by construction is satisfied by (X_k, Y_k)), we obtain $d(v) \leq 2\, d_{Y_k}(v) + 132$, where $d_{Y_k}(v)$ denotes the number of neighbors of v in Y_k. Since v is a big vertex, $d(v) \geq 133$ and so $d_{Y_k}(v) \geq \frac{1}{2}(133 - 132) > 0$. This means that we must have $Y_k \neq \varnothing$, which concludes the proof of Lemma 2.12. ∎

2.3.2 Proof of Lemma 2.19

We recall the hypotheses of the lemma: We have a real number γ; H is a multigraph; each vertex v of H has an associated integer $D(v)$; and for each edge e a positive number b_e is given. The following three conditions are satisfied:

(H1') For all vertices v in H: $d(v) \leq D(v) \leq \beta$.

(H2') For all edges $e = uv$ in H: $b_e \geq \left(\frac{3}{2}\beta + K\right) - (D(u) - d(u)) - (D(v) - d(v))$.

(H3') For all non-empty subsets $W \subseteq V(H)$: $\sum_{w \in W} (D(w) - d(w)) \leq e_H(W, V(H) \setminus W) + \gamma\,|W|$.

In the proof that follows, at certain moments we will give lower bounds for K so that any K satisfying all these lower bounds will satisfy the lemma, i.e., such that the vector $\vec{x} = (x_e)$, $x_e = \frac{1}{b_e}$ will be in $\mathcal{MP}(H)$.

For an edge $e = uv$ in H, define

$$a_e = \left(\tfrac{3}{2}\,\beta + K\right) - (D(u) - d(u)) - (D(v) - d(v)) \qquad \text{and} \qquad y_e = \frac{1}{a_e}. \tag{2.1}$$

We will in fact prove that the vector $\vec{y} = (y_e)$ is in the matching polytope $\mathcal{MP}(H)$. Since $b_e \geq a_e$, we have $x_e = \dfrac{1}{b_e} \leq \dfrac{1}{a_e} = y_e$. So, by Edmonds' characterisation of the matching polytope, if $\vec{y} \in \mathcal{MP}(H)$, this guarantees that $\vec{x} \in \mathcal{MP}(H)$, as required.

Applying condition (H3') to the set $W = \{v\}$ gives $D(v) - d(v) \leq d(v) + \gamma$, which implies:

(a) *For all vertices $v \in V(H)$ we have $d(v) \geq \frac{1}{2}\,(D(v) - \gamma)$.*

Let $e = uv$ be an edge of H. If we use the estimate above for both u and v in the definition of a_e in (2.1), and recalling that $D(u), D(v) \leq \beta$, we obtain

$$a_e \;\geq\; \tfrac{3}{2}\,\beta - \tfrac{1}{2}\,D(u) - \tfrac{1}{2}\,D(v) + K - \gamma \;\geq\; \tfrac{1}{2}\beta + K - \gamma.$$

On the other hand, if we use observation (a) to u only we get

$$a_e \;\geq\; d(v) + \tfrac{3}{2}\,\beta - \tfrac{1}{2}\,D(u) - D(v) + K - \tfrac{1}{2}\,\gamma \;\geq\; d(v) + K - \tfrac{1}{2}\,\gamma.$$

So if we make sure that $K \geq 2\,\gamma$, then the following two conclusions hold.

(b) *For all edges $e = uv$ in $E(H)$ we have $a_e \geq d(v) + \frac{1}{2}K$.*

(c) *For all edges $e \in E(H)$ we have $a_e \geq \frac{1}{2}\,\beta + \frac{1}{2}\,K$.*

Note that observation (c) also gives $b_e \geq a_e \geq \frac{1}{2}\,\beta$ for all $e \in E(H)$, as required.

Next notice that for any $\kappa > 0$, the function $x \mapsto \frac{x}{x+\kappa}$ is increasing in x. Together with the fact that $d(v) \leq \beta$ for all $v \in V(H)$ and observation (b), we find

$$\sum_{e \ni v} \frac{1}{a_e} \;\leq\; d(v) \cdot \frac{1}{d(v) + \frac{1}{2}\,K} \;\leq\; 1, \text{ which shows that}$$

Claim 2.20 *For all vertices $v \in V(H)$ we have $\displaystyle\sum_{e \ni v} y_e \leq 1$.*

Using Theorem 2.17, all that remains is to prove that for all $W \subseteq V(H)$ with $|W| \geq 3$ and $|W|$ odd we have $\sum\limits_{e \in E(W)} y_e \leq \frac{1}{2}\,(|W| - 1)$. We actually will prove this for all $|W| \geq 3$. Note that we certainly can assume $E(W) \neq \varnothing$.

Using observation (b), we infer that:

$$\sum_{e \in E(W)} \frac{1}{a_e} \leq \frac{1}{2} \sum_{u \in W} \frac{d_{H[W]}(u)}{d(u) + \frac{1}{2}\,K} = \frac{1}{2} \sum_{u \in W} \Bigl(\frac{d(u)}{d(u) + \frac{1}{2}\,K} - \frac{d(u) - d_{H[W]}(u)}{d(u) + \frac{1}{2}\,K}\Bigr).$$

Since $\dfrac{d(u)}{d(u)+\frac{1}{2}K} \le \dfrac{\beta}{\beta+\frac{1}{2}K}$ and $\dfrac{d(u)-d_{H[W]}(u)}{d(u)+\frac{1}{2}K} \ge \dfrac{d(u)-d_{H[W]}(u)}{\beta+\frac{1}{2}K}$, this implies

$$\sum_{e\in E(W)} \frac{1}{a_e} \;\le\; \frac{1}{2}\,|W|\,\frac{\beta}{\beta+\frac{1}{2}K} - \frac{1}{2}\,\frac{e(W,W^c)}{\beta+\frac{1}{2}K}.$$

Here we used that $\sum_{u\in W}\big(d(u)-d_{H[W]}(u)\big) = e(W,W^c)$, where $W^c = V(H)\setminus W$.

If $e(W,W^c) \ge \beta$, we obtain

$$\sum_{e\in E(W)} y_e \;\le\; \frac{1}{2}\,(|W|-1)\cdot\frac{\beta}{\beta+\frac{1}{2}K} \;\le\; \frac{1}{2}\,(|W|-1),$$

provided that $K \ge 0$.

So we can assume in the following that $e(W,W^c) \le \beta$, in which case Condition (H3') of Lemma 2.19 implies

$$\sum_{u\in W}(D(u)-d(u)) \;\le\; e(W,W^c)+\gamma\,|W| \;\le\; \beta+\gamma\,|W|.$$

For a vertex u set $c(u) = D(u)-d(u)$, and for a set of vertices U we define $c(U) = \sum_{u\in U} c(u)$. So we can write the above as $c(W) \le \beta+\gamma\,|W|$.

In the following we use the fact that all a_e are large enough to find a bound for the sum $\sum_{e\in E(W)} a_e^{-1}$. To this aim, recall from definition (2.1) that $a_e = \big(\frac{3}{2}\beta+K\big)-c(u)-c(v)$ for all edges $e=uv$ in H. This gives

$$\sum_{e\in E(W)} a_e \;\ge\; \big(\tfrac{3}{2}\beta+K\big)\,|E(W)| - \sum_{u\in W} c(u)\,d_{H[W]}(u).$$

Since $d_{H[W]}(u) \le d(u) = D(u)-c(u) \le \beta - c(u)$, we have

$$\sum_{e\in E(W)} a_e \;\ge\; \big(\tfrac{3}{2}\beta+K\big)\,|E(W)| - \beta\,c(W) + \sum_{u\in W} c(u)^2.$$

Set $p = \min_{uv\in E(W)}\big\{\big(\frac{3}{2}\beta+K\big)-c(u)-c(v)\big\}$ and $q = \frac{3}{2}\beta+K$. This means that $q-p = \max_{uv\in E(W)}\big\{c(u)+c(v)\big\}$. Let $e=uv$ be an edge in $E(W)$ so that $c(u)+c(v) = q-p$. Then $c(u)^2+c(v)^2 \ge \frac{1}{2}\,(q-p)^2$, and hence we can be sure that

$$\sum_{e\in E(W)} a_e \;\ge\; q\,|E(W)| - \beta\,c(W) + \tfrac{1}{2}\,(q-p)^2.$$

We now use this inequality and the following claim to bound $\sum_{e\in E(W)} a_e^{-1}$.

Claim 2.21 *Let $r_1, \dots, r_m$ be m real numbers so that $1 < p \le r_1, \dots, r_m \le q$ and $\sum_{1\le i\le m} r_i \ge q\,m - (q-p)\,S$, for some $S \ge 0$. Then we have $\sum_{1\le i\le m} r_i^{-1} \le \frac{S}{p} + \frac{m-S}{q}$.*

Proof The result is trivial if $p = q$, so suppose $p < q$. For any $1 \le i \le m$, set $c_i = \frac{q-r_i}{q-p}$. Now we have $0 \le c_i \le 1$ for all $1 \le i \le m$, and $\sum_{1\le i\le m} c_i \le S$. Since the function $x \mapsto \frac{1}{x}$ is convex, we have that for $1 \le i \le m$,

$$\frac{1}{r_i} = \frac{1}{q - c_i\,(q-p)} = \frac{1}{c_i\,p + (1-c_i)\,q} \le c_i\,\frac{1}{p} + (1-c_i)\,\frac{1}{q} = c_i\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{1}{q}.$$

As a consequence,

$$\sum_{1\le i\le m} \frac{1}{r_i} \le \left(\frac{1}{p} - \frac{1}{q}\right) \sum_{1\le i\le m} c_i + \frac{m}{q} \le \left(\frac{1}{p} - \frac{1}{q}\right) S + \frac{m}{q} \le \frac{S}{p} + \frac{m-S}{q}. \qquad \square$$

We set $R = \beta\,c(W) - \frac{1}{2}\,(q-p)^2$ and $S = \frac{R}{q-p}$. Using Claim 2.21, at this point we have

$$\sum_{e\in E(W)} \frac{1}{a_e} \le \frac{S}{p} + \frac{|E(W)| - S}{q} = \frac{S\,(q-p)}{p\,q} + \frac{|E(W)|}{q} = \frac{R}{p\,q} + \frac{2\,|E(W)|}{3\,\beta + 2K}.$$

Notice that by condition (H3') of Lemma 2.19, $2\,|E(W)| \le \sum_{u\in W} D(u) - 2\,c(W) + \gamma\,|W| \le \beta\,|W| - 2\,c(W) + \gamma\,|W|$. Hence we find

$$\sum_{e\in E(W)} \frac{1}{a_e} \le \frac{\beta\,|W|}{3\beta + 2K} + \frac{R}{p\,q} - \frac{2\,c(W)}{3\beta + 2K} + \frac{\gamma\,|W|}{3\,\beta + 2K}. \tag{2.2}$$

Claim 2.22 *For K large enough we have $\frac{R}{p\,q} - \frac{2\,c(W)}{3\,\beta + 2K} + \frac{\gamma\,|W|}{3\,\beta + 2K} \le \frac{2K}{3\,(3\,\beta + 2K)}\,|W|$.*

Proof Since $q = \frac{3}{2}\,\beta + K$, we only have to prove that $\frac{2\,R}{p} - 2\,c(W) + \gamma\,|W| \le \frac{1}{3}\,K\,|W|$.

Let us write $q - p = \alpha\,\beta$, and so $p = \frac{1}{2}\,(3 - 2\,\alpha)\,\beta + K$ and $R = \beta\,c(W) - \frac{1}{2}\,\alpha^2\,\beta^2$. Using that $c(W) \le \beta + \gamma\,|W|$, we have

$$\begin{aligned}
\frac{2\,R}{p} - 2\,c(W) + \gamma\,|W| &= \frac{2\,\beta\,c(W)}{p} - \frac{\alpha^2\,\beta^2}{p} - 2\,c(W) + \gamma\,|W| \\
&= c(W)\,\frac{2\,\beta - 2\,p}{p} - \frac{\alpha^2\,\beta^2}{p} + \gamma\,|W| \\
&\le \frac{\beta}{p}\,(2\,\beta - 2\,p - \alpha^2\,\beta) + \gamma\,|W|\,\frac{2\,\beta - p}{p}.
\end{aligned}$$

As $2\,p\,=\,(3-2\,\alpha)\,\beta+2K$, we have $2\,\beta-2\,p-\alpha^2\,\beta\,=\,(-1+2\,\alpha-\alpha^2)\,\beta-2K\,=\,-(\alpha-1)^2\,\beta-2K\,<\,0$. Note that by observation (a) and condition (H1') we have $q-p\leq\beta+\gamma$, hence if we choose $K\geq\gamma$, then we have $q-p\leq\ \beta+K$, and hence $p\geq\frac{1}{2}\,\beta$. We can conclude $\dfrac{2\,R}{p}-2\,c(W)+\gamma\,|W|\leq 3\,\gamma\,|W|$. As soon as $K\geq\frac{9}{2}\,\gamma$, we have $3\,\gamma\,|W|\leq\frac{2}{3}\,K\,|W|$, which completes the proof of the claim. □

Combining (2.2) and Claim 2.22, we obtain that for any $K\geq\frac{9}{2}\gamma$:

$$\begin{aligned}\sum_{e\in E(W)} y_e \;=\; \sum_{e\in E(W)}\frac{1}{a_e} \;&\leq\; \frac{\beta|W|}{3\beta+2K}+\frac{2K|W|}{3\,(3\beta+2K)}\\ &=\; \frac{(\beta+\frac{2}{3}K)}{(3\,\beta+2K)}\,|W| \;=\; \tfrac{1}{3}\,|W|.\end{aligned}$$

Since $|W|\geq 3$, $\frac{1}{3}\,|W|\leq\frac{1}{2}\,(|W|-1)$, which completes the proof of the lemma. ■

2.4 Proof of Theorem 2.11

Let γ and γ' be as given in Lemma 2.12, and take $\gamma_1=\max\bigl\{\bigl\lceil\frac{1}{4}\,(3\,\gamma+37)\bigr\rceil,11\bigr\}$ and $\beta_1\,=\,\gamma'$. Next take $\beta\,\geq\,\gamma'$. Suppose the theorem is false, and let the planar graph G be a counterexample with the minimum number of vertices, for some $B\subseteq V$.

Suppose G contains vertices u,v that are incident with a common face, and so that $d(u)+d(v)\leq\beta$. Construct a new planar graph G_1 by identifying u and v into a new vertex w. Set $V_1=(V\setminus\{u,v\})\cup\{w\}$, and notice that G_1 has strictly fewer vertices than G, and w has degree at most $d_G(u)+d_G(v)\leq\beta$ in G_1. In other words, $w\in V_1^\beta$. If $v\notin B$, then set $B_1=B$; otherwise, set $B_1=(B\setminus\{u,v\})\cup\{w\}$.

Every (B^β,B)-clique in G not containing u corresponds to a (B_1^β,B_1)-clique in G of the same size. Since G was chosen as the smallest counterexample to Theorem 2.11, this means that every (B^β,B)-clique in G of size larger than $\frac{3}{2}\,\beta+\gamma_1$ must contain u. On the other hand, any (B^β,B)-clique in G containing u has size at most $1+d^\beta(u)$.

We can conclude that for all pairs of vertices u,v in G incident with a common face and with $d(u)+d(v)\leq\beta$, we have that u and v are in every (B^β,B)-clique of size larger than $\frac{3}{2}\,\beta+\gamma_1$, and these vertices satisfy $d^\beta(u),d^\beta(v)\geq\frac{3}{2}\,\beta+\gamma_1$.

Since $\beta\geq\gamma'$, we can apply Lemma 2.12. We use the notation from the lemma. Because of the observation above, conclusions (S1) and (S2) of that lemma are not possible. Hence we know that G contains $X,Y\subseteq V^\beta$ satisfying (S3) from the lemma. We recall the crucial properties :

(i) *Every vertex $y \in Y$ has degree at most four. Moreover, y is adjacent to exactly two vertices of X and the other neighbors of y have degree at most four as well.*

For $y \in Y$, let X^y be the set of its two neighbors in X. And for $W \subseteq X$, let Y^W be the set of vertices $y \in Y$ with $X^y \subseteq W$ (that is, the set of vertices of Y having their two neighbors from X in W).

(ii) *For all pairs of vertices $y, z \in Y$, if y and z are adjacent or have a common neighbor $w \notin X$, then $X^y = X^z$.*

(iii) *For all nonempty subsets $W \subseteq X$, we have the following inequality:*

$$e(W, V \setminus W) \;\le\; e(W, Y) + e(W, Y \setminus Y^W) + \gamma\,|W|.$$

By (i), it follows that all vertices in Y are in every (B^β, B)-clique of size larger than $\frac{3}{2}\beta + \gamma_1$. Hence in particular:

(a) *For every $y \in Y$ we have $d^\beta(y) \ge \frac{3}{2}\beta + \gamma_1$.*

Also by the properties of the vertices in Y according to (i) and (ii) we have for all $y \in Y$ and $X^y = \{x_1, x_2\}$:

$$\begin{aligned} d^\beta(y) \;&\le\; 4 + 2\cdot(4-1) + (d(x_1) - 1) + (d(x_2) - 1) - |Y^{\{x_1,x_2\}} \setminus \{y\}| \\ &=\; 9 + d(x_1) + d(x_2) - |Y^{\{x_1,x_2\}}| \end{aligned}$$

(the term $|Y^{\{x_1,x_2\}} \setminus \{y\}|$ is subtracted, since these vertices are counted twice in $(d(x_1) - 1) + (d(x_2) - 1)$). Since $d(x_1), d(x_2) \le \beta$, from (a) we can conclude that

(b) *for every pair $x_1, x_2 \in X$ we have $|Y^{\{x_1,x_2\}}| \le \frac{1}{2}\beta - \gamma_1 + 9$.*

We also must have that all pairs of vertices from Y are adjacent or have a common neighbor from B^β. By (ii), this proves that for every two vertices $y_1, y_2 \in Y$ we have $X^{y_1} \cap X^{y_2} \ne \varnothing$. As a consequence, if X' denotes the set of vertices of X with at least one neighbor in Y, and H denotes the graph with vertex set X' in which two vertices are adjacent if they have a common neighbor in Y, then H is either a triangle or a star.

Case 1. H is a triangle or H is a star with at most two leaves.
First suppose H is a triangle. Let $y \in Y$ with $(X')^y = \{x_1, x_2\}$, where $X' = \{x_1, x_2, x_3\}$. Then $Y = Y^{\{x_1,x_2\}} \cup Y^{\{x_1,x_3\}} \cup Y^{\{x_2,x_3\}}$, hence by (b), $|Y| \le \frac{3}{2}\beta - 3\gamma_1 + 27$. Since all vertices in Y are in every (B^β, B)-clique of size larger than $\frac{3}{2}\beta + \gamma_1$, we can estimate, using (i):

$$\begin{aligned} d^\beta(y) \;&\le\; 2\cdot(4-1) + |X'| + |Y| + e(\{x_1, x_2\}, V \setminus (X' \cup Y)) \\ &\le\; \tfrac{3}{2}\beta - 3\gamma_1 + 36 + e(\{x_1, x_2\}, V \setminus (X' \cup Y)). \end{aligned}$$

By (iii) we have, using that $Y^{X'} = Y$ by definition of X',

$$e(\{x_1, x_2\}, V \setminus (X' \cup Y)) \le e(X', V \setminus (X' \cup Y)) = e(X', V \setminus X') - e(X', Y) \le 3\gamma.$$

These two estimates give $d^\beta(y) \le \frac{3}{2}\beta + 3\gamma + 36 - 3\gamma_1$, which contradicts (a), since $4\gamma_1 \ge 3\gamma + 37$.

If H is a star with at most two leaves, then similar arguments will give a contradiction.

Case 2. H is a star with at least three leaves.
For any $y \in Y$, the β-neighbors of y in G are the neighbors of y, the neighbors of y's neighbors of degree four, the neighbors of the centre of the star (there are at most β of these), or the vertices adjacent to all the leaves of the star. Since H has at least three leaves and G is planar, there is at most one vertex of the last type. Subtracting one when y itself appears as one of the types above, we can estimate

$$d^\beta(y) \;\le\; 4 + 2\cdot(4-1) + (\beta - 1) + 1 \;\le\; \beta + 10.$$

Again we find a contradiction with (a), which completes the proof of the theorem. ∎

Lemma 2.12 was proved with $\gamma = 132$ and $\gamma' = 1060$. Following the proof above means we can obtain $\gamma_1 = 109$ and $\beta_1 = 1060$ in Theorem 2.11. But it is clear that these values are far from best possible. Using more elaborate discharging arguments and more careful reasoning in the final parts of the proof of Lemma 2.12 can give significantly smaller values. Since our first goal is to show that we can obtain constant values for these results, we do not pursue this further.

2.5 Conclusion

2.5.1 About the Proof

The proof of our main theorem in general follows the same lines as the proof of Theorem 2.3 in [HHM$^+$07]. In particular, the proof of that theorem also starts with a structural lemma comparable to Lemma 2.12, uses the structure of the graph to reduce the problem to edge-coloring a specific multigraph, and then apply (and extend) Kahn's approach to that multigraph. Of course, a difference is that Theorem 2.3 only deals with list coloring the square of a graph, but it is probably quite straightforward to generalize the whole proof to the case of list (B^β, B)-coloring. Nevertheless, there are some important differences in the proofs we feel deserve highlighting.

Lemma 2.12 is stronger than the comparable lemma in [HHM$^+$07]. The properties of the set Y in Lemma 2.12 mean that in our proof we can construct a multigraph H so that a standard list edge-coloring of H provides the information to color the vertices in Y (see Lemma 2.14). In the lemma in [HHM$^+$07], the translation to a list-edge coloring of a multigraph is not so clean; apart from the normal condition in the list edge-coloring of H (that adjacent edges need different colors), for each edge there may be up to $O(\Delta^{1/2})$ non-adjacent edges that also need to get a different color. In particular this means that in [HHM+07], Kahn's result in Theorem 2.18 cannot be used directly. Instead, a new, stronger, version has to be proved that can deal with a certain number of non-adjacent edges that need to be colored differently. Lemma 2.12 allows us to use Kahn's Theorem directly.

A second aspect in which our Lemma 2.12 is stronger is that in the final condition (S3)(iii), we have an "error term" that is a constant times $|W|$. In [HHM+07] the comparable term is $\Delta^{9/10}\,|W|$, where Δ is the maximum degree of the graph. This in itself already means that the approach in [HHM+07] at best can give a bound of the type $\bigl(\frac{3}{2}+o(1)\bigr)\,\Delta$. The fact that we cannot do better with the stronger structural result is because of the limitations of Kahn's Theorem, Theorem 2.18. If it would be possible to replace the condition in that theorem by a condition of the form "the vector $\vec{x}=(x_e)$ with $x_e = \dfrac{1}{|L(e)|-K}$ for all $e \in E(H)$ is an element of $\mathcal{MP}(H)$", where K is some positive constant, the work in this paper would give an improvement for the bound in Theorem 2.10 to $\frac{3}{2}\,\beta + O(1)$ (because our version of Lemma 2.19 is strong enough to also support that case).

Lemma 2.12 also allows us to prove a bound $\frac{3}{2}\,\beta{+}O(1)$ for the (B^{β}, B)-clique number in Theorem 2.11. The important corollary that the square of a planar graph has clique number at most $\frac{3}{2}\,\Delta+O(1)$ would have been impossible without the improved bound in the lemma.

Also Lemma 2.19 is stronger than its compatriot in [HHM+07]. The lemma in [HHM+07] only deals with the case $D(v)=\beta$ for all vertices v in H. Because of this, it can only be applied to the case that all vertices in H have maximum degree Δ. Some non-trivial trickery then has to be used to deal with the case that there are vertices in H of degree less than Δ. Apart from that difference, the proof of Lemma 2.19 is completely different from the proof in [HHM+07]. We feel that our new proof is more natural and intuitive, giving a clear relation between the lower bounds on the sizes of the lists and the upper bound of the sum of their inverses. The proof in [HHM+07] is more ad-hoc, using some non-obvious distinction in a number of different cases, depending on the size of W and the degrees of some vertices in W.

2.5.2 Further Work

A natural way to extend Wegner's and Borodin's conjectures to (A, B)-colorings is the following:

Conjecture 2.23 *There exist constants c_1, c_2, c_3 such that for all planar graphs G and $A, B \subseteq V$ we have*

$$\begin{aligned} \chi(G;A,B) &\le \left\lfloor \tfrac{3}{2}\,\Delta(G;A,B)\right\rfloor + c_1; \\ ch(G;A,B) &\le \left\lfloor \tfrac{3}{2}\,\Delta(G;A,B)\right\rfloor + c_2; \\ ch(G;A,B) &\le \left\lfloor \tfrac{3}{2}\,\Delta(G;A,B)\right\rfloor + 1, \qquad \text{if } \Delta(G;A,B) \ge c_3. \end{aligned}$$

If $A = \varnothing$ (hence $\Delta(G;A,B) = 0$) and $B = V$, then the Four Color Theorem means that the smallest possible value for c_1 is four; while the fact that planar graphs are always 5-list colorable but not always 4-list colorable, shows the smallest possible value for c_2 is five.

We feel that our work is just the beginning of the study of general (A,B)-coloring. It should be possible to obtain deeper results taking into account the structure of the two sets A and B, and not just the degrees of the vertices. The following easy result is an example of this.

Theorem 2.24 *Let $G = (V,E)$ be a planar graph and $A, B \subseteq V$. Suppose that for every two distinct vertices in A we have that their distance in G is at least three. Then $ch(G;A,B) \le \Delta(G;A,B) + 5$.*

Proof. Since G is planar, there exists an ordering $v_1, \ldots, v_n$ of the vertices so that each v_i has at most five neighbors in $\{v_1, \ldots, v_{i-1}\}$. We greedily color the vertices $v_1, \ldots, v_n$ that are in B in that order. Note that each vertex has at most one neighbor from A.

When coloring the vertex v_i, we need to take into account its neighbors in $\{v_1, \ldots, v_{i-1}\}$, plus the neighbors in $\{v_1, \ldots, v_{i-1}\}$ of a vertex $a \in A$ adjacent to v_i (where that vertex a can be in $\{v_{1+1}, \ldots, v_n\}$). By construction of the ordering, there are at most five neighbors of v_i in $\{v_1, \ldots, v_{i-1}\}$. And a neighbor $a \in A$ has at most $d_B(a) - 1 \le \Delta(G;A,B) - 1$ neighbors in $\{v_1, \ldots, v_{i-1}\}$ different from v_i. So the total number of forbidden colors when coloring v_i is at most $\Delta(G;A,B) + 4$. Since each vertex has $\Delta(G;A,B) + 5$ colors available, the greedy algorithm will always find a free color. ∎

Note that saying that the vertices in A have distance at least three is the same as saying that two different vertices in A have no common neighbor. We think that it is possible to generalize our main theorem and the theorem above in the following way. For $A, B \subseteq V$, let $k(G;A,B)$ be the maximum of $|N_B(a_1) \cap N_B(a_2)|$ over all $a_1, a_2 \in A$, $a_1 \ne a_2$.

Conjecture 2.25 *There exists a constant c so that for all planar graphs G and $A, B \subseteq V$ we have*

$$ch(G;A,B) \le \Delta(G;A,B) + k(G;A,B) + c.$$

This conjecture would fit with our current proof of Theorem 2.10, the main part of which is a reduction of the original problem to a list edge-coloring problem. For this approach, Shannon's Theorem [Sha49] that a multigraph with maximum degree Δ has an edge-coloring using at most $\left\lfloor \frac{3}{2}\,\Delta(G) \right\rfloor$ colors, forms a natural base for the bounds conjectured in Conjecture 2.23. If the relation between coloring the square of planar graphs and edge-coloring multigraphs holds in a stronger sense, then Conjecture 2.25 forms a logical extension of Vizing's Theorem [Viz64] that a multigraph with maximum degree Δ and maximum edge-multiplicity μ has an edge-coloring with at most $\Delta + \mu$ colors.

In Borodin *et al.* [BBG$^+$07], a weaker version of Conjecture 2.25 for cyclic coloring was proved. Recall that if G is a plane graph, then Δ^* is the maximum number of vertices in a face. Let k^* denote the maximum number of vertices that two faces of G have in common.

Theorem 2.26 [BBG$^+$07] *For a plane graph G with $\Delta^* \geq 4$ and $k^* \geq 4$ we have $\chi^*(G) \leq \Delta^* + 3\,k^* + 2$.*

Chapter 3

Frugal coloring

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In the previous chapter, we studied the coloring of the square of graphs (every pair of vertices at distance at most two must be assigned distinct colors). Another way to look at this coloring is to say that it is proper (no two adjacent vertices have the same color), and no color appears more than once in every neighborhood.

A natural way to generalize this is to consider a proper coloring such that no color appears more than p times in every neighborhood, for some given p. This coloring was introduced under the name of p-frugal coloring by Hind, Molloy and Reed [HMR97].

In this chapter, we study the frugal coloring of planar graphs, planar graphs with large girth, and outerplanar graphs, and relate this coloring with $L(p, q)$-labelling and cyclic coloring, both seen in the previous chapter. We also study frugal edge-colorings of multigraphs.

3.1 Introduction

For an integer $p \geq 1$, a *p-frugal coloring* of a graph G is a proper vertex coloring of G such that no color appears more than p times in the neighborhood of any vertex. Alternatively, a p-frugal coloring can be defined as a proper coloring in which every pair of color classes induces a subgraph with maximum degree at most k. The least number of colors in a p-frugal coloring of G is called the *p-frugal chromatic number* of G, denoted $\chi_p(G)$. Clearly, $\chi_1(G)$ is the chromatic number of G^2; and for p at least the maximum degree of G, $\chi_p(G)$ is the usual chromatic number of G. An easy consequence of the definition is that for any graph G with maximum degree Δ, we have $\chi_p(G) \geq \lceil \frac{\Delta}{p} \rceil + 1$.

$\chi_p(G)$

Let L be a list assignment for the vertices of a graph G. A p-frugal coloring c of G is called a *p-frugal L-coloring* if for any vertex v of G, $c(v) \in L(v)$. The smallest integer t, such that for any t-list assignment L, the graph G has a p-frugal L-coloring, is called the *p-frugal choice number* of G, denoted by $ch_p(G)$.

$ch_p(G)$

Recall that a *multigraph* is a graph which can have multiple edges (loops are not allowed). A *p-frugal edge coloring* of a multigraph G is a (possibly improper) coloring of the edges of G such that no color appears more than p times on the edges incident with a vertex. The least number of colors in a p-frugal edge coloring of G, the *p-frugal chromatic index* of G, is denoted by $\chi'_p(G)$. Observe that for $p = 1$ we have $\chi'_1(G) = \chi'(G)$, the usual chromatic index of G. We can also define the *p-frugal edge choice number* in the same way (see Section 3.6). Again, a straightforward consequence of the definition is that for any graph G with maximum degree Δ, we have $\chi'_p(G) \geq \lceil \frac{\Delta}{p} \rceil$.

$\chi'_p(G)$

Frugal vertex colorings were introduced by Hind *et al.* [HMR97], as a tool towards improving results about the *total chromatic number* of a graph. One of their results is that a graph with large enough maximum degree Δ has a $(\log^8 \Delta)$-frugal coloring using at most $\Delta + 1$ colors. They also show that there exist graphs for which a $\left(\frac{\log \Delta}{\log \log \Delta}\right)$-frugal coloring cannot be achieved using only $O(\Delta)$ colors.

Our aim in this chapter is to study some aspects of frugal colorings and frugal list colorings in their own right. In the first part we consider frugal vertex colorings of planar graphs. We show that frugal coloring is related with $L(p, q)$-labellings in general, and with cyclic coloring in the case of planar graphs (these two notions have been introduced in the previous chapter).

In the final section we derive some results on frugal edge colorings of multigraphs in general.

3.2 Frugal coloring of planar graphs

In the next four sections we consider p-frugal (list) colorings of planar graphs. For a large part, our work in that area is inspired by Wegner's conjecture mentionned in the previous chapter.

Conjecture 3.1 [Weg77] *For any planar graph G of maximum degree $\Delta(G) \geq 8$ we have $\chi(G^2) \leq \left\lfloor \frac{3}{2}\,\Delta(G) \right\rfloor + 1$.*

Wegner also conjectured maximum values for the chromatic number of the square of planar graph with maximum degree less than eight and gave examples showing that his bounds would be tight. For even $\Delta \geq 8$, these examples are sketched in Figure 3.1.

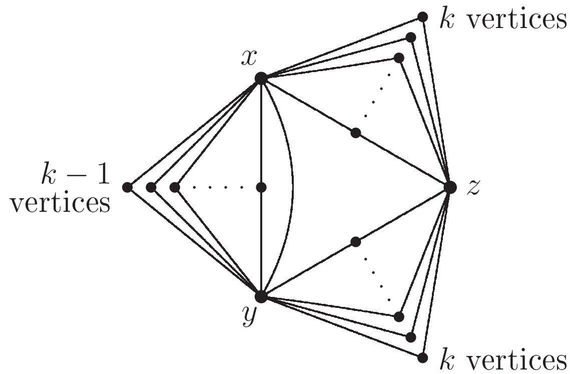

Figure 3.1: The planar graphs G_k.

Inspired by Wegner's Conjecture, we conjecture the following bounds for the p-frugal chromatic number of planar graphs.

Conjecture 3.2 [AEH07] *For any integer $p \geq 1$ and any planar graph G with maximum degree $\Delta(G) \geq \max\{\,2\,p,\ 8\,\}$ we have*

$$\chi_p(G) \;\leq\; \begin{cases} \left\lfloor \frac{\Delta(G)-1}{p} \right\rfloor + 2, & \text{if } p \text{ is even;} \\ \left\lfloor \frac{3\,\Delta(G)-2}{3\,p-1} \right\rfloor + 2, & \text{if } p \text{ is odd.} \end{cases}$$

Note that the graphs G_k in Figure 3.1 also show that the bounds in this conjecture cannot be decreased. The graph G_k has maximum degree $2\,k$. First consider a p-frugal coloring with $p = 2\,\ell$ even. We can use the same color at most $\frac{3}{2}\,p$ times on the vertices of G_k, and every color that appears exactly $\frac{3}{2}\,p = 2\,\ell$ times must appear exactly ℓ times on each of the three sets of common neighbors of x and y, of x and z, and of y and z. So we can take at most $\frac{1}{\ell}\,(k-1) = \frac{1}{p}\,(\Delta(G_k)-1)$ colors that are used $\frac{3}{2}\,p$ times. In this case, x and y must then be colored with two new colors, since otherwise the neighborhood of x or y contains more than p times the same color.

If $p = 2\ell + 1$ is odd, then each color can appear at most $3\ell + 1 = \frac{1}{2}(3p-1)$ times, and the only way to use a color so many times is by using it on the vertices in $V(G_k) \setminus \{x, y, z\}$. Doing this at most $\frac{3k-1}{(3p-1)/2} = \frac{3\Delta(G)-2}{3p-1}$ times, we are left with a graph that requires at least two new colors.

We next derive some upper bounds on the p-frugal chromatic number of planar graphs. The first one is a simple extension of the approach from [HM03]. In that paper, Van den Heuvel and McGuinness prove the following structural lemma:

Lemma 3.3 [HM03] *Let G be a planar simple graph. Then there exists a vertex v with m neighbors $v_1, \ldots, v_k$ with $d(v_1) \le \cdots \le d(v_k)$ such that one of the following holds:*

(i) $k \le 2$*;*

(ii) $k = 3$ *with* $d(v_1) \le 11$*;*

(iii) $k = 4$ *with* $d(v_1) \le 7$ *and* $d(v_2) \le 11$*;*

(iv) $k = 5$ *with* $d(v_1) \le 6$*,* $d(v_2) \le 7$*, and* $d(v_3) \le 11$.

In [HM03], this structural lemma is used to prove that the chromatic number of the square of a planar graph is at most $2\Delta + 25$. Making slight changes in their proof, it is not difficult to obtain a first bound on ch_p (and hence on χ_p) for planar graphs.

Theorem 3.4 [AEH07] *For any planar graph G with $\Delta(G) \ge 12$ and integer $p \ge 1$ we have $ch_p(G) \le \left\lfloor \frac{2\Delta(G)+19}{p} \right\rfloor + 6$.*

Proof. We will prove that if a planar graph satisfies $\Delta(G) \le C$ for some $C \ge 12$, then $ch_p(G) \le \left\lfloor \frac{2C+19}{p} \right\rfloor + 6$. We use induction on the number of vertices, noting that the result is obvious for small graphs. So let G be a graph with $|V(G)| > 1$, choose $C \ge 12$ so that $\Delta(G) \le C$, and assume each vertex v has a list $L(v)$ of $\left\lfloor \frac{2C+19}{p} \right\rfloor + 6$ colors. Take $v, v_1, \ldots, v_k$ as in Lemma 3.3. Contracting the edge vv_1 to a new vertex v' will result in a planar graph G' in which all vertices except v' have degree at most as much as they had in G, while v' has degree at most $\Delta(G)$ (for case (i)) or at most 12. (for the cases (ii) – (iv)). In particular we have that $\Delta(G') \le C$. If we give v' the same list of colors as v_1 had (all vertices in $V(G) \setminus \{v, v_1\}$ keep their list), then, using induction, G' has a p-frugal coloring. Using the same coloring for G, where v_1 gets the color v' had in G', we obtain a p-frugal coloring of G with the one deficit that v has no color yet. But the colors forbidden for v are the colors on its neighbors, and for each neighbor v_i, the colors that already appear p times around v_i.

So the number of forbidden colors is at most $k + \sum_{i=1}^{k} \left\lfloor \frac{d(v_i)-1}{p} \right\rfloor$. Using the knowledge from the cases (i) – (iv), we get that $|L(v)| = \left\lfloor \frac{2C+19}{p} \right\rfloor + 6$ is at least one more than this number of forbidden colors, hence we can always find an allowed color for v. ∎

In the next section we will obtain (asymptotically) better results based on more recent work on special labellings of planar graphs.

3.3 Frugal coloring and $L(p, q)$-labelling

In this section, we relate frugal colorings with $L(p, q)$-labelling, a generalization of the coloring of the square of graphs seen in the previous chapter. Our main tool is the following proposition:

Proposition 3.5 *For any graph G and integer $p \geq 1$ we have $\chi_p(G) \leq \left\lceil \frac{1}{p} \lambda_{p,1}(G) \right\rceil$ and $ch_p(G) \leq \left\lceil \frac{1}{p} \lambda^l_{p,1}(G) \right\rceil$.*

Proof. We only prove the second part, the first one can be done in a similar way. Set $\ell = \left\lceil \frac{1}{p} \lambda^l_{p,1}(G) \right\rceil$, and let L be an ℓ-list assignment on the vertices of G. Using that all elements in the lists are integers, we can define a new list assignment L^* by setting $L^*(v) = \bigcup_{x \in L(v)} \{p\,x,\, p\,x+1,\, \ldots,\, p\,x+p-1\}$. Then L^* is a $(p\,\ell)$-list assignment. Since $p\,\ell \geq \lambda^l_{p,1}(G)$, there exists an $L(p, 1)$-labelling f^* of G with $f^*(v) \in L^*(v)$ for all vertices v. Define a new labelling f of G by taking $f(v) = \left\lfloor \frac{1}{p} f^*(v) \right\rfloor$. We immediately get that $f(v) \in L(v)$ for all v. Since adjacent vertices received an f^*-label at least p apart, their f-labels are different. Also, all vertices in a neighborhood of a vertex v received a different f^*-label. Since the map $x \mapsto \left\lfloor \frac{1}{p} x \right\rfloor$ maps at most p different integers x to the same image, each f-label can appear at most p times in each neighborhood. So f is a p-frugal coloring using labels from each vertex' list. This proves that $ch_p(G) \leq \ell$, as required. ∎

We will combine this proposition with the following recent result from Havet *et al.*, already mentionned in the previous chapter.

Theorem 3.6 [HHM[+]07] *For any fixed p, and any planar graph G with maximum degree Δ, we have $\lambda^l_{p,1}(G) \leq \left(\frac{3}{2} + o(1)\right)\Delta$.*

Combining this with Proposition 3.5 gives the asymptotically best upper bound for χ_p and ch_p for planar graphs we currently have.

Corollary 3.7 *Fix $\varepsilon > 0$ and an integer $p \geq 1$. Then there exists an integer $\Delta_{\varepsilon,p}$ so that if G is a planar graph with maximum degree $\Delta(G) \geq \Delta_{\varepsilon,p}$, then $ch_p(G) \leq \frac{(3+\varepsilon)\,\Delta(G)}{2\,p}$.*

In [MS05], Molloy and Salavatipour proved that for any planar graph G and any integer $p \geq 1$, we have $\lambda_{p,1}(G) \leq \left\lceil \frac{5}{3}\,\Delta(G) \right\rceil + 18\,p + 60$. Together with Proposition 3.5, this refines the result of Theorem 3.4 and gives a better bound than Corollary 3.7 for small values of Δ. Note that this corollary only concerns frugal coloring, and not frugal list coloring.

Corollary 3.8 *For any planar graph G and integer $p \geq 1$, we have $\chi_p(G) \leq \left\lceil \frac{5\,\Delta(G)+180}{3\,p} \right\rceil + 18$.*

Proposition 3.5 has another corollary for planar graphs of large girth that we describe below. Recall that the *girth* of a graph is the length of a shortest cycle in the graph.

In [LW03], Lih and Wang proved that for planar graphs of large girth the following holds:

- $\lambda_{p,q}(G) \leq (2\,q-1)\,\Delta(G) + 6\,p + 12\,q - 8$ for planar graphs of girth at least six, and
- $\lambda_{p,q}(G) \leq (2\,q-1)\,\Delta(G) + 6\,p + 24\,q - 14$ for planar graphs of girth at least five.

Furthermore, Dvořák *et al.* [DKN+08] proved the following tight bound for $L(p,1)$-labellings of planar graphs of girth at least seven, and of large degree.

Theorem 3.9 [DKN+08] *Let G be a planar graph of girth at least seven, and maximum degree $\Delta(G) \geq 190 + 2\,p$, for some integer $p \geq 1$. Then we have $\lambda_{p,1}(G) \leq \Delta(G) + 2\,p - 1$.*

Moreover, this bound is tight, i.e., there exist planar graphs which achieve the upper bound.

A direct corollary of these results are the following bounds for planar graphs with large girth.

Corollary 3.10 *Let G be a planar graph with girth g and maximum degree $\Delta(G)$. For any integer $p \geq 1$, we have*

$$\chi_p(G) \leq \begin{cases} \left\lceil \frac{\Delta(G)-1}{p} \right\rceil + 2, & \text{if } g \geq 7 \text{ and } \Delta(G) \geq 190 + 2\,p; \\ \left\lceil \frac{\Delta(G)+4}{p} \right\rceil + 6, & \text{if } g \geq 6; \\ \left\lceil \frac{\Delta(G)+10}{p} \right\rceil + 6, & \text{if } g \geq 5. \end{cases}$$

3.4 Frugal coloring of outerplanar graphs

We now prove a variant of Conjecture 3.2 for outerplanar graphs (graphs that can be drawn in the plane so that all vertices are lying on the outside face). For $p = 1$, i.e., if we are coloring the square of the graph, Hetherington and Woodall [HW06] proved the best possible bound for outerplanar graphs G: $ch_1(G) \leq \Delta(G) + 2$ if $\Delta(G) \geq 3$, and $ch_1(G) = \Delta(G) + 1$ if $\Delta(G) \geq 6$.

Theorem 3.11 [AEH07] *For any integer $p \geq 2$ and any outerplanar graph G with maximum degree $\Delta(G) \geq 3$, we have $\chi_p(G) \leq ch_p(G) \leq \lfloor \frac{\Delta(G)-1}{p} \rfloor + 3$.*

Proof. In [EO07a] (see Appendix A for further details), we proved a result implying that any outerplanar graph contains a vertex u such that one of the following holds: (i) u has degree at most one; (ii) u has degree two and is adjacent to another vertex of degree two; or (iii) u has degree two and its neighbors v and w are adjacent, and either v has degree three or v has degree four and its two other neighbors (i.e., distinct from u and w) are adjacent (see Figure 3.2).

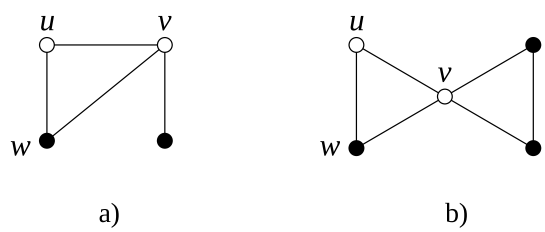

Figure 3.2: Unavoidable configurations in an outerplanar graph without 1-vertices and without two adjacent 2-vertices.

We prove the theorem by induction on the number of vertices, observing that it is trivial for graphs with at most two vertices. If G has at least three vertices, let u be a vertex of G having one of the properties described above. By the induction hypothesis, there exists a p-frugal list coloring c of $G - u$ if the lists $L(v)$ contain at least $\lfloor \frac{\Delta(G)-1}{p} \rfloor + 3$ colors. If u has property (i) or (ii), let t be the neighbor of u whose degree is not necessarily bounded by two. It is easy to see that at most $2 + \lfloor \frac{\Delta(G)-1}{p} \rfloor$ colors are forbidden for u: the colors of the neighbors of u and the colors appearing p times in the neighborhood of t. If u has property (iii), at most $2 + \lfloor \frac{\Delta(G)-2}{p} \rfloor$ colors are forbidden for u: the colors of the neighbors of u and the colors appearing p times in the neighborhood of w. Note that if v has degree four, its two other neighbors are adjacent and the p-frugality of v is respected since $p \geq 2$. In all cases we found that at most $\lfloor \frac{\Delta(G)-1}{p} \rfloor + 2$ colors are forbidden for u. If u has a list with one more color, we can extend c to a p-frugal list coloring of G, which completes the induction. ∎

3.5 Frugal coloring and cyclic coloring

In this section, we discuss the link between frugal coloring and cyclic coloring of plane graphs. Recall that a plane graph is a planar graph with a prescribed planar embedding, and that the size (number of vertices in its boundary) of a largest face of a plane graph G is denoted by $\Delta^*(G)$.

The previous chapter was devoted to the study of *cyclic coloring* of plane graphs: a vertex coloring such that any two vertices incident to the same face have distinct colors. Recall that Borodin [Bor84] (see also Jensen and Toft [JT95, page 37]) conjectured that any plane graph G has a cyclic coloring with $\lfloor \frac{3}{2}\,\Delta^*(G) \rfloor$ colors, and proved this conjecture for $\Delta^*(G) = 4$.

In this section we show that if there is an even $p \geq 4$ such that Borodin's conjecture holds for all plane graphs with $\Delta^* \leq p$, and if our Conjecture 3.2 is true for the same value p, then Wegner's conjecture is true up to an additive constant factor.

Theorem 3.12 [AEH07] *Let $p \geq 4$ be an even integer such that every plane graph H with $\Delta^*(H) \leq p$ has a cyclic coloring using at most $\frac{3}{2}\,p$ colors. Then, if G is a planar graph satisfying $\chi_p(G) \leq \lfloor \frac{\Delta(G)-1}{p} \rfloor + 2$, we also have $\chi(G^2) = \chi_1(G) \leq \lfloor \frac{3}{2}\,\Delta(G) \rfloor + 3p$.*

Proof. Let G be a planar graph with a given embedding and let $p \geq 4$ be an even integer such that $t = \chi_p(G) \leq \lfloor \frac{\Delta(G)-1}{p} \rfloor + 2$. Consider an optimal p-frugal coloring c of G, with color classes $C_1, \ldots, C_t$. For $i = 1, \ldots, t$, construct the graph G_i as follows: Firstly, G_i has vertex set C_i, which we assume to be embedded in the plane in the same way they were for G. For each vertex $v \in V(G) \setminus C_i$ with exactly two neighbors in C_i, we add an edge in G_i between these two neighbors. For a vertex $v \in V(G) \setminus C_i$ with $\ell \geq 3$ neighbors in C_i, let $x_1, \ldots, x_\ell$ be those neighbors in C_i in a cyclic order around v (determined by the plane embedding of G). Now add edges $x_1x_2, x_2x_3, \ldots, x_{\ell-1}x_\ell$ and $x_\ell x_1$ to G_i. These edges will form a face of size ℓ in the graph we have constructed so far. Call such a face a *special face*. Note that since C_i is a color class in a p-frugal coloring, this face has size at most p.

Do the above for all vertices $v \in V(G) \setminus C_i$ that have at least two neighbors in C_i. The resulting graph is a plane graph with some faces labelled special. Add edges to triangulate all faces that are not special. The resulting graph is a plane graph with vertex set G_i and every face size at most p. From the first hypothesis it follows that there is a cyclic coloring of each G_i with $\frac{3}{2}\,p$ new colors. Since every two vertices in C_i that have a common neighbor in G are adjacent in G_i or are incident to the same (special) face, vertices in C_i that are adjacent in the square of G receive different colors. Hence, combining these t colorings, using different colors for each G_i, we obtain a coloring of the square of G, using at most $\frac{3}{2}\,p \cdot \left(\lfloor \frac{\Delta(G)-1}{p} \rfloor + 2\right) \leq \lfloor \frac{3}{2}\,\Delta \rfloor + 3p$ colors. ∎

Since Borodin [Bor84] proved his cyclic coloring conjecture in the case $\Delta^* = 4$, we have the following corollary.

Corollary 3.13 *If G is a planar graph so that $\chi_4(G) \le \left\lfloor \frac{\Delta(G)-1}{4} \right\rfloor + 2$, then $\chi(G^2) \le \left\lfloor \frac{3}{2}\,\Delta(G) \right\rfloor + 12$.*

3.6 Frugal edge coloring

An important element in the proof of Theorem 2.10 in the previous chapter is the derivation of a relation between (list) coloring square of planar graphs and (list) edge colorings of multigraphs. Because of this, it seems to be opportune to have a short look at a frugal variant of edge colorings of multigraphs in general.

Edge colorings of multigraphs have the same definitions as for simple graphs: given a multigraph G, the minimum number of colors required is the *chromatic index*, denoted $\chi'(G)$. The *list chromatic index $ch'(G)$* is defined analogously as the minimum length of lists that needs to be given to each edge so that we can use colors from each edge's list to obtain a proper coloring. $\chi'(G)$ $ch'(G)$

A *p-frugal edge coloring* of a multigraph G is a (possibly improper) coloring of the edges of G such that no color appears more than p times on the edges incident with a vertex. The least number of colors in a p-frugal edge coloring of G, the *p-frugal edge chromatic number* (or *p-frugal chromatic index*), is denoted by $\chi'_p(G)$. $\chi'_p(G)$

Note that a p-frugal edge coloring of G is not the same as a p-frugal coloring of the vertices of the line graph $L(G)$ of G. Since the neighborhood of any vertex in the line graph $L(G)$ can be partitioned into at most two cliques, every proper coloring of $L(G)$ is also a p-frugal coloring for $p \ge 2$. A 1-frugal coloring of $L(G)$ (i.e., a vertex coloring of the square of $L(G)$) would correspond to a proper edge coloring of G in which each color class induces a matching. Such colorings are known as *strong edge colorings*, see, e.g., [FF83].

The list version of p-frugal edge coloring can also be defined in the same way: given lists of size t for each edge of G, one should be able to find a p-frugal edge coloring such that the color of each edge belongs to its list. The smallest t with this property is called the *p-frugal edge choice number*, denoted $ch'_p(G)$. $ch'_p(G)$

Frugal edge colorings and their list version were studied under the name *improper edge-colorings* and *improper L-edge-colorings* by Hilton *et al.* [HSS01].

It is obvious that the chromatic index and the edge choice numbers are always at least the maximum degree Δ. The best possible upper bounds in terms of the maximum degree only are given by the following results.

Theorem 3.14
(a) **[Viz64]** *For any simple graph G we have $\chi'(G) \le \Delta(G) + 1$.*

(b) **[Sha49]** *For any multigraph G we have $\chi'(G) \le \left\lfloor \frac{3}{2}\,\Delta(G) \right\rfloor$.*

(c) **[Gal95]** *For any bipartite multigraph G we have $ch'(G) = \Delta(G)$.*

(d) **[BKW97]** *For any multigraph G we have $ch'(G) \le \left\lfloor \frac{3}{2}\,\Delta(G) \right\rfloor$.*

We will use Theorem 3.14 (c) and (d) to prove two results on the p-frugal chromatic index and the p-frugal edge choice number. The first result shows that for even p, the maximum degree completely determines the values of these two numbers. This result was earlier proved by Hilton *et al* [HSS01] in a slightly more general setting, involving a more complicated proof. We now give a short proof of this theorem:

Theorem 3.15 [HSS01] *Let G be a multigraph, and let p be an even integer. Then we have $\chi'_p(G) = ch'_p(G) = \left\lceil \frac{1}{p}\,\Delta(G) \right\rceil$.*

Proof. It is obvious that $ch'_p(G) \ge \chi'_p(G) \ge \left\lceil \frac{1}{p}\,\Delta \right\rceil$, so it suffices to prove $ch'_p(G) \le \left\lceil \frac{1}{p}\,\Delta \right\rceil$.

Let $p = 2\,\ell$. Without loss of generality, we can assume that Δ is a multiple of p and G is a Δ-regular multigraph. (Otherwise, we can add some new edges and, if necessary, some new vertices. If this larger multigraph is p-frugal edge choosable with lists of size $\left\lceil \frac{1}{p}\,\Delta \right\rceil$, then so is G.) As p, and hence Δ, is even, we can find an Euler tour in each component of G. By giving these tours a direction, we obtain an orientation D of the edges of G such that the in-degree and the out-degree of every vertex is $\frac{1}{2}\,\Delta$. Let us define the bipartite multigraph $H = (V_1 \cup V_2, E)$ as follows: V_1, V_2 are both copies of $V(G)$. For every arc (a, b) in D, we add an edge between $a \in V_1$ and $b \in V_2$.

Since D is a directed multigraph with in- and out-degree equal to $\frac{1}{2}\,\Delta$, H is a $(\frac{1}{2}\,\Delta)$-regular bipartite multigraph. This means that we can decompose the edges of H into $\frac{1}{2}\,\Delta$ perfect matchings $M_1, M_2, \ldots, M_{\Delta/2}$. Define disjoint subgraphs $H_1, H_2, \ldots, H_\ell$ as follows: for $i = 0, 1, \ldots, \ell-1$ set $H_{i+1} = M_{\frac{i}{p}\Delta+1} \cup M_{\frac{i}{p}\Delta+2} \cdots \cup M_{\frac{i+1}{p}\Delta}$. Notice that each H_i is a $\left(\frac{1}{p}\,\Delta\right)$-regular bipartite multigraph.

Now, suppose that each edge comes with a list of colors of size $\frac{1}{p}\,\Delta$. Each subgraph H_i has maximum degree $\frac{1}{p}\,\Delta$, so by Theorem 3.14(c) we can find a proper edge coloring of each H_i such that the color of each edge is inside its list. We claim that the same coloring of edges in G is p-frugal. For this we need the following observation:

Observation *Let M be a matching in H. Then the set of corresponding edges in G form a subgraph of maximum degree at most two.*

To see this, remark that each vertex has two copies in H: one in V_1 and one in V_2. The contribution of the edges of M to a vertex v in the original multigraph is then at most two, at most one from each copy of v.

To conclude, we observe that each color class in H is the union of at most ℓ matchings, one in each H_i. So at each vertex, each color class appears at most two times the number of H_i's, i.e., at most $2\,\ell = p$ times. This is exactly the p-frugality condition we set out to satisfy. ∎

For odd values of p we give a tight upper bound of the p-frugal edge chromatic number.

Theorem 3.16 [AEH07] *Let p be an odd integer. Then we have* $\left\lceil \frac{\Delta(G)}{p} \right\rceil \leq \chi'_p(G) \leq ch'_p(G) \leq \left\lceil \frac{3\,\Delta(G)}{3\,p-1} \right\rceil$.

Proof. Again, all we have to prove is $ch'_p(G) \leq \left\lceil \frac{3\,\Delta(G)}{3\,p-1} \right\rceil$.

Let $p = 2\,\ell + 1$. Since $3\,p - 1$ is even and not divisible by three, we can again assume, without loss of generality, that Δ is even and divisible by $3\,p - 1$, and that G is Δ-regular. Set $\Delta = m\,(3\,p - 1) = 6\,\ell\,m + 2\,m$. Using the same idea as in the previous proof, we can decompose G into two subgraphs G_1, G_2, where G_1 is $(6\,\ell\,m)$-regular and G_2 is $(2\,m)$-regular. (Alternatively, we can use Petersen's Theorem [Pet91] that every even regular multigraph has a 2-factor, to decompose the edge set in 2-factors, and combine these 2-factors appropriately.) Since $\frac{1}{2\,\ell} \cdot 6\,\ell\,m = \frac{3}{3\,p-1}\,\Delta$, by Theorem 3.15 we know that G_1 has a $2\,\ell$-frugal edge coloring using the colors from each edge's lists. Similarly we have $\frac{3}{2} \cdot 2\,m = \frac{3}{3\,p-1}\,\Delta$, and hence Theorem 3.14 (d) guarantees that we can properly color the edges of G_2 using colors from those edges' lists. The combination of these two colorings is a $(2\,\ell + 1)$-frugal list edge coloring, as required. ∎

Note that Theorem 3.16 is best possible: For $k \geq 1$, let $T^{(k)}$ be the multigraph with three vertices and k parallel edges between each pair. If $p = 2\,\ell + 1$ is odd, then the maximum number of edges with the same color a p-frugal edge coloring of $T^{(k)}$ can have is $3\,\ell + 1$. Hence the minimum number of colors needed for a p-frugal edge coloring of $T^{(k)}$ is $\left\lceil \frac{3\,k}{3\,\ell+1} \right\rceil = \left\lceil \frac{3}{3\,p-1}\,\Delta(T^{(k)}) \right\rceil$.

3.7 Conclusion

We sum up the upper bounds obtained for the frugal choice number of graphs with maximum degree Δ in Table 3.1, where Δ is supposed to be *large enough*.

Many possible directions for future research are still open. An intriguing question is inspired by the results on frugal edge coloring in the previous section. These results demonstrate an essential difference between even and odd p as far as p-frugal edge coloring is concerned. Based on what we think are the extremal examples of planar graphs for p-frugal *vertex* coloring, also our Conjecture 3.2 gives different values for

G	$ch_p(G)$	conjecture (p even | odd)
planar	$\frac{(3+\varepsilon)\Delta}{2p}$	$\left\lfloor\frac{\Delta-1}{p}\right\rfloor+2$ | $\left\lfloor\frac{3\Delta-2}{3p-1}\right\rfloor+2$
planar with $g(G)\geq 5$	$\left\lceil\frac{\Delta+10}{p}\right\rceil+6$	$\left\lceil\frac{\Delta}{p}\right\rceil+1$
planar with $g(G)\geq 6$	$\left\lceil\frac{\Delta+4}{p}\right\rceil+6$	
planar with $g(G)\geq 7$	$\left\lceil\frac{\Delta(G)-1}{p}\right\rceil+2$	
outerplanar	$\left\lfloor\frac{\Delta-1}{p}\right\rfloor+3$	

Table 3.1: $ch_p(G)$ for G with large enough maximum degree Δ.

even and odd p. But for frugal vertex colorings of planar graphs in general we have not been able to obtain results that are different for even and odd p. Most of our results for vertex coloring of planar graphs are consequences of Proposition 3.5 and known results on $L(p,1)$-labelling of planar graphs, for which no fundamental difference between odd and even p has ever been demonstrated. Hence, a major step would be to prove that Proposition 3.5 is far from tight when p is even.

A second line of future research could be to investigate which classes of graphs have p-frugal chromatic number equal to the minimum possible value $\left\lceil\frac{\Delta}{p}\right\rceil+1$. Corollary 3.10 and Theorem 3.11 give bounds for planar graphs with large girth and outerplanar graphs that are very close to the best possible bound. We conjecture that, in fact, planar graphs with large enough girth and outerplanar graphs of large enough maximum degree do satisfy $\chi_p(G)=\left\lceil\frac{\Delta(G)}{p}\right\rceil+1$ for all $p\geq 1$. A step toward this conjecture would be to minimize the value g^* (resp. to maximize the value d^*) such that for some constant C, every planar graph G with $g(G)\geq g^*$ (resp. every graph G with $\text{mad}(G)<d^*$) satisfies $\chi_p(G)\leq\left\lceil\frac{\Delta(G)}{p}\right\rceil+C$ for all $p\geq 1$.

In [KW01], Kostochka and Woodall conjectured that for any graph G, the chromatic number and the list chromatic number of G^2 are the same. We conjecture the following, which corresponds to the conjecture of [KW01] when $p=1$.

Conjecture 3.17 *For any multigraph G and any integer $p\geq 1$, we have $\chi_p(G)=ch_p(G)$.*

The famous *List Coloring Conjecture* (see, e.g., the book of Jensen and Toft [JT95]) states that for any multigraph G the chromatic index and the list chromatic index of G are the same. Again, this can be seen as a special case of the following conjecture:

Conjecture 3.18 *For any multigraph G and any integer $p \geq 1$, we have $\chi'_p(G) = ch'_p(G)$.*

When p is even, this has already been proved in [HSS01], as explained in Section 3.6. On the other hand, Galvin [Gal95] proved the List Coloring Conjecture for bipartite multigraphs. It could be interesting to see whether Conjecture 3.18 (when $p \geq 3$ is odd) is easier to solve when G is a bipartite multigraph.

Chapter 4

Linear choosability

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In the previous chapter, we studied p-frugal colorings, that is proper colorings such that no color appears more than p times in the neighborhood of a vertex. This is equivalent to a proper coloring such that the union of any two color classes induces a subgraph of maximum degree at most p. We saw that a 1-frugal coloring of a graph G was a proper coloring of G^2. A 2-frugal coloring is by definition a proper coloring such that the union of any two color classes induces a disjoint union of cycles and paths. If instead of this, we require that the union of any two color classes induces a forest of paths, we obtain a *linear* coloring, introduced by Yuster [Yus98].

Out aim in this chapter is to investigate linear colorings and show that most of the results we can obtain for certain families of graphs (outerplanar and planar graphs, graphs with small maximum degree, and graphs with bounded maximum average degree) are close from the results we obtained for 2-frugal colorings in the previous chapter.

4.1 Introduction

The notion of acyclic colorings was introduced by Grünbaum [Gru73]: a vertex coloring is said to be acyclic if it is proper (no two adjacent vertices have the same color), and if there is no bicolored cycle (the subgraph induced by the union of any two color classes is a forest).

Yuster [Yus98] mixed this notion and the concept of frugal colorings seen in the previous chapter, while introducing the concept of linear coloring. A *linear coloring* of a graph is an acyclic and 2-frugal coloring. It can also be seen as a coloring such that the subgraph induced by the union of any two color classes is a forest of paths (an acyclic graph with maximum degree at most two). The *linear chromatic number* of a graph

$\Lambda(G)$ G, denoted by $\Lambda(G)$, is the minimum number of colors in a linear coloring of G.

A graph G is *linearly L-colorable* if for a given list assignment $L = \{L(v) : v \in V(G)\}$, there exists a linear coloring c of G such that $c(v) \in L(v)$ for each vertex v. Such a coloring is called a *linear L-coloring* of G. If G is linearly L-colorable for any k-list assignment L, then G is said to be *linearly k-choosable*. The smallest integer k such that the graph G is linearly k-choosable is called the *linear choice number*,

$\Lambda^l(G)$ denoted by $\Lambda^l(G)$.

Using Lovász Local Lemma (see Lemma 1.5 in Chapter 1), Yuster proved that $\Lambda(G) = O(\Delta(G)^{3/2})$ in the general case, and he constructed graphs for which $\Lambda(G) = \Omega(\Delta(G)^{3/2})$.

We begin with some basic results (Section 4.2). In Section 4.3, we show that every outerplanar graph G with maximum degree Δ verifies $\Lambda^l(G) \leq \lceil \Delta/2 \rceil + 2$. In Section 4.4, we prove that every planar graph of maximum degree $\Delta \geq 12$ has linear choice number at most $\Delta + 26$. Section 4.5 is dedicated to the study of graphs with small maximum degree: we prove that $\Lambda^l(G) \leq 5$ when $\Delta(G) \leq 3$, and $\Lambda^l(G) \leq 9$ when $\Delta(G) \leq 4$. In Section 4.6, we give bounds for graphs with bounded maximum average degree. Finally, in Section 4.7, we prove that determining whether a bipartite subcubic planar graph is linearly 3-colorable is an NP-complete problem.

In the following, we will use a slight abuse of terminology, by saying that the 2-frugality of a vertex v is *respected* or *preserved*, when no color appears more than twice in $N(v)$.

4.2 First results

A linear coloring is a 2-frugal coloring, so there are at least $\lceil d/2 \rceil$ distinct colors in the neighborhood of each d-vertex. Hence, for any graph G with maximum degree Δ, we have $\Lambda^l(G) \geq \Lambda(G) \geq \lceil \Delta/2 \rceil + 1$. The following proposition shows that this bound is tight for some families of graphs, such as trees.

Proposition 4.1 *If G is a tree with maximum degree Δ, then $\Lambda^l(G) = \lceil \Delta/2 \rceil + 1$.*

Proof. Let L be a $(\lceil \Delta/2 \rceil + 1)$-list assignment to the vertices of G. We proceed by induction on the order of the graph. Let v be a leaf of G, and let u be its unique neighbor. By the induction assumption, there exists a linear L-coloring c of $G - v$. We now extend c to v by finding a color $c(v) \in L(v)$ such that the coloring obtained is linear. We only forbid to v the color $c(u)$ and the colors appearing at least twice in u's neighborhood. This is sufficient to obtain a proper and 2-frugal coloring, and thus a linear coloring of the tree G. There are at most $1 + \lfloor \frac{\Delta - 1}{2} \rfloor = \lceil \Delta/2 \rceil$ forbidden colors. Since $|L(v)| \geq \lceil \Delta/2 \rceil + 1$, it is possible to color v with a color from its list. ∎

Let $K_{m,n}$ be the complete bipartite graph with stable sets V and V' of size m and n respectively. We show the following result:

Proposition 4.2 *If $m \geq n$, $\Lambda(K_{m,n}) = \lceil m/2 \rceil + n$.*

Proof. To prove that $\Lambda(K_{m,n}) \geq \lceil m/2 \rceil + n$, observe that if two vertices of a same set V or V' have the same color, then all the vertices of the other set must have distinct colors (otherwise there would be a bicolored cycle of length four). Moreover a given color cannot appear more than twice in $V \cup V'$ since otherwise the 2-frugality would not be respected. Hence, the best solution is to assign each color to a pair of vertices in the largest set, and to color all the remaining vertices with distinct colors (see Figure 4.1).

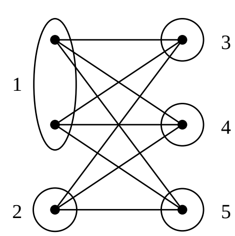

Figure 4.1: A linear coloring of $K_{3,3}$.

∎

Observe that the linear chromatic number of $K_{n,n}$ is asymptotically equivalent to $\frac{3\Delta}{2}$.

Recall that a 2-degenerate graph is a graph every subgraph of which contains a vertex of degree at most two. We prove the following proposition:

Proposition 4.3 *If G is a 2-degenerate graph of maximum degree Δ, then $\Lambda^l(G) \leq \Delta + 2$.*

Proof. We prove the theorem by induction on the order of G. Let L be a $(\Delta+2)$-list assignment for the vertices of G. Since G is 2-degenerate, it contains a vertex v with degree at most two. Consider the graph $H = G - v$. H is a proper subgraph of G, thus it is a 2-degenerate graph with order strictly less than that of G. By the induction hypothesis, there exist a linear L-coloring c of H.

Assume that the vertex v has degree one. To extend the coloring c to the whole graph G, we shall choose for v a color distinct from the color of its neighbor w and from the colors appearing twice in w's neighborhood. At most $\lfloor \frac{\Delta-1}{2} \rfloor + 1 = \lceil \Delta/2 \rceil$ colors are forbidden to v, so it is possible to color it with a color from its list $L(v)$, since $|L(v)| \geq \Delta + 2$.

If the vertex v has degree two, let u and w be its neighbors. We forbid to v the colors belonging to the set $\mathcal{C}$ defined as follows. A color a is in $\mathcal{C}$ if one of the following conditions is verified:

- one neighbor of u and one neighbor of w are both colored with a (a bicolored cycle could be created if v was also colored with a);
- two neighbors of u are colored with a (the 2-frugality of u would not be preserved if v was also colored with a);
- two neighbors of w are colored with a (2-frugality of w).

Observe that $|\mathcal{C}| \leq \Delta - 1$, since any color of $\mathcal{C}$ appears at least twice in $(N(u) \cup N(w)) \setminus \{v\}$. Since v must receive a color distinct from the colors of u and w, there are at most $\Delta - 1 + 2 = \Delta + 1$ forbidden colors for v. Since $|L(v)| \geq \Delta + 2$, there remains at least one color in $L(v)$ that can be assigned to v. We obtain a linear L-coloring of G, which completes the induction. ∎

4.3 Outerplanar graphs

Since outerplanar graphs are 2-degenerate, it follows from Proposition 4.3 that outerplanar graphs with maximum degree Δ have linear choice number at most $\Delta+2$. In this section, we improve this bound by proving the following theorem:

Theorem 4.4 [EMR08] *If G is an outerplanar graph with maximum degree Δ, then $\Lambda^l(G) \leq \lceil \Delta/2 \rceil + 2$.*

Proof. We prove the theorem by induction on the order of G. Let L be a $(\lceil \Delta/2 \rceil + 2)$-list assignment for the vertices of G. As in the previous chapter, we use a result from [EO07a] (see Appendix A for further details), which states that any outerplanar graph contains a vertex u such that one of the following holds: (i) u has degree at most one; (ii) u has degree two and is adjacent to another vertex of degree two; or (iii) u has degree two and its neighbors v and w are adjacent, and either v has degree three or v has degree four and its two other neighbors (i.e., distinct from u and w) are adjacent (see Figure 3.2).

Let u be as described above. If (i) u has degree at most one, let v be the neighbor of u, if it exists, and let c be a linear L-coloring of $G-u$. Color u with a color distinct from $c(v)$ and the colors appearing twice in $N(v)$. At most $\lfloor \frac{\Delta-1}{2} \rfloor + 1 = \lceil \Delta/2 \rceil$ are forbidden for u, and the coloring obtained is linear.

If (ii) u has degree two and is adjacent with a 2-vertex, say v, let c be a linear L-coloring of $G-\{u,v\}$. Let u' be the neighbor of u distinct from v and let v' be the neighbor of v distinct from u. Choose for v a color $c(v)$ distinct from $c(u')$, $c(v')$, and the colors appearing twice in $N(v')$. Then color u with a color distinct from $c(u')$, $c(v)$, and the colors appearing twice in $N(u')$. At most $\lfloor \frac{\Delta-1}{2} \rfloor + 2 \leq \lceil \Delta/2 \rceil + 1$ are forbidden for u and v, and the coloring obtained is linear (having $c(v) \neq c(u')$ ensures that the coloring c is acyclic).

If (iii) u has degree two and its neighbors v and w are adjacent, and either v has degree three or v has degree four and its two other neighbors (i.e., distinct from u and w) are adjacent, let c be a linear L-coloring of $G-u$. Take $c(u)$ distinct from $c(v)$ and $c(w)$, and from the colors appearing twice in $N(w)\backslash\{v\}$. At most $\lfloor \frac{\Delta-2}{2} \rfloor + 2 \leq \lceil \Delta/2 \rceil + 1$ are forbidden for u, and the coloring obtained is linear: since v and w are adjacent, the coloring is acyclic, and (iii) ensures that the only color that may appears twice in $N(v)$ is $c(v)$ (wich is forbidden for u).

In any case, it is possible to color the uncolored vertices given lists of size at least $\lceil \Delta/2 \rceil + 2$, in order to obtain a linear L-coloring of G, which completes the induction. ∎

4.4 Planar graphs

As in Chapter 3, we use Lemma 3.3 from Van den Heuvel and McGuinness [HM03] to prove the following result.

Theorem 4.5 [EMR08] *If G is a planar graph with maximum degree $\Delta \geq 12$, then $\Lambda^l(G) \leq \Delta + 26$.*

Proof. We prove the theorem by induction on the order of G. Let L be a $(\Delta + 26)$-list assignment to the vertices of G.

Let $k, v, v_1, \ldots, v_k$ be as in Lemma 3.3, and let H be the graph obtained from G by contracting the edge vv_1 into the vertex v_1. This graph has maximum degree 12 (case (ii)) or Δ, so by induction, there exists a linear coloring c of H such that any vertex $u \in V(H)$ is colored with a color $c(u) \in L(u)$. In order to extend c to G, we only need to color v with a color from its list $L(v)$. Choose the color of v different from the colors of $v_1, \ldots, v_k$ as well as the colors of the neighbors of $v_1, \ldots, v_{k-2}$ if $k \geq 3$. Choose it also different from the colors appearing twice among the vertices adjacent to v_{k-1} or v_k. In total we forbid at most $5 + 5 + 6 + 10 + (2\Delta - 2)/2 = \Delta + 25$ colors to v. Since $|L(v)| \geq \Delta + 26$, it is possible to find an appropriate color for this vertex.

We now prove that the coloring obtained is linear. Since the coloring c of H is linear, no color appears more than twice in the neighborhood of v in G. If $k \geq 3$, the colors of the neighbors of $v_1, \ldots, v_{k-2}$ are forbidden to v, so the 2-frugality of $v_1, \ldots, v_{k-2}$ is preserved and any bicolored cycle passing through v contains v_{k-1} and v_k. The colors appearing twice in $N(v_{k-1})$ or twice in $N(v_k)$ are forbidden, so the 2-frugality of v_{k-1} and v_k is preserved. The colors appearing in $N(v_{k-1})$ and $N(v_k)$ are also forbidden, so v cannot belong to any bicolored cycle. We thus obtain a linear L-coloring of G, which completes the induction. ∎

4.5 Graphs with small maximum degree

4.5.1 Subcubic graphs

As seen in Section 4.2, the graph $K_{3,3}$ is not linearly 4-colorable. Let G be a graph with maximum degree three, containing at least one $^{\leq}2$-vertex. Then G is 2-degenerate and we have $\Lambda^l(G) \leq 5$ by Proposition 4.3. So the hardest part is to prove that 3-regular graphs have linear choice number at most five. To show this, we prove a slightly stronger statement:

Theorem 4.6 [EMR08] *Let G be a graph with maximum degree $\Delta \leq 3$, and L be a 5-list-assignment to the vertices of G. Then there exists a* *linear L-coloring of G such that the two neighbors of any 2-vertex have distinct colors.*

Proof. We prove the theorem by induction on the order of G. let L be a 5-list-assignment to the vertices of G. We can assume that G is connected, otherwise we can color each connected component by induction and obtain a linear list L-coloring of G with the desired property.

If G contains a 1-vertex v adjacent to a vertex u, then by induction, the graph $G - v$ has a linear L-coloring c such that the neighbors of any 2-vertex have distinct colors. By coloring v with a color distinct from $c(u)$ and from the colors of the neighbors of u, we obtain a linear L-coloring of G such that the neighbors of any 2-vertex have distinct colors.

If G contains a 2-vertex v with neighbors u and w, let H be the graph obtained from G by removing the vertex v and adding an edge uw if it does not already exist. H has maximum degree at most three and is smaller than G, so there exists a linear L-coloring c of H, such that the neighbors of any 2-vertex have distinct colors. We choose for v a color distinct from $c(u)$, $c(w)$, and from the colors appearing twice in the neighborhood of u, or twice in the neighborhood of w. Since $c(u) \neq c(w)$, we do not create any bicolored cycle. We forbid at most four colors to v, so we can choose a color for v and obtain a linear L-coloring of G such that the neighbors of any 2-vertex have distinct colors.

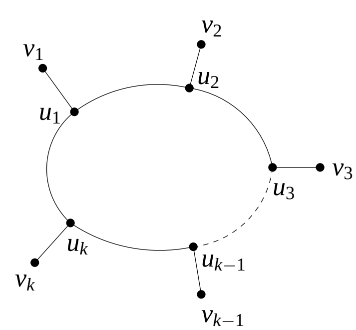

Figure 4.2: A shortest cycle in a minimum counterexample.

Otherwise the graph G is 3-regular. Let $u_1, \ldots, u_k$, with $k \geq 3$ be a shortest cycle (see Figure 4.2). For all $1 \leq i \leq k$, we denote by v_i the neighbor of u_i outside the cycle (that is, distinct from u_{i-1} and u_{i+1}, where all values are taken modulo k). Observe that two vertices v_i and v_j could be the same vertex, but that each v_i is distinct from all the vertices u_j, since otherwise there would be a cycle with less than k vertices. Let H be the graph obtained from G by removing the vertices $u_1, \ldots, u_k$. By induction there exists a linear L-coloring c of H, such that the neighbors of any 2-vertex have distinct colors. In particular, each vertex v_i has degree at most two in H, so its neighbors have distinct colors and the 2-frugality of v_i will be preserved regardless of the color we assign to u_i.

We now color the vertices $u_1, \ldots, u_k$ in this order. We choose for u_1 a color distinct from $c(v_1)$ and $c(v_2)$. For any $2 \leq i \leq k-1$, we choose for u_i a color distinct from $c(u_{i-1})$, $c(v_i)$, and $c(v_{i+1})$. For u_k, we choose a color distinct from $c(u_1)$, $c(u_{k-1})$, $c(v_k)$, and $c(v_1)$. By doing so, we prevent any bicolored cycle containing a vertex v_i, and the 2-frugality of every vertex u_i is respected. But at this point, the cycle $u_1, \ldots, u_k$ could still be a bicolored cycle. Hence, if $k \geq 4$, we also forbid the color of u_1 to u_3 while we are coloring this vertex (if $k = 3$ the cycle is a triangle and it cannot be properly bicolored). At most four colors are forbidden to each vertex u_i, so we can choose a color $c(u_i) \in L(u_i)$ for any of them, and the coloring obtained is a linear L-coloring of G. Since G is 3-regular, the additional property that the neighbors of any 2-vertex have distinct colors is trivially verified. ∎

Since $K_{3,3}$ seems to be the only subcubic graph which linear choice number is equal to 5, we propose the following conjecture :

Conjecture 4.7 *If G has maximum degree 3, and is different from $K_{3,3}$, then $\Lambda^l(G) \leq 4$.*

4.5.2 Graphs with maximum degree 4

According to Proposition 4.2, we have $\Lambda^l(K_{4,4}) \geq 6$. Applying the same method of reducible configurations to graphs with maximum degree 4, we obtain the following theorem, which we suspect not to be tight.

Theorem 4.8 [EMR08] *If G is a graph with maximum degree $\Delta \leq 4$, then $\Lambda^l(G) \leq 9$.*

Proof. Let G be a counterexample of minimum order: there exists a 9-list-assignment L such that G is not linearly L-colorable. Using the same arguments as in the previous proof, we show that G does not contain any $^{\leq}3$-vertex. Hence, the graph is 4-regular. We now show that G does not contain any 4-vertices.

Let u be a 4-vertex and let v, w, x, and y be its neighbors. Let G' be the graph obtained from $G - v$ by adding the edges vw and xy if they are not already there (see Figure 4.3). Let c be a linear L-coloring of G'. We now extend c to the initial graph G: we only have to color the vertex u with a color from its list $L(u)$. We have to choose a color distinct from the colors of v, w, x, and y. The condition of 2-frugality for these four vertices forbids at most four additional colors. If v, w, x, and y have distinct colors, it is impossible to create a bicolored cycle, so we can color u with the ninth color of $L(u)$, and thus obtain a linear L-coloring of G.

Otherwise, we have for example $c(v) = c(y)$ and $c(w) \neq c(x)$. The neighbors of u forbid only three colors, and their 2-frugality forbids at most 4 colors. But it is possible to create a bicolored cycle passing through v and y. To avoid this, we forbid to u the colors of v's neighbors. This makes only two additional colors, as the third one was already counted to ensure v's 2-frugality. There are still at most eight forbidden colors for the choice of $c(u)$.

In the last case, we have without loss of generality $c(v) = c(x)$ and $c(w) = c(y)$. The neighbors of u forbid two colors to this vertex. To ensure the 2-frugality of v, w, x, and y we forbid at most four other colors to u. To prevent any bicolored cycle it suffices to forbid to u the colors of v's and w's neighbors (six colors, among which two have already been counted). This makes at most eight forbidden colors for the choice of u. So it is possible to color this vertex with a color of its list, and to obtain a linear L-coloring of G. This completes the proof.

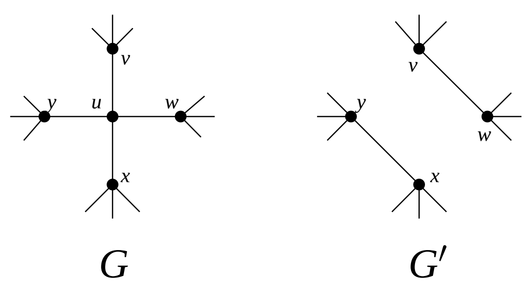

Figure 4.3: Elimination of a 4-vertex.

∎

As noticed by Frédéric Havet, there exists a simpler way to prove Theorem 4.8 when we restrict ourselves to linear coloring (instead of linear list coloring): since G is 4-regular, it is the union of two cycle-factors F_1 and F_2. Each F_i admits a linear coloring c_i with three colors, and the product of c_1 and c_2 gives a linear coloring of G with 9 colors.

4.6 Graphs with bounded maximum average degree

Recall that the maximum average degree of a graph G, denoted by $\mathrm{mad}(G)$ is defined by:

$$\text{mad}(G) = \max\{2|E(H)|/|V(H)|, H \subseteq G\}.$$

Theorem 4.9 [EMR08] *Let G be a graph with maximum degree Δ:*

1. *If $\Delta \geq 3$ and $mad(G) < \frac{16}{7}$, then $\Lambda^l(G) = \lceil \frac{\Delta}{2} \rceil + 1$.*

2. *If $mad(G) < \frac{5}{2}$, then $\Lambda^l(G) \leq \lceil \frac{\Delta}{2} \rceil + 2$.*

3. *If $mad(G) < \frac{8}{3}$, then $\Lambda^l(G) \leq \lceil \frac{\Delta}{2} \rceil + 3$.*

Since every planar or projective-planar graph G with girth $g(G)$ verifies $\text{mad}(G) < \frac{2g(G)}{g(G)-2}$, we obtain the following corollary:

Corollary 4.10 *Let G be a planar or projective-planar graph with maximum degree Δ:*

1. *If $\Delta \geq 3$ and $g(G) \geq 16$, then $\Lambda^l(G) = \lceil \frac{\Delta}{2} \rceil + 1$.*

2. *If $g(G) \geq 10$, then $\Lambda^l(G) \leq \lceil \frac{\Delta}{2} \rceil + 2$.*

3. *If $g(G) \geq 8$, then $\Lambda^l(G) \leq \lceil \frac{\Delta}{2} \rceil + 3$.*

Observe that cycles are linearly 3-choosable; hence, we cannot remove the condition on Δ in Theorem 4.9.1 and Corollary 4.10.1.

Proof of Theorem 4.9.1 Let G be a counterexample of minimum order, with $\Delta \geq 3$ and $\text{mad}(G) < \frac{16}{7}$. There exists an assignment of lists of size at least $\lceil \frac{\Delta}{2} \rceil + 1$ such that G is not linearly L-colorable. Using the method of reducible configurations, we first prove that G satisfies the following claim:

Claim 4.11 *G does not contain any of the following configurations:*

(C4.11.1) *a 1-vertex,*

(C4.11.2) *a 2-vertex adjacent to two 2-vertices,*

(C4.11.3) *a 3-vertex adjacent to three 2-vertices, each of them adjacent to a 2-vertex.*

Proof.

(C4.11.1) If G contains a 1-vertex v, let c be a linear L-coloring of $G-v$ (which exists as $G-v$ is a subgraph of G and thus verifies $\mathrm{mad}(G-v) < \frac{16}{7}$). We now extend c to v: the neighbor u of v forbids one color; we also have to preserve u's 2-frugality: among its d already colored neighbors ($d \leq \Delta - 1$), there are at worst $\lceil \frac{\Delta}{2} \rceil - 1$ pairs of vertices having the same color. This forbids at most $\lceil \frac{\Delta}{2} \rceil$ colors to v. Thus v can be colored with a remaining color in its list $L(v)$, and the coloring obtained is a linear L-coloring of G, which is a contradiction.

(C4.11.2) If G contains a 2-vertex v adjacent to two 2-vertices u and w, we color the graph $G-v$ linearly with colors belonging to the lists of L (it is possible by the minimality of G). If u and w have distinct colors, we choose for v a color distinct from the colors of its neighbors, and it is impossible to create a bicolored cycle. If u and w have the same color, we forbid it to v, as well as the color of the second neighbor of u. This prevents the creation of any bicolored cycle. There are at most two forbidden colors, what enables us to color v since $\lceil \frac{\Delta}{2} \rceil + 1 \geq 3$ when $\Delta \geq 3$.

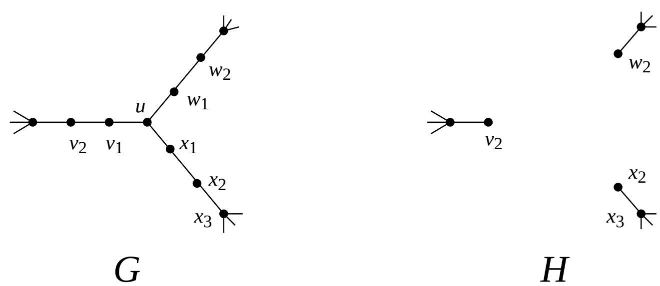

Figure 4.4: Elimination of Configuration (C4.11.3).

(C4.11.3) If G contains a 3-vertex adjacent to three 2-vertices, each of them being adjacent to another 2-vertex, then we color the reduced graph H obtained from G by removing the vertices u, v_1, w_1, and x_1 (see Figure 4.4). This reduced graph H is a subgraph of G, and so $\mathrm{mad}(H) < 16/7$. We now have to color the vertices u, v_1, w_1, and x_1. For v_1, we choose a color different from the color of v_2. For w_1 we take a color different from those of w_2 and v_1. We color u with a color different from those of v_1 and w_1. For the last vertex, we have to handle two different cases: if u and x_2 have different colors it is impossible to create a bicolored cycle, so we can take for x_1 a color different from those of u and x_2. If u and x_2 have the same color, we choose for x_1 a color different from those of x_2 and x_3 (what prevents bicolored cycles coming from x_3). As in the previous situation, there are at most two forbidden colors for each vertex, what enables us to color each of them with a color of its own list. We then obtain a linear L-coloring of G, which is a contradiction.

∎

We complete the proof of Theorem 4.9.1 with a discharging procedure. First, we assign to each vertex v a *charge* $\omega(v)$ equal to its degree. We then apply the following discharging rules:

Rule 1. Each $^{\geq}4$-vertex gives $\frac{2}{7}$ to each adjacent 2-vertex.

Rule 2. Each 3-vertex gives $\frac{2}{7}$ to each adjacent 2-vertex neighbor of another 2-vertex, and $\frac{1}{7}$ to each adjacent 2-vertex which is not neighbor of a 2-vertex.

Let $\omega^*(v)$ be the charge of v after the procedure. Let v be a k-vertex ($k \geq 2$, as G does not contain Configuration (C4.11.1)).

- If $k = 2$, v receives $\frac{2}{7}$ if it is adjacent to a $^{\geq}4$-vertex or to a 3-vertex and a 2-vertex. Otherwise v must be adjacent to two 3-vertices (Configuration (C4.11.2) does not appear in the graph), and will receive two times $\frac{1}{7}$, so $\omega^*(v) \geq 2 + \frac{2}{7} = \frac{16}{7}$.
- If $k = 3$, v gives at most $\frac{2}{7} + \frac{2}{7} + \frac{1}{7}$ (the graph does not contain Configuration (C4.11.3)), thus $\omega^*(v) \geq 3 - \frac{5}{7} = \frac{16}{7}$.
- If $k \geq 4$, then by Rule 1 $\omega^*(v) \geq k - k \times \frac{2}{7} \geq \frac{20}{7}$.

In any case, $\omega^*(v) \geq \frac{16}{7}$, so $\sum_{v\in V(G)} \omega^*(v) \geq \frac{16n}{7}$. Since $\sum_{v\in V(G)} \omega^*(v) = \sum_{v\in V(G)} \omega(v) = \sum_{v\in V(G)} d(v) = 2|E(G)|$, we have:

$$\mathrm{mad}(G) \geq \frac{2|E(G)|}{|V(G)|} = \frac{\sum_{v\in V(G)} \omega^*(v)}{|V(G)|} \geq \frac{16/7|V(G)|}{|V(G)|} = \frac{16}{7}$$

We obtain a contradiction, since $\mathrm{mad}(G) < \frac{16}{7}$ according to the the definition of G.

Proof of Theorem 4.9.2 Let G be a counterexample of minimum order, with $\text{mad}(G) < \frac{5}{2}$. There exists an assignment L of lists of size $\lceil\frac{\Delta}{2}\rceil + 2$ such that G is not linearly L-colorable. Using the method of reducible configurations, we first prove that G satisfies the following claim:

Claim 4.12 *G does not contain any of the following configurations:*

(C4.12.1) a 1-vertex,

(C4.12.2) two adjacent 2-vertices,

(C4.12.3) a 3-vertex adjacent to three 2-vertices.

Proof.

(C4.12.1) The case of the 1-vertex has already been handled in the previous proof (see Configuration (C4.11.1)).

(C4.12.2) If G contains two adjacent 2-vertices v and w, let c be a linear L-coloring of $G - \{v, w\}$. We extend c to the whole graph by finding colors $c(v) \in L(v)$ and $c(w) \in L(w)$ for v and w such that the new coloring c is a linear coloring of G. Let u be the neighbor of v in G distinct from w and let x be the neighbor of w in G distinct from v. For v, we choose a color distinct from those of u and x. We also need to preserve u's 2-frugality; to do this we forbid at most $\lceil\frac{\Delta}{2}\rceil - 1$ other colors to v. We take for w a color different from those of v and x; x's 2-frugality also forbids at most $\lceil\frac{\Delta}{2}\rceil - 1$ other colors to w. At most $\lceil\frac{\Delta}{2}\rceil + 1$ colors are forbidden to v and w, so it is possible to color them with colors from their own lists. We obtain a linear L-coloring of G, which is a contradiction.

(C4.12.3) If G contains a 3-vertex adjacent to three 2-vertices, let c be a linear L-coloring of the reduced graph H obtained from G by removing the vertices u, x_1, and w_1 (see Figure 4.5). In order to extend c to the whole graph G, we have to find colors for the remaining vertices: w_1, x_1, and u. We choose for w_1 a color distinct from the colors of w_2 and v_1, and from the at most $\lceil\frac{\Delta}{2}\rceil - 1$ colors appearing twice in w_2's neighborhood. We take for u a color different from those of v_1, w_1, and x_2. Finally we forbid to x_1 the colors of x_2 and u, as well as most $\lceil\frac{\Delta}{2}\rceil - 1$ colors appearing twice in x_2's neighborhood. Such a coloring preserves the property of 2-frugality of all the vertices, and since $c(w_1) \neq c(v_1)$ and $c(u) \neq c(x_2)$ no bicolored cycle can be created. So we can color each of these vertices with a color from its own list in order to obtain a linear L-coloring of G, which is a contradiction.

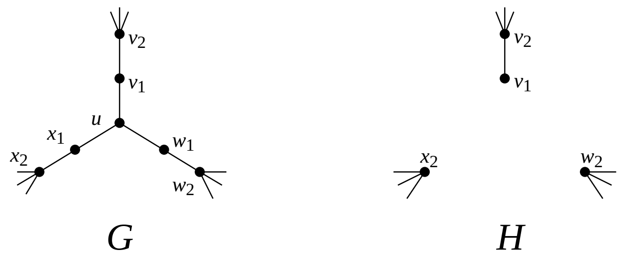

Figure 4.5: Elimination of Configuration (C4.12.3).

■

We complete the proof of Theorem 4.9.2 with a discharging procedure. First, we assign to each vertex v a charge $\omega(v)$ equal to its degree. We then apply the following discharging rule:

Rule. Each $^{\geq}$3-vertex gives $\frac{1}{4}$ to each adjacent 2-vertex.

Let $\omega^*(v)$ be the charge of v after the procedure. Let v be a k-vertex of G ($k \geq 2$, as G does not contain Configuration (C4.12.1)).

- If $k = 2$, v is adjacent to two $^{\geq}$3-vertices (the graph does not contain Configuration (C4.12.2)), thus $\omega^*(v) \geq 2 + 2 \times \frac{1}{4} = \frac{5}{2}$.
- If $k = 3$, v is adjacent to at most two 2-vertices (the graph does not contain Configuration (C4.12.3)), thus $\omega^*(v) \geq 3 - 2 \times \frac{1}{4} = \frac{5}{2}$.
- If $k \geq 4$, v can be adjacent to k 2-vertices, so $\omega^*(v) \geq k - k \times \frac{1}{4} \geq 3$.

In any case, $\omega^*(v) \geq \frac{5}{2}$, so $\sum_{v \in V(G)} \omega^*(v) \geq \frac{5n}{2}$. Since $\sum_{v \in V(G)} \omega^*(v) = \sum_{v \in V(G)} \omega(v) = \sum_{v \in V(G)} d(v) = 2|E(G)|$, we have:

$$\text{mad}(G) \geq \frac{2|E(G)|}{|V(G)|} = \frac{\sum_{v \in V(G)} \omega^*(v)}{|V(G)|} \geq \frac{5/2|V(G)|}{|V(G)|} = \frac{5}{2}$$

We obtain a contradiction, since $\text{mad}(G) < \frac{5}{2}$ according to the the definition of G.

Proof of Theorem 4.9.3 Let G be a counterexample of minimum order, with $\mathrm{mad}(G) < \frac{8}{3}$. There exists an assignment L of lists of size $\lceil\frac{\Delta}{2}\rceil + 3$ such that G is not linearly L-colorable. Using the method of reducible configurations, we first show that G satisfied the following claim:

Claim 4.13 *G does not contain any of the following configurations:*

(C4.13.1) a 1-vertex,

(C4.13.2) two adjacent 2-vertices,

(C4.13.3) a 3-vertex adjacent to two 2-vertices.

Proof.

(C4.13.1) see Configuration (C4.11.1).

(C4.13.2) see Configuration (C4.12.2).

(C4.13.3) If G contains a 3-vertex adjacent to two 2-vertices, let c be a linear L-coloring of the reduced graph H obtained from G by removing the vertices u, x_1, and w_1 (see Figure 4.6. This coloring exists, as H is a subgraph of G, and thus $\mathrm{mad}(H) \leq \mathrm{mad}(G) < \frac{8}{3}$. We extend c to the whole graph G, by coloring w_1, x_1, and u with colors of $L(w_1)$, $L(x_1)$, and $L(u)$ respectively. We take for w_1 a color different from the colors of v and w_2, and from the $\lceil\frac{\Delta}{2}\rceil - 1$ colors appearing twice in w_2's neighborhood. We then color u with a color different from those of w_1, v, x_2, and from the $\lceil\frac{\Delta}{2}\rceil - 1$ colors appearing twice in v's neighbors (2-frugality of v). Finally, we color x_1 with a color different from those of u, x_2, and from at most $\lceil\frac{\Delta}{2}\rceil - 1$ colors among the colors of x_2's neighbors. So we can color each vertex with a color from its list, and we obtain a linear L-coloring of G, which is a contradiction.

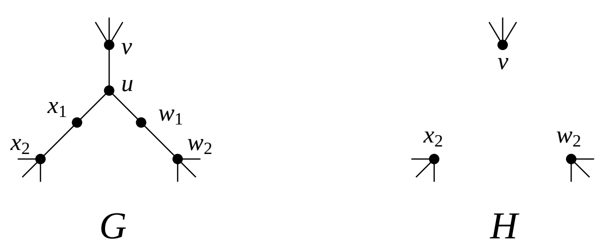

Figure 4.6: Elimination of Configuration (C4.13.3).

∎

We complete the proof of Theorem 4.9.3 with a discharging procedure. First, we assign to each vertex v a charge $\omega(v)$ equal to its degree. We then apply the following discharging rule:

Rule. Each $^{\geq}3$-vertex gives $\frac{1}{3}$ to each adjacent 2-vertex.

Let $\omega^*(v)$ be the charge of v after the procedure. Let v be a k-vertex of G ($k \geq 2$, as G does not contain Configuration (C4.13.1)).

- If $k = 2$, v is adjacent to two $^{\geq}3$-vertices (G does not contain Configuration (C4.13.2)), thus $\omega^*(v) \geq 2 + 2 \times \frac{1}{3} = \frac{8}{3}$.
- If $k = 3$, v is adjacent to at most one 2-vertex (G does not contain Configuration (C4.13.3)), thus $\omega^*(v) \geq 3 - \frac{1}{3} = \frac{8}{3}$.
- If $k \geq 4$, v can be adjacent to k 2-vertices, thus $\omega^*(v) \geq k - k \times \frac{1}{3} \geq \frac{8}{3}$.

In any case, $\omega^*(v) \geq \frac{8}{3}$, so $\sum_{v \in V(G)} \omega^*(v) \geq \frac{8n}{3}$. Since $\sum_{v \in V(G)} \omega^*(v) = \sum_{v \in V(G)} \omega(v) = \sum_{v \in V(G)} d(v) = 2|E(G)|$, we have:

$$\mathrm{mad}(G) \geq \frac{2|E(G)|}{|V(G)|} = \frac{\sum_{v \in V(G)} \omega^*(v)}{|V(G)|} \geq \frac{8/3|V(G)|}{|V(G)|} = \frac{8}{3}$$

We obtain a contradiction, since $\mathrm{mad}(G) < \frac{8}{3}$ according to the the definition of G.

4.7 NP-completeness

Theorem 4.14 [EMR08] *Deciding whether a bipartite subcubic planar graph is linearly 3-colorable is an NP-complete problem.*

Proof. The proof of the NP-completeness proceeds by a reduction to the problem of 3-coloring of planar graphs, which is an NP-complete problem [GJS76]. Given an instance of this problem –a planar graph H, we need to create a bipartite subcubic planar graph G of a size polynomial in $|V(H)|$ such that G is linearly 3-colorable if and only if H is 3-colorable.

Let $\mathcal{M}$ be the 7×2 grid (see Figure 4.7). Observe that in any linear 3-coloring c of $\mathcal{M}$, we have $c(x_1) = c(x_2)$ and $c(y_1) = c(y_2)$.

Let $\mathcal{N}(z_1, z_2)$ be the graph depicted in Figure 4.8. This graph is bipartite, subcubic, planar, and linearly 3-colorable. Moreover, by the property of $\mathcal{M}$ we have $c(z_1) = c(z_2)$ in any linear 3-coloring c of $\mathcal{N}$.

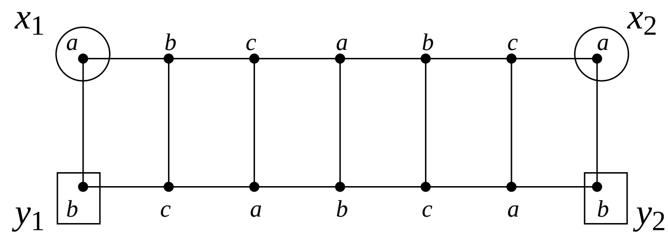

Figure 4.7: A linear 3-coloring of the graph $\mathcal{M}$.

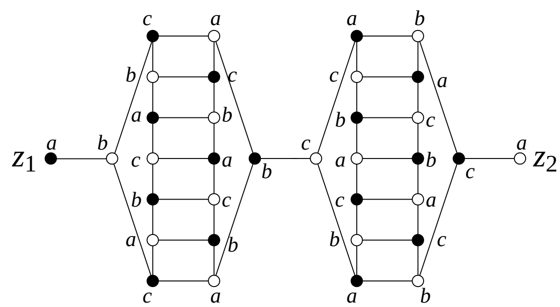

Figure 4.8: The graph $\mathcal{N}(z_1, z_2)$. The two stable sets are represented with white and black dots respectively.

To make the reduction, we first replace each d-vertex $u \in V(H)$ by a tree T_u with maximum degree at most 3, having d leaves (each leaf u_v corresponds to a link to a neighbor v of u in H). We then replace each edge xy of these trees by the graph $\mathcal{N}(x, y)$. We then link each vertex u_v to the vertex v_u by an edge (see Figure 4.9). Each tree is bipartite, but our construction may not be bipartite at this point: if we color each tree T_u properly with the colors *black* and *white*, two leaves v_w and w_v may be colored with the same color. If this is the case, we subdivide the edge $v_w w_v$, thus creating a new vertex m_{vw} adjacent to v_w and w_v. We then replace the edge $v_w m_{vw}$ by the graph $\mathcal{N}(v_w, m_{vw})$. We repeat this process until the graph obtained is properly 2-colorable, and thus bipartite.

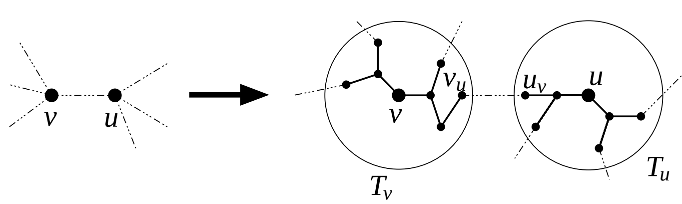

Figure 4.9: Transformation of the planar graph into a subcubic bipartite planar graph.

The graph G obtained is planar, bipartite, and subcubic. Each vertex of the tree T_u receives the color of u in the 3-coloring of H. This 3-coloring of the graph G is linear : there is no problem of 2-frugality in the trees, and there are no bicolored cycles (there are no bicolored paths of size at least four in the widgets).

Conversely, in a linear 3-coloring of G, the vertices of a given tree T_u have the same color, which can be used to color u in H. So we easily obtain a 3-coloring of H. ∎

We could have used a 4×2 grid instead of a 7×2 grid to build the widget. All the properties would have been conserved, but the widget would not have been bipartite (it would have contained some C_5). The theorem of NP-completeness would have been a little weaker.

4.8 Conclusion

Table 4.1 sums up the upper bounds obtained for the linear choice number of graphs with maximum degree Δ.

G	$\Lambda_k^l(G)$
$\Delta \leq 3$	5
$\Delta \leq 4$	9
$\Delta \geq 3$ and $\mathrm{mad}(G) < \frac{16}{7}$	$\lceil \frac{\Delta}{2} \rceil + 1$
$\mathrm{mad}(G) < \frac{5}{2}$	$\lceil \frac{\Delta}{2} \rceil + 2$
$\mathrm{mad}(G) < \frac{8}{3}$	$\lceil \frac{\Delta}{2} \rceil + 3$
outerplanar	$\lceil \frac{\Delta}{2} \rceil + 2$
planar with $\Delta \geq 12$	$\Delta + 26$

Table 4.1: $\Lambda^l(G)$ for G with maximum degree Δ.

Since this work has been written, the bound of Theorem 4.5 has been reduced from $\Delta + 26$ down to $\frac{9}{10}\Delta + 5$ (when $\Delta \geq 85$) by Raspaud and Wang [RW06]. It is believed that the right bound should be $\Delta/2 + C$, where C is an absolute constant, but this seems to be a difficult problem. It is also an open problem to know whether $\Lambda^l(G) = \Lambda(G)$ for every graph G.

A generalization of linear coloring can be made, by replacing the condition of 2-frugality by a condition of k-frugality. More precisely, we define the *k-forested coloring* of a graph G as a proper coloring of the vertices of G such that the union of any two color classes is a forest of maximum degree at most k. The *k-forested number* of a graph G, denoted by $\Lambda_k(G)$, is the smallest number of colors appearing in a k-forested coloring of G.

The lower bound $\Lambda(G) \geq \lceil \frac{\Delta(G)}{2} \rceil + 1$ can be easily generalized to $\Lambda_k(G) \geq \lceil \frac{\Delta}{k} \rceil + 1$ for all graph G of maximum degree Δ. The example described by Yuster in [Yus98] can also be generalized in k dimensions in order to prove that $\Lambda_k(G) = \Omega(\Delta^{\frac{k+1}{k}})$. However, as soon as $k \geq 4$, this construction is less interesting than the probabilistic bound of $\Omega\left(\frac{\Delta^{4/3}}{(\log \Delta)^{1/3}}\right)$ given by Alon, McDiarmid and Reed [AMR91] for the acyclic chromatic number.

Recently, Kang and Müller [KM07] investigated this coloring and found some connections with t-improper colorings (colorings such that every color class induces a graph with maximum degree t).

Chapter 5

$(p, 1)$-total labelling

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In the previous chapters, we investigated distance-two colorings of specific families of graphs: graphs with bounded maximum degree, with bounded maximum average degree, forests, outerplanar graphs, planar graphs, and planar graphs with large girth. In this chapter, we study *incidence graphs*, for which distance-two colorings are of particular interest.

5.1 Introduction

For a graph G, let us define the *incidence graph* $G^\star$ of G as the graph obtained from G by replacing every edge by a path of length two (see Figure 5.1 for an example). Observe that for a graph G, coloring the square of $G^\star$ is equivalent to coloring the vertices and edges of G such that: $G^\star$

(*i*) the edge-coloring is proper, i.e. no two incident edges receive the same color;

(*ii*) the vertex-coloring is proper, i.e. no two adjacent vertices receive the same color;

(*iii*) every edge has a color distinct from the colors of its end vertices.

$\chi^T(G)$

Such a coloring is called a *total coloring* of G, and the smallest number of colors in a total coloring of G is the *total chromatic number* of G, denoted by $\chi^T(G)$. By the observation above, $\chi^T(G)$ is equal to the chromatic number of the square of $G^\star$. In the late sixties, Behzad [Beh65] and Vizing [Viz68] independently proposed the following conjecture, which is still an open problem:

Conjecture 5.1 (The Total Coloring Conjecture) *For any graph G with maximum degree Δ, $\chi^T(G) \leq \Delta + 2$.*

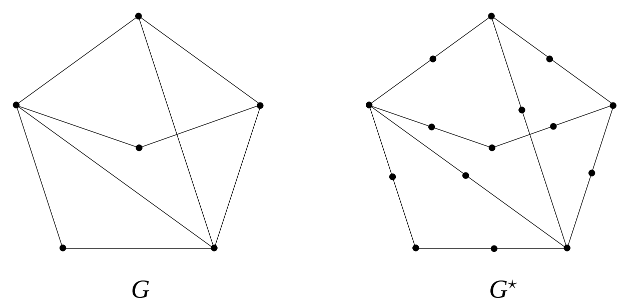

Figure 5.1: An example of incidence graph.

Kostochka [Kos77] proved that for a graph G with maximum degree Δ, we have $\chi^T(G) \leq \lfloor \frac{3}{2}\Delta \rfloor$. The first bound in $\Delta + o(\Delta)$ was given by Hind [Hin90], who proved that $\chi^T(G) \leq \Delta + 2\sqrt{\Delta}$. This was later improved to $\Delta + 18\Delta^{1/3}\log(3\Delta)$ by Häggkvist and Chetwynd [HC92]. A significant step was then made by Hind, Molloy and Reed [HMR99], who proved a bound of $\Delta + \mathrm{poly}(\log \Delta)$ using frugal colorings (see Chapter 3). The best bound so far is due to Molloy and Reed [MR98], who proved that the total chromatic number of any graph with maximum degree Δ is at most Δ plus an absolute constant.

Recall that for integers $p, q \geq 0$, an *$L(p, q)$-labelling of G* is an assignment f of integers to the vertices of G such that :

- $|f(u) - f(v)| \geq p$, if $d_G(u, v) = 1$,
- $|f(u) - f(v)| \geq q$, if $d_G(u, v) = 2$.

In 1995, Georges, Mauro, and Whittlesey [GMW95] studied the $L(2,1)$-labelling of incidence graphs. An $L(2,1)$-labelling of the incidence graph of G is equivalent to an assignment of integers to each element of $V(G) \cup E(G)$ such that :

(i) the edge-coloring is proper,

(ii) the vertex-coloring is proper,

(iii) the difference between the integer assigned to a vertex and those assigned to its incident edges is at least 2.

This labelling is called a $(2,1)$-total labelling of G. Havet and Yu [HY08] generalized it to the $(p,1)$-total labelling of a graph: a *$(p,1)$-total labelling* of a graph $G=(V,E)$ is a map $c: V \cup E \to \mathbb{N}$ verifying:

(i) $\forall (u,v) \in V^2 : uv \in E \Rightarrow c(u) \neq c(v)$,

(ii) $\forall (u,v,w) \in V^3 : uv \in E, uw \in E \Rightarrow c(uv) \neq c(uw)$,

(iii) $\forall (u,v) \in V^2 : uv \in E \Rightarrow |c(u) - c(uv)| \geq p$.

The *$(p,1)$-total number* of a graph G, denoted by $\lambda_p^T(G)$, is the minimum integer k such that G has a $(p,1)$-total labelling[1] with labels from $\{1,\ldots,k\}$. Figure 5.2 gives an example of a $(2,1)$-total labelling with 6 colors.

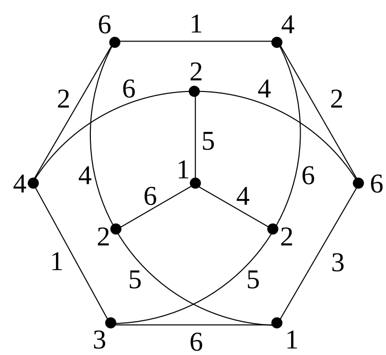

Figure 5.2: A $(2,1)$-total labelling of Petersen's graph.

Observe that $(1,1)$-total labelling is the usual total coloring (which, again, is basically the same as coloring the square of the incidence graph): for any graph G, $\lambda_1^T(G) = \chi^T(G) = \chi(G^{\star 2})$.

We recall some bounds and a conjecture for the $(p,1)$-total number:

Theorem 5.2 [HY08] *Let G be a graph with maximum degree Δ, then:*

(i) $\lambda_p^T(G) \geq \Delta + p$.

(ii) *If G is Δ-regular,* $\lambda_p^T(G) \geq \Delta + p + 1$.

[1] As in Chapter 3, our definition of $\lambda_p^T(G)$ may differ by one from some of the definitions found in the literature, since we consider labels from $\{1,\ldots k\}$ instead of $\{0,\ldots k\}$. We choose this convention in order to be coherent with the definition of $L(p,q)$-labelling given in Chapter 3 and to have $\lambda_1^T(G)$ equal to the total chromatic number of G.

(iii) *If* $p \geq \Delta$, $\lambda_p^T(G) \geq \Delta + p + 1$.

Observe that if we color the vertices properly with colors belonging to an interval I_V containing $\chi(G)$ colors and the edges with colors belonging to an interval I_E containing $\chi'(G)$ colors, I_V and I_E being separated by an interval of size $p-1$, we obtain a $(p,1)$-total labelling of the graph G. Theorem 5.3 is deduced from this observation :

Theorem 5.3 [HY08] *Let G be a graph, then*

(i) $\lambda_p^T(G) \leq \chi(G) + \chi'(G) + p - 1$

(ii) $\lambda_p^T(G) \leq 2\Delta + p$

Observe that the following conjecture is a generalization of the Total Coloring Conjecture:

Conjecture 5.4 [HY08] *Let G be a graph with maximum degree Δ, then $\lambda_p^T(G) \leq \Delta + 2p$.*

Montassier and Raspaud [MR03] proved this conjecture for graphs with large maximum degree and small maximum average degree.

Theorem 5.5 [MR03] *Let G be a connected graph with maximum degree Δ, and let $p \geq 2$ be an integer, then $\lambda_p^T(G) \leq \Delta + 2p - 1$ in the following cases :*

(i) $\Delta \geq 2p + 1$ *and* $mad(G) < \frac{5}{2}$;

(ii) $\Delta \geq 2p + 2$ *and* $mad(G) < 3$;

(iii) $\Delta \geq 2p + 3$ *and* $mad(G) < \frac{10}{3}$.

As mentionned above, Molloy and Reed [MR98] proved that the total chromatic number of any graph with maximum degree Δ is at most Δ plus an absolute constant. Moreover, in [MR02], they gave a slightly simpler proof of this result for sparse graphs. In this chapter, our aim is to generalize their approach to the $(p,1)$-total number. Our proof follows the lines of the proof in [MR02], but the analysis is significantly more complex. Besides, we fill in some blanks of [MR02], which is more a sketch than a complete proof.

A vertex $v \in V(G)$ is said to be α-*sparse* if the subgraph of G induced by $N(v)$ contains at most $\binom{\Delta}{2} - \alpha\Delta$ edges. An α-sparse graph is a graph in which all the vertices are α-sparse. In this chapter, we will consider $\varepsilon\Delta$-sparse graph for fixed $0 < \varepsilon < \frac{1}{2}$, in other words, graphs such that the (subgraph induced by the) neighborhood of any vertex contains at most $c\binom{\Delta}{2}$ edges, for some absolute constant $c < 1$. Note that every $G_{n,p}$ with $p < 1$ is asymptotically almost surely (that is, with probability tending to 1 when n tends to infinity) $\varepsilon\Delta$-sparse for some $0 < \varepsilon < \frac{1}{2}$.

Our main result is the following :

Theorem 5.6 [EMR06] *For any $0 < \varepsilon < \frac{1}{2}$, and any positive integer p, there exists a constant $C_{p,\varepsilon}$ such that for any $\varepsilon\Delta$-sparse graph G with maximum degree Δ, we have $\lambda_p^T(G) \leq \Delta + C_{p,\varepsilon}$.*

The proof of Theorem 5.6 is based on a probabilistic approach. It uses intensively concentration inequalities and Lovász Local Lemma. We also conjecture the following, which is a weakening of Conjecture 5.4:

Conjecture 5.7 *For any positive integer p, there exists a constant C_p, such that for any graph G with maximum degree Δ, we have $\lambda_p^T(G) \leq \Delta + C_p$.*

In Section 5.2, we present the procedure used to prove Theorem 5.6 and in Section 5.3, we analyze this procedure. The probabilistic tools used in the proof are described in Chapter 1 (for further details, see [MR02]).

5.2 Proof of Theorem 5.6

Since $\lambda_p^T(G) \leq 2\Delta + p$, if we prove that for some $\Delta_0(p, \varepsilon)$ and some $D_{p,\varepsilon}$, any $\varepsilon\Delta$-sparse graph G of maximum degree $\Delta \geq \Delta_0$ verifies $\lambda_p^T(G) \leq \Delta + D_{p,\varepsilon}$, then Theorem 5.6 will be proved.

The second observation is that it suffices to prove the theorem for Δ-regular graphs (graphs in which all the vertices have degree Δ). If G is not Δ-regular, take two copies of G and join the two copies of any vertex with degree less than Δ (see Figure 5.3 for an example). Since the minimum degree increases by one, by repeating this process we eventually obtain a Δ-regular graph containing G. Moreover it is easy to see that if G is $\varepsilon\Delta$-sparse, then the graph obtained from the construction is also $\varepsilon\Delta$-sparse. Hence, we can assume from now on that the graph G is Δ-regular.

Let ϕ be a full or partial coloring of G. Any edge $e = uv$ such that $|\phi(u) - \phi(e)| < p$ or/and $|\phi(v) - \phi(e)| < p$ is called a *reject edge*. The graph R induced by the reject edges is called the *reject graph*. It will be convenient for us to consider the *reject degree* of a vertex v, which is the number of edges $e = uv$ such that $|\phi(u) - \phi(e)| < p$. Observe that $d_R(v)$ is at most the reject degree of v plus $2p - 1$.

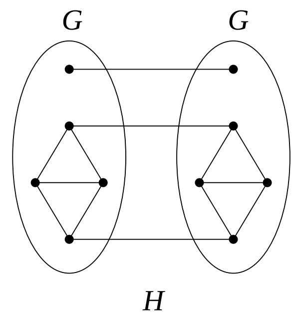

Figure 5.3: $G \subseteq H$, $\Delta(G) = \Delta(H)$, and $\delta(H) = \delta(G) + 1$.

5.2.1 Sketch of Proof

Set $\mathcal{C} = \Delta + 1$. To prove Theorem 5.6, we apply the following steps :

Step 1. First, we will color the edges by Vizing's Theorem using colors from $\{1, \dots, \mathcal{C}\}$.

Step 2. Then we will use the *Naive Coloring Procedure* to color the vertices with colors $\{1, \dots, \mathcal{C}\}$. This procedure creates reject edges. However, we can prove that after the procedure, the maximum degree of the reject graph R is a constant $D_{p,\varepsilon}$ which does not depend on Δ.

Step 3. Finally, we remove the color of the vertices of R and recolor these vertices greedily with the colors from $\{\Delta+p+1, \dots, \Delta+p+2+D_{p,\varepsilon}\}$. Taking $C_{p,\varepsilon} = D_{p,\varepsilon} + p - 2$, this proves that $\lambda_p^T(G) \leq \Delta + C_{p,\varepsilon}$.

We now present the Naive Coloring Procedure.

5.2.2 The Naive Coloring Procedure

For each vertex v, we maintain two lists of colors: L_v and F_v. L_v is the set of colors which do not appear in the neighborhood of v. Initially, $L_v = \{1, \dots, \mathcal{C}\}$. After iteration I (specified later), F_v will be a set of forbidden colors. Until iteration I, $F_v = \varnothing$.

During the Naive Coloring Procedure, we will perform i^* (specified later) iterations of the following procedure :

Step 1. Assign to each uncolored vertex v a color chosen uniformly at random in L_v.

Step 2. Uncolor any vertex which receives the same color as a neighbor in this iteration.

Step 3. **Iteration $i \leq I$.** Let v be a vertex having more than T (specified later) neighbors u which are assigned a color $c(u)$ such that $|c(uv) - c(u)| < p$ in this iteration. For any v, we uncolor all such neighbors.

Iteration $i > I$.

(a) Uncolor any vertex v which receives a color from F_v in this iteration.

(b) Let v be a vertex having more than one neighbor u which is assigned a color such that $|c(uv) - c(u)| < p$ in this iteration. For any v, we uncolor all such neighbor.

(c) Let v be a vertex having at least one neighbor u such that $|c(uv) - c(u)| < p$ in this iteration. For any v, we place $\{c(vw) - p + 1, \ldots, c(vw), \ldots, c(vw) + p - 1\}$ in F_w for every $w \in N(v)$.

Step 4. For any vertex v which retained its color (*i.e.* which was not uncolored during a previous step), we remove $c(v)$ from L_u for any $u \in N(v)$.

After i^* iterations of this procedure, we have a partial coloring of G. We complete this coloring in order to obtain a reject graph R with a bounded maximum degree which does not depend on Δ.

5.3 Analysis of the procedure

5.3.1 The first iteration

Let $\zeta = \frac{\varepsilon}{2e^3}$. In this subsection, we prove that:

Claim 5.8 *The first iteration produces a partial coloring with bounded reject degree for which every vertex has at least $\frac{\zeta}{2}\Delta$ repeated colors in its neighborhood.*

We recall that $\mathcal{C} = \Delta + 1$ is the initial size of each color list L_v. Let A_v be the number of colors c such that at least two neighbors of v receive the color c and all such vertices retain their color during Step 2. Let B_v be the number of neighbors of v which are uncolored at Step 3. Notice that vertices are uncolored at Step 3 regardless of what happened at Step 2. Let X_v be the event that "$A_v < \zeta\Delta$". Let Y_v be the event that "$B_v \geq \frac{\zeta}{2}\Delta$". If no type X event occurs, every vertex has at least $\zeta\Delta$ repeated colors in its neighborhood at the end of Step 2. If no type Y event occurs, less than $\frac{\zeta}{2}\Delta$ vertices are uncolored in each neighborhood. As a consequence, if we show that with positive probability, no type X or Y event occurs, Claim 5.8 will be proved.

Claim 5.9 $\mathbf{Pr}(X_v) < e^{-\alpha \log^2 \Delta}$, *for a particular constant* $\alpha > 0$.

Proof. We first bound the expected value of A_v. Let A'_v be the number of colors c such that exactly two neighbors of v receive the color c and are not uncolored during Step 2. Notice that $A_v \geq A'_v$, and thus $\mathbf{E}(A_v) \geq \mathbf{E}(A'_v)$. Let u and w be two non adjacent neighbors of v. The probability that u and w are colored with c, while no other neighbor of v is colored with c, and while no neighbor of u or w is colored with c is exactly $\left(\frac{1}{\mathcal{C}}\right)^2 \left(1 - \frac{1}{\mathcal{C}}\right)^{3\Delta-3} > \left(\frac{1}{\mathcal{C}}\right)^2 \left(1 - \frac{1}{\mathcal{C}}\right)^{3\Delta}$. Since G is $\varepsilon\Delta$-sparse, $|E(N(v))| \leq \binom{\Delta}{2} - \varepsilon\Delta^2$. We assumed without loss of generality that G was Δ-regular, so there are at least $\varepsilon\Delta^2$ pairs of non adjacent vertices among the neighbors of v. There are $\mathcal{C}$ choices for the color c, thus

$$\mathbf{E}(A'_v) > \mathcal{C}\varepsilon\Delta^2 \left(\frac{1}{\mathcal{C}}\right)^2 \left(1 - \frac{1}{\mathcal{C}}\right)^{3\Delta} = \frac{\varepsilon\Delta^2}{\mathcal{C}} \left(1 - \frac{1}{\mathcal{C}}\right)^{3\Delta}$$

For $\Delta > 2$, we have $\ln(1-\frac{1}{\mathcal{C}}) \geq -\frac{1}{\mathcal{C}} - \frac{1}{\mathcal{C}^2}$, and thus $\left(1 - \frac{1}{\mathcal{C}}\right)^{3\Delta} \geq e^{-3}e^{-\frac{3}{\mathcal{C}}}$. For Δ large enough, $\Delta/\mathcal{C} > \sqrt{3}/2$ and $e^{-\frac{3}{\mathcal{C}}} > \sqrt{3}/2$, so:

$$\mathbf{E}(A'_v) > \frac{3\varepsilon\Delta}{4e^3} = \frac{3}{2}\zeta\Delta$$

Since $\mathbf{E}(A_v) \geq \mathbf{E}(A'_v)$, we also have $\mathbf{E}(A_v) > \frac{3}{2}\zeta\Delta$. Let AT_v be the number of colors assigned to at least two neighbors of v, and let Del_v be the number of colors assigned to at least two neighbors of v and not retained by at least one of them. Note that $A_v = AT_v - \mathrm{Del}_v$, and by linearity of expectation, $\mathbf{E}(A_v) = \mathbf{E}(AT_v) - \mathbf{E}(\mathrm{Del}_v)$. The random variable AT_v only depends on the Δ colors assigned to the neighbors of v. Moreover, changing one of these colors can only affect AT_v by at most 1. Using the Simple Concentration bound, we obtain:

$$\mathbf{Pr}\left(|AT_v - \mathbf{E}(AT_v)| > t\right) < 2e^{-\frac{t^2}{2\Delta}}. \tag{5.1}$$

The random variable Del_v only depends on the nearly Δ^2 colors assigned to the vertices at distance at most 2 from v. As previously, changing one of these colors can only affect Del_v by at most 1. Furthermore, if $\mathrm{Del}_v \geq s$, we can find at most $3s$ vertices, whose colors certify that $\mathrm{Del}_v \geq s$ (for each color α counted by $\mathrm{Del}_v \geq s$, we take two neighbors x and y of v colored with α and a neighbor z of x or y also colored with α). Applying Talagrand's Inequality with $c = 1$ and $r = 3$, we obtain for all $t \geq \sqrt{\Delta \log \Delta}$

$$\mathbf{Pr}\left(|\mathrm{Del}_v - \mathbf{E}(\mathrm{Del}_v)| > t\right) < 4e^{-\frac{\left(t - 60\sqrt{3\mathbf{E}(\mathrm{Del}_v)}\right)^2}{24\mathbf{E}(\mathrm{Del}_v)}} < 4e^{-\frac{t^2}{25\Delta}}, \tag{5.2}$$

since $\mathbf{E}(\mathrm{Del}_v) \leq \Delta$. Recall that $\mathbf{E}(A_v) = \mathbf{E}(AT_v) - \mathbf{E}(\mathrm{Del}_v)$. Let $t = \frac{1}{2}\log\Delta\sqrt{\mathbf{E}(A_v)}$. If $|A_v - \mathbf{E}(A_v)| > \log\Delta\sqrt{\mathbf{E}(A_v)}$ we have either $|AT_v - \mathbf{E}(AT_v)| > t$ or $|\mathrm{Del}_v - \mathbf{E}(\mathrm{Del}_v)| > t$. Using (5.1) and (5.2), the probability that this happens is at most

$$2e^{-\frac{t^2}{2\Delta}} + 4e^{-\frac{t^2}{25\Delta}} < 2e^{-\frac{3}{16}\zeta\log^2\Delta} + 4e^{-\frac{3}{200}\zeta\log^2\Delta} < e^{-\frac{\zeta}{100}\log^2\Delta}$$

So, for Δ large enough, $\mathbf{Pr}\left(|A_v - \mathbf{E}(A_v)| > \log\Delta\sqrt{\mathbf{E}(A_v)}\right) < e^{-\frac{\zeta}{100}\log^2\Delta}$.

$$\begin{aligned}
\mathbf{Pr}\left(|A_v - \mathbf{E}(A_v)| > \log\Delta\sqrt{\mathbf{E}(A_v)}\right) &\geq \mathbf{Pr}\left(A_v < \mathbf{E}(A_v) - \log\Delta\sqrt{\mathbf{E}(A_v)}\right) \\
&\geq \mathbf{Pr}\left(A_v < \frac{3}{2}\zeta\Delta - \log\Delta\sqrt{\Delta}\right) \\
&\geq \mathbf{Pr}\left(A_v < \zeta\Delta\right)
\end{aligned}$$

Since $\mathbf{Pr}(X_v) = \mathbf{Pr}(A_v < \zeta\Delta)$, we proved that $\mathbf{Pr}(X_v) < e^{-\frac{\zeta}{100}\log^2\Delta}$. ∎

Claim 5.10 $\mathbf{Pr}(Y_v) < e^{-\beta\Delta}$, *for a particular constant* $\beta > 0$.

Proof. Let u be a neighbor of v. The vertex u will be uncolored in Step 3 if for some neighbor w of u, u and T other neighbors $x_1, \ldots, x_T$ of w are each assigned a color $c(x_i)$ such that $|c(u) - c(wu)| < p$ and $|c(x_i) - c(wx_i)| < p$ for all $1 \leq i \leq T$. The probability that this happens is at most

$$\Delta\binom{\Delta-1}{T}\left(\frac{2p-1}{\mathcal{C}}\right)^{T+1} < \frac{(2p-1)^{T+1}}{T!}$$

For T large enough, $(2p-1)^{T+1}/T! < \zeta/4$, and thus $\mathbf{E}(B_v) < \frac{\zeta\Delta}{4}$. The random variable B_v only depends on the nearly Δ^3 colors assigned to the vertices at distance at most 3 from v. Changing one of these colors can affect B_v by at most $T+1$. Moreover, if $B_v \geq s$ there is a set of at most $(T+1)s$ vertices whose colors certify that $B_v \geq s$ (for each uncolored neighbor u of v, take u and T other neighbors $x_1, \ldots, x_T$ of some neighbor w of u, such that $|c(u) - c(wu)| < p$ and $|c(x_i) - c(wx_i)| < p$ for all $1 \leq i \leq T$). Applying Talagrand's Inequality to B_v with $c = T+1$ and $r = T+1$, we obtain for all $t \geq \sqrt{\Delta\log\Delta}$

$$\mathbf{Pr}\left(|B_v - \mathbf{E}(B_v)| > t\right) < 4e^{-\frac{\left(t - 60(T+1)\sqrt{(T+1)\mathbf{E}(B_v)}\right)^2}{8(T+1)^3\mathbf{E}(B_v)}} < 4e^{-\frac{t^2}{9(T+1)^3\Delta}}.$$

Taking $t = \frac{\zeta\Delta}{8}$, we obtain $\mathbf{Pr}\left(|B_v - \mathbf{E}(B_v)| > \frac{\zeta\Delta}{8}\right) < 4e^{-\frac{\zeta^2\Delta}{576(T+1)^3}} < e^{-\frac{\zeta^2\Delta}{577(T+1)^3}}$. Now, since

$$\begin{aligned}\mathbf{Pr}\left(|B_v - \mathbf{E}(B_v)| > \frac{\zeta\Delta}{8}\right) &\geq \mathbf{Pr}\left(B_v > \mathbf{E}(B_v) + \frac{\zeta\Delta}{8}\right)\\ &\geq \mathbf{Pr}\left(B_v > \frac{3}{8}\zeta\Delta\right)\\ &\geq \mathbf{Pr}\left(B_v \geq \frac{\zeta\Delta}{2}\right)\end{aligned}$$

we have $\mathbf{Pr}(Y_v) < e^{-\frac{\zeta^2}{577(T+1)^3}\Delta}$. ∎

We now use Lovász Local Lemma to prove Claim 5.8. Each event X_v only depends on the colors assigned to the vertices at distance at most 2 from v, and each event Y_v depends on the colors assigned to the vertices at distance at most 3 from v. Hence, each event is mutually independent of all but at most $2\Delta^6$ other events. For Δ sufficiently large, $\mathbf{Pr}(X_v) < \frac{1}{8\Delta^6}$ and $\mathbf{Pr}(Y_v) < \frac{1}{8\Delta^6}$. Using Lovász Local Lemma, this proves that with positive probability no type X or Y event happens. Thus with positive probability, the first iteration produces a partial coloring with bounded reject degree, such that each vertex has at least $\frac{\zeta\Delta}{2}$ repeated colors in its neighborhood.

5.3.2 The next iterations

Let $d_i = \left(1 - \frac{1}{4}e^{-\frac{2}{\zeta}}\right)^i \Delta$ and $f_i = \frac{4(2p-1)}{\zeta}\sum_{j=I+1}^{i-1} d_j$. Let i^* be the smallest integer i such that $d_i \leq \sqrt{\Delta}$. Observe that for any $i \leq i^*$, we have $d_i \geq (1 - \frac{1}{4}e^{-\frac{2}{\zeta}})\sqrt{\Delta}$.

Claim 5.11 *At the end of each iteration $1 \leq i \leq i^*$, with positive probability every vertex has at most d_i uncolored neighbors, and each list F_v has size at most f_i.*

Proof. We prove Claim 5.11 by induction on i. At the end of the first iteration, every vertex has at least $\frac{\zeta\Delta}{2}$ repeated colors in its neighborhood. So the number of uncolored vertices in the neighborhood of any vertex is at most $(1-\zeta)\Delta$, which is less than $d_1 = \left(1 - \frac{1}{4}e^{-\frac{2}{\zeta}}\right)\Delta$. Morever, for any vertex v, the list F_v is still empty at the end of the first iteration, thus $|L_v| = 0 = f_1$.

Suppose $i > 1$. By induction, there are at most d_{i-1} uncolored vertices in each neighborhood at the beginning of iteration i, and each F_v has size at most f_{i-1}. We define the random variable D_v^i as the number of uncolored neighbors of v after iteration i, and the random variable F_v^i as the size of the list F_v after iteration i. To complete the induction, we show that with positive probability, $D_v^i \le d_i$ and $F_v^i \le f_i$ for any vertex v. Since every vertex v has at least $\frac{\zeta\Delta}{2}$ repeated colors in its neighborhood, every list L_v has size at least $\frac{\zeta\Delta}{2}$. Thus, the probability that a newly colored vertex is not uncolored during Step 2 is at least $\left(1-\frac{2}{\zeta\Delta}\right)^{\Delta}$. So the probability that a newly colored vertex is uncolored during Step 2 is at most:

$$1-\left(1-\frac{2}{\zeta\Delta}\right)^{\Delta} \le 1-\frac{3}{4}e^{-\frac{2}{\zeta}}$$

For $i \le I$, the probability that the newly colored vertex v is uncolored during Step 3 is at most:

$$\Delta\binom{d_{i-1}}{T}\left(\frac{2p-1}{\zeta\Delta/2}\right)^{T+1} \le \left(\frac{2(2p-1)}{\zeta\Delta}\right)^{T+1}\frac{1}{T!} \le \frac{1}{4}e^{-\frac{2}{\zeta}}$$

Observe that for I sufficiently large in terms of ζ and p, we have

$$\begin{aligned} f_i = \frac{4(2p-1)\Delta}{\zeta}\sum_{j=I+1}^{i-1}\left(1-\frac{1}{4}e^{-\frac{2}{\zeta}}\right)^{j} &\le \frac{4(2p-1)\Delta}{\zeta}\times 4e^{\frac{2}{\zeta}}\left(1-\frac{1}{4}e^{-\frac{2}{\zeta}}\right)^{I+1} \\ &< \frac{\zeta\Delta}{16}e^{-\frac{2}{\zeta}}. \end{aligned}$$

Thus, for $i > I$, the probability that the vertex v is uncolored during Step $3(a)$ is at most:

$$\frac{|F_v|}{|L_v|} \le \frac{2}{\zeta\Delta}f_{i-1} < \frac{1}{8}e^{-\frac{2}{\zeta}}$$

And the probability that v is uncolored during Step $3(b)$ is at most:

$$\Delta d_{i-1}\left(\frac{2(2p-1)}{\zeta\Delta}\right)^2 \le \left(1-\frac{1}{4}e^{-\frac{2}{\zeta}}\right)^{I}\left(\frac{2(2p-1)}{\zeta}\right)^2 \le \frac{1}{8}e^{-\frac{2}{\zeta}}$$

Combining these results, the probability that a newly colored vertex is uncolored during Step 2 or Step 3 is at most $1-\frac{3}{4}e^{-\frac{2}{\zeta}}+\frac{1}{4}e^{-\frac{2}{\zeta}} = 1-\frac{1}{2}e^{-\frac{2}{\zeta}}$. As a consequence,

$$\mathbf{E}(D_v^i) \le \left(1-\frac{1}{2}e^{-\frac{2}{\zeta}}\right)d_{i-1}$$

Let X_v^i be the event that $D_v^i > \left(1-\frac{1}{4}e^{-\frac{2}{\zeta}}\right)d_{i-1}$. We define the random variable NF_v^i as the number of colors added to F_v during iteration i. Let Y_v^i be the event that $NF_v^i > \frac{4(2p-1)}{\zeta}d_{i-1}$. Using Lovász Local Lemma, we prove that with positive probability none of the type X or Y events occurs.

Claim 5.12 $\mathbf{Pr}(X_v^i) < e^{-\delta \log^2 d_{i-1}}$, *for a particular constant* $\delta > 0$.

Proof. Let v be a vertex of G. Let A be the number of neighbors of v that are uncolored during Step 2. For $i \leq I$ we define B as the number of neighbors of v that are uncolored during Step 3. For $i > I$ we define C (resp. D) as the number of neighbors of v that are uncolored during Step 3.(a) (resp. 3.(b)). Using the Simple Concentration Bound on A, Talagrand's Inequality on B and D, and Chernoff Bound on C, combined with $\mathbf{E}(D_v^i) \leq (1 - \frac{1}{2}e^{-\frac{2}{\zeta}})d_{i-1}$, we prove the following inequalities:

$$\mathbf{Pr}\left(|A - \mathbf{E}(A)| > \frac{1}{2}\log d_{i-1}\sqrt{\mathbf{E}(A+B)}\right) < 2e^{-\frac{e^{-\frac{2}{\zeta}}}{64}\log^2 d_{i-1}} \quad (5.3)$$

$$\mathbf{Pr}\left(|B - \mathbf{E}(B)| > \frac{1}{2}\log d_{i-1}\sqrt{\mathbf{E}(A+B)}\right) < 4e^{-\frac{e^{-\frac{2}{\zeta}}}{64(T+1)^3}\log^2 d_{i-1}} \quad (5.4)$$

$$\mathbf{Pr}\left(|A - \mathbf{E}(A)| > \frac{1}{3}\log d_{i-1}\sqrt{\mathbf{E}(A+C+D)}\right) < 2e^{-\frac{e^{-\frac{2}{\zeta}}}{144}\log^2 d_{i-1}} \quad (5.5)$$

$$\mathbf{Pr}\left(|C - \mathbf{E}(C)| > \frac{1}{3}\log d_{i-1}\sqrt{\mathbf{E}(A+C+D)}\right) < 2e^{-\frac{1}{144}\log^2 d_{i-1}} \quad (5.6)$$

$$\mathbf{Pr}\left(|D - \mathbf{E}(D)| > \frac{1}{3}\log d_{i-1}\sqrt{\mathbf{E}(A+C+D)}\right) < 2e^{-\frac{e^{-\frac{2}{\zeta}}}{1152}\log^2 d_{i-1}} \quad (5.7)$$

The proof of these results is very close from the proofs of Claims 5.9 and 5.10. Combining (5.3), (5.4), (5.5), (5.6) and (5.7), we obtain for T and Δ large enough :

$$\mathbf{Pr}(X_v^i) < e^{-\frac{e^{-\frac{2}{\zeta}}}{65(T+1)^3}\log^2 d_{i-1}}$$

∎

Claim 5.13 $\mathbf{Pr}(Y_v^i) < e^{-\gamma d_{i-1}}$, *for a particular constant* $\gamma > 0$.

Proof. The probability that a neighbor u of v is assigned a color $c(u)$ such that $|c(u) - c(uv)| < p$ is $\frac{2p-1}{|L_u|} \leq \frac{2(2p-1)}{\zeta\Delta}$. Thus $\mathbf{E}(NF_v) \leq \frac{2(2p-1)}{\zeta\Delta}d_{i-1}$. Applying Talagrand's Inequality to the random variable NF_v with $c = (2p-1)^2$ and $r = 1$, we obtain :

$$\mathbf{Pr}\left(|NF_v - \mathbf{E}(NF_v)| > t\right) < 4e^{-\frac{\zeta t^2}{16(2p-1)^5 d_{i-1}}}$$

for any $t > \log d_{i-1}\sqrt{d_{i-1}}$. Taking $t = \frac{2p-1}{\zeta}d_{i-1}$, we obtain :

$$\begin{aligned}\mathbf{Pr}\left(NF_v > \tfrac{4(2p-1)}{\zeta}d_{i-1}\right) &\leq \mathbf{Pr}\left(|NF_v - \mathbf{E}(NF_v)| > \tfrac{2p-1}{\zeta}d_{i-1}\right)\\ &< 4e^{-\frac{d_{i-1}}{2\zeta(2p-1)^3}}.\end{aligned}$$

∎

The variable X_v^i only depends on the colors assigned to the vertices at distance at most 3 from v during iteration i, while the variable Y_v^i depends on the colors assigned to the vertices at distance at most 2 from v during iteration i. Thus, each type X or Y event is mutually independent from all but at most $2d_{i-1}^6$ other events. Using Claims 5.12 and 5.13, we have $\mathbf{Pr}(X_v^i) < \frac{1}{8d_{i-1}^6}$ and $\mathbf{Pr}(Y_v^i) < \frac{1}{8d_{i-1}^6}$ for Δ large enough (recall that according to our choice of i^* we always have $d_i \geq (1 - \frac{1}{4}e^{-\frac{2}{\zeta}})\sqrt{\Delta})$. Lovász Local Lemma completes the induction. ∎

5.3.3 The final phase

At this point, we have a partial coloring such that:

- each vertex v has at most $\sqrt{\Delta}$ uncolored neighbors;
- the reject degree of each vertex is at most $IT + 1$;
- each uncolored vertex has a list of at least $\frac{\zeta\Delta}{2}$ available colors.

It will be more convenient to use lists of equal sizes. So we arbitrarily remove colors from each list, so that for every uncolored vertex v, we have $|L_v| = \lceil \frac{\zeta\Delta}{2} \rceil$. For each uncolored vertex, we choose a subset of colors from L_v which will be *candidates* for v and we prove that with positive probability, there exists a candidate for each uncolored vertex, such that we can complete our partial coloring of G.

A candidate a for v is said to be *good* if:

Condition 1 for every neighbor u of v, a is not candidate for u;

Condition 2 for every neighbor u of v, and every neighbor w of u, there is no candidate b of w such that $|c(uv) - a| < p$ and $|c(uw) - b| < p$.

If we find a good candidate for every uncolored vertex, Condition 1 ensures that the vertex coloring obtained is proper, and Condition 2 ensures that no reject degree increases by more than one.

Claim 5.14 *There exists a set of candidates S_v for each uncolored vertex v, such that each set contains at least one good candidate.*

Proof. For each uncolored vertex v, we choose a random permutation of L_v, and take the first twenty colors of the list as set of candidates for v. Let C_v be the event that none of the candidates for v is a good candidate. Each event C_v depends on at most Δ^4 other events. We now show that

$\mathbf{Pr}(C_v) < \frac{1}{4\Delta^4}$. Lovász Local Lemma will complete the proof.

Let v be an uncolored vertex of G. We define:

$Bad_1 = \{c \in L_v : c \text{ is candidate for some neighbor of } v\}$

$Bad_2 = \{c \in L_v : \text{choosing } c \text{ for } v \text{ violates Condition 2}\}$

$Bad = Bad_1 \cup Bad_2$

Note that a candidate for v is good if and only if it does not belong to Bad. Let D be the event that $|Bad| \leq 60(2p-1)^2\sqrt{\Delta}$. Observe that :

$$\mathbf{Pr}(C_v|D) \leq \left(\frac{|Bad|}{|L_v|}\right)^{20} \leq \left(\frac{60(2p-1)^2\sqrt{\Delta}}{\lceil\frac{\zeta\Delta}{2}\rceil}\right)^{20} \leq \frac{120^{20}(2p-1)^{40}}{\zeta^{20}\Delta^{10}}$$

So for Δ sufficiently large, $\mathbf{Pr}(C_v|D) < \frac{1}{8\Delta^4}$.

Each vertex has at most $\sqrt{\Delta}$ uncolored neighbors, thus $|Bad_1| \leq 20\sqrt{\Delta} \leq 20(2p-1)^2\sqrt{\Delta}$. We now show that with high probability, the size of Bad_2 is at most $40(2p-1)^2\sqrt{\Delta}$. A color c belongs to Bad_2 if for some neighbor u of v such that $|c(uv) - c| < p$, there is a neighbor w of u and a candidate a for w such that $|c(uw) - a| < p$. Thus we obtain:

$$\mathbf{Pr}(c \in Bad_2) \leq (2p-1) \times 20\sqrt{\Delta} \times \frac{2p-1}{\lceil\frac{\zeta\Delta}{2}\rceil}$$

$$\mathbf{E}(|Bad_2|) \leq \left\lceil\frac{\zeta\Delta}{2}\right\rceil \times \mathbf{Pr}(c \in Bad_2) \leq 20(2p-1)^2\sqrt{\Delta}$$

The random variable $|Bad_2|$ only depends on at most Δ^2 permutations of color lists of uncolored vertices at distance at most 2 from v. Moreover, exchanging two members of one of the permutations can affect $|Bad_2|$ by at most $2p-1$. If $|Bad_2| \geq s$, we can certify this by giving, for each color $\alpha \in Bad_2$, a neighbor u of v such that $|c(uv) - \alpha| < p$, as well as a neighbor w of u having a candidate a such that $|c(uw) - a| < p$. Recall that a is a candidate for w if it belongs to the first twenty positions of the permutation of L_w. So we only need to give s choices of candidates to certify that $|Bad_2| \geq s$. We apply McDiarmid's Inequality to $X = |Bad_2|$ with $n = 0$, $m = \Delta^2$, $c = 2p-1$, $r = 1$, and $t = 10(2p-1)^2\sqrt{\Delta}$:

$$\mathbf{Pr}\left(|X - \mathbf{E}(X)| > 10(2p-1)^2\sqrt{\Delta} + 60(2p-1)\sqrt{\mathbf{E}(X)}\right) < 4e^{-\frac{100(2p-1)^4\Delta}{8(2p-1)^2\mathbf{E}(X)}}$$

Since $\mathbf{E}(X) \leq 20(2p-1)^2\sqrt{\Delta}$, this implies for Δ sufficiently large:

$$\mathbf{Pr}\left(|Bad_2| > 40(2p-1)^2\sqrt{\Delta}\right) < 4e^{-\frac{5}{8}\sqrt{\Delta}}$$

So for Δ large enough, $\mathbf{Pr}\left(\overline{D}\right) < \frac{1}{8\Delta^4}$. We can express the probability of C_v as $\mathbf{Pr}(C_v) = \mathbf{Pr}(C_v|D)\mathbf{Pr}(D) + \mathbf{Pr}(C_v|\overline{D})\mathbf{Pr}(\overline{D})$. Hence,

$$\mathbf{Pr}(C_v) \leq \mathbf{Pr}(C_v|D) + \mathbf{Pr}(\overline{D}) < \frac{1}{4\Delta^4}$$

∎

We obtain a coloring of G with maximum reject degree at most $IT+2$. So the reject graph R obtained has maximum degree at most $IT+2p+1$. We uncolor the vertices of R and recolor them greedily with colors from $\{\Delta+p+1, \ldots, \Delta+IT+3p+3\}$ using Brooks theorem. This final coloring is a $(p,1)$-total labelling of G. Since I and T are independent of Δ, we proved that $\lambda_p^T(G) \leq \Delta + C_{p,\varepsilon}$.

5.4 Conclusion

Using general ideas from [MR02], Theorem 5.6 can be seen as a first step to prove Conjecture 5.7, which would be the closest result from Conjecture 5.4 so far.

Indeed, we only use the sparseness of G to prove that after the first iteration, we obtain a partial coloring with many repeated colors in each neighborhood. So the proof of Theorem 5.6 also implies the following lemma:

Lemma 5.15 *For every $\varepsilon, \zeta > 0$ and every integer p, there exists two constants $C(\zeta, p, \varepsilon)$ and $\Delta(\zeta, p, \varepsilon)$ such that the following holds : consider any graph G with maximum degree $\Delta \geq \Delta(\zeta, p, \varepsilon)$, any edge coloring of G, and any partial vertex coloring of G such that every uncolored vertex has $\zeta\Delta$ colors appearing at least twice in its neighborhood. The partial vertex coloring can be completed in order to obtain a $(p,1)$-total labelling of G such that the maximum reject degree does not increase by more than $C(\zeta, p, \varepsilon)$.*

It seems that Lemma 5.15 could be used to prove Conjecture 5.7, by only modifying the first iteration of the procedure (for example, by coloring first the dense components, and then apply the lemma to the remaining vertices). However, this would require much deeper probabilistic techniques and tools.

Chapter 6

Game coloring

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In this chapter, we look at distance-two coloring through a different angle. We study a two-player game in which the first player (Alice) tries to color the square of a graph with a given set of colors, whereas the second player (Bob) tries to prevent her from succeeding. The aim is to understand why acyclic game coloring is so different from the usual game coloring. To obtain bounds on the size of the color sets for which Alice has a winning strategy, we refine the usual activation strategy and adapt it to the case of distance-two colorings.

6.1 Introduction

The game coloring number of a simple graph G is defined through a two-player game. Alice and Bob take turns marking unmarked vertices of G, with Alice having the first move. Each move marks one unmarked vertex. The *game coloring number* $\mathrm{col}_g(G)$ of G is the smallest integer k such that Alice has a strategy to ensure that at any step of the game, every unmarked vertex is adjacent to at most $k-1$ marked vertices.

The game coloring number was first explicitly introduced by Zhu [Zhu99] as a tool in the study of the game chromatic number of graphs, which is also defined through a two-player game: let G be a graph and C

be a set of colors. Alice and Bob take turns coloring unmarked vertices of G, with Alice having the first move. Each move colors one unmarked vertex, subject to the condition that two adjacent vertices cannot be marked with the same color. Alice wins the game if eventually every vertex is marked. Bob wins the game if some unmarked vertex x cannot be marked anymore (each color in C has been assigned to some neighbor of x). The *game chromatic number* $\chi_g(G)$ of G is the minimum k for which Alice has a winning strategy with colors from $\{1, \ldots, k\}$ in this game.

The game chromatic number was introduced by Bodlaender [Bod91], and has been widely studied over the last ten years. The question of determining the game chromatic number of planar graphs has raised particular interest [Bod91, DZ99, Kie00, KT94, Zhu99]. Recently, Wu and Zhu [WZ08] proved that there exist planar graphs with game coloring number at least 11, and Zhu [Zhu08] proved that every planar graph has game chromatic number at most 17.

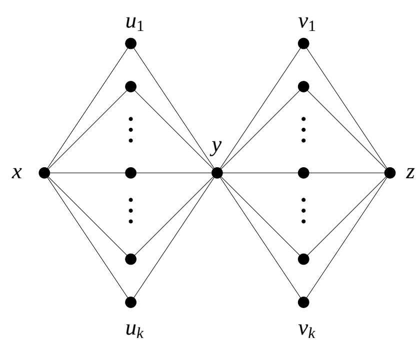

Figure 6.1: A partial 2-tree G with $\chi_{a,g}(G) \geq \Delta(G)/2$.

In his Ph.D Thesis, Chang [Cha07] recently investigated acyclic game colorings. The only difference with the definition above is that, at any step, the partial coloring has to be acyclic (that is, a proper coloring without bicolored cycles). The acyclic game chromatic number of a graph G is denoted by $\chi_{a,g}(G)$. Surprisingly, while the acyclic chromatic number of planar graphs is at most 5 [Bor79], their acyclic game chromatic number is not bounded. Chang [Cha07] gave an example of a partial 2-tree (with acyclic chromatic number at most three) with acyclic game chromatic number at least $\Delta/2$ (see Figure 6.1). It is easy to check that during his first two moves, Bob can color x and y with the same color, or y and z with the same color (depending on Alice's first moves). Then, either $u_1, \ldots u_k$, or $v_1, \ldots v_k$ must have distinct colors, and the acyclic game chromatic number is at least $\Delta/2$.

It is easy to prove that $\chi(G) \leq \chi_g(G) \leq \mathrm{col}_g(G) \leq \Delta + 1$ for any graph G with maximum degree Δ. Unfortunately, obtaining good upper bounds for the acyclic game chromatic number seems difficult in general. However, we can use the following observation, which is one of the main reasons why we studied distance-two game colorings.

Observation 6.1 *For every graph G, $\chi_{a,g}(G) \leq col_g(G^2)$.*

If Alice has a strategy to win the marking game in G^2 with k colors, then by using the same strategy she can win the acyclic game with k colors. When playing, Alice picks a vertex v such that at any step of the game, any unmarked vertex has at most $k-1$ marked vertices at distance one or two. She then colors v with a color distinct from all the colors at distance at most two from v. She eventually obtains a proper coloring of G^2, which is also an acyclic coloring of G.

It is very important to observe that $\chi_{a,g}(G) \leq \chi_g(G^2)$ may not be true in general, since Bob has more freedom in the ayclic game than in the game coloring of the square (which prevents Alice from using the exactly the same strategy).

Also note that if we have a winning strategy for a graph G, we cannot necessarily use it to obtain a winning strategy in a subgraph H of G. Furthermore, having a winning strategy with k colors for a graph G does not mean that we have a strategy with $k+1$ colors for G. As a consequence, it seems difficult to use proofs by induction or with minimum counterexamples as in Chapters 2, 3, and 4.

The following is an easy observation about the game chromatic number of the square of graphs with bounded maximum degree (and as a consequence, about their acyclic game chromatic number).

Observation 6.2 *If G has game colouring number k and maximum degree Δ, then $\chi_g(G^2) \leq col_g(G^2) \leq (k-1)(2\Delta - k + 1) + 1$.*

Assume that Alice has a strategy for the marking game on G to ensure that at any moment of the game, any unmarked vertex has at most $k-1$ marked neighbours in G. We shall show that by using the same strategy, Alice can ensure that at any moment of the game, any unmarked vertex has at most $(k-1)(2\Delta-k+1)$ marked vertices at distance at most 2 in G. Indeed, if v is an unmarked vertex, then let $N_M(v)$ be the set of marked neighbours of v in G, and $N_U(v)$ be the set of unmarked neighbours of v in G. Each vertex of $N_M(v)$ has at most $\Delta-1$ marked neighbours, and each vertex of $N_U(v)$ has at most $k-1$ marked neighbours. It is obvious that $k \leq \Delta+1$. If $k = \Delta+1$, then G^2 has maximum degree at most $(k-1)\Delta$, and the conclusion holds trivially. If $k \leq \Delta$, then since $|N_M(v)| \leq k-1$, the number of marked vertices at distance at most two from v in G is at most $|N_M(v)|(\Delta-1)+|N_M(v)|+(k-1)(\Delta-|N_M(v)|) \leq (k-1)(2\Delta-k+1)$.

6.2 Game coloring of the square of forests

For special classes of graphs, the upper bound for $\chi_g(G^2)$ in Observation 6.2 can usually be improved. This section proves a better upper bound for $\chi_g(G^2)$ when G is a forest.

Theorem 6.3 *If G is a forest with maximum degree $\Delta \geq 9$, then $\Delta+1 \leq \chi_g(G^2) \leq col_g(G^2) \leq \Delta + 3$.*

For any forest G, $\omega(G^2) = \Delta + 1$. Therefore $\chi_g(G^2) \geq \Delta + 1$. Assume $G = (V, E)$ is a forest with $\Delta \geq 9$. To prove that $\mathrm{col}_g(G^2) \leq \Delta + 3$, we shall give a strategy for Alice for the marking game on G^2, so that at any moment of the game, each unmarked vertex has at most $\Delta + 2$ marked neighbors in G^2.

If G is not a tree, then we may add some edges to G to obtain a tree. Thus we may assume that G is a tree. Alice's strategy is a variation of the activation strategy, which is widely used in the study of coloring game and marking game. She keeps track of a set $V_a \subseteq V$ of active vertices, which always induces a subtree of G. When a vertex v is added to V_a, we say that v is *activated*. Vertices in V_a are called *active* vertices, and other vertices are called *inactive*.

Choose a vertex r of G as the *root*, and view G as a rooted tree. For a vertex x, $f^1(x)$ (abbreviated as $f(x)$) is the father of x and for $i \geq 2$, let $f^i(x) = f(f^{i-1}(x))$. For convenience, we let $f(r) = r$. The vertices in $\{f^i(x) : i \geq 1\}$ are called the *ancestors* of x. Let $S(x)$ be the set of sons of x, and let $S^2(x) = \cup_{y \in S(x)} S(y)$ be the set of grandsons of x.

Alice's strategy:

- Initially she sets $V_a = \{r\}$, and marks r.
- Assume Bob has just marked a vertex x and there are still unmarked vertices. Let P_x be the unique path from x to the nearest vertex y of V_a. In particular, if $x \in V_a$, then $x = y$ and P_x consists of the single vertex x. Alice adds all the vertices of P_x to V_a, and marks the first unmarked vertex from the sequence: $f^2(y), f(y), y, z^*, v$, where v is an unmarked vertex with no unmarked ancestors, and z^* is defined as follows: Let $Z = \{z \in S(y) : |(S(z) \cup S^2(z)) \cap V_a|$ is maximum among all unmarked sons of $y\}$. Let M be the set of marked vertices. Then z^* is a vertex in Z for which $|(S(z^*) \cup S^2(z^*)) \cap M|$ is maximum. In case $Z = \varnothing$, then ignore the vertex z^* in the sequence.

This completes the description of Alice's strategy. In the following, we shall show that by using this strategy, each unmarked vertex has at most

$\Delta + 2$ marked neighbors in G^2 (or equivalently, each unmarked vertex has at most $\Delta + 2$ marked vertices at distance one or two in G).

For each vertex x marked by Bob, there is a path P_x defined as above. If $(w, f(w))$ is an edge in P_x for some P_x, then we say that w *made a contribution to* $f(w)$ and $f(w)$ *received a contribution from* w. Let x' be the last vertex of P_x. We also say that w made a contribution to $f(w)$ if one of the following holds:

1. If $w = x'$ and Alice marked $f(x')$.
2. If $w = x'$ or $w = f(x')$ and Alice marked $f^2(x')$.

Lemma 6.4 *Assume Alice has just finished a move and y has two active sons. Then $f^2(y)$ is marked.*

Proof. When the first son of y is activated, then y and all its ancestors are activated. When the second son of y is activated, then the corresponding path P_x ends at y, and by the strategy, Alice marks $f^2(y)$, provided that $f^2(y)$ was not marked earlier. ∎

Lemma 6.5 *Assume Alice has just finished a move, and one of $y, f(y)$ is an unmarked vertex. Then the following holds:*

(1) y has at most 3 active sons.

(2) $S(y) \cup S^2(y)$ contains at most 6 active vertices. Moreover, if $S(y) \cup S^2(y)$ does contain 6 active vertices, then y has 3 active sons, each of which has one active son.

Proof. Assume y or $f(y)$ is unmarked. According to the strategy, if in a move of Alice, a vertex in $S(y) \cup S^2(y)$ is activated, then the corresponding path P_x either goes through y, or ends at y or ends at a vertex $z \in S(y)$. As $y, f(y)$ are not both marked, whenever a vertex in $S(y) \cup S^2(y)$ is activated, y receives a contribution. When y receives the first contribution, $y, f(y), f^2(y)$ are all activated. When y receives the second contribution, if $f(y)$ was not marked earlier, one of $f(y), f^2(y)$ is marked. When y receives the third contribution, one of $y, f(y)$ is marked. When it receives the fourth contribution, y must be marked. Since y or $f(y)$ is unmarked, y received at most three contributions. During each of the three corresponding moves of Alice, at most one vertex of $S(y)$ and at most one vertex of $S^2(y)$ are activated. So $S(y)$ contains at most three active vertices and $S^2(y)$ contains at most three active vertices. In case $S(y) \cup S^2(y)$ does contain 6 active vertices, then y has three active sons, each of which has one active son. ∎

Lemma 6.6 *Assume Alice has just finished a move, and one of $y, f(y)$ is an unmarked vertex. Then y has at most one unmarked son x such that $S(x) \cup S^2(x)$ contains more than 2 active vertices.*

Proof. Assume to the contrary that y and $f(y)$ are not both marked and y has two unmarked sons x_1, x_2 such that for each $j = 1, 2$, $S(x_j) \cup S^2(x_j)$ contains more than 2 active vertices. For $j = 1, 2$, if a vertex in $S(x_j) \cup S^2(x_j)$ is activated, the corresponding path P_x ends at x_j or a vertex $z \in S(x_j)$. Hence x_j receives a contribution. Since x_j is unmarked, x_j passes the contribution to y. As $S(x_j) \cup S^2(x_j)$ contains more than 2 active vertices, there are at least two steps in which some vertex in $S(x_j) \cup S^2(x_j)$ is activated. Hence y received at least 4 contributions. As remarked in the proof of Lemma 6.5, if y received 4 contributions, then both $y, f(y)$ are marked. ∎

Lemma 6.7 *Assume Alice has just finished a move. Then the following holds:*

- *y has at most two unmarked sons x for which $S(x) \cup S^2(x)$ contains more than 2 active vertices.*
- *If y has 3 active sons, then y has at most one unmarked son x for which $S(x) \cup S^2(x)$ contains more than 2 active vertices. If y has 4 or more active sons, then for each unmarked $x \in S(y)$, $S(x) \cup S^2(x)$ contains at most two active vertices and contains at most one marked vertex.*

Proof. By Lemma 6.6, before y and $f(y)$ are both marked, y has at most one unmarked son x such that $S(x) \cup S^2(x)$ contains more than 2 active vertices. Therefore at the moment the last of the two vertices y and $f(y)$ is marked, y has at most two unmarked sons x for which $S(x) \cup S^2(x)$ has more than 2 active vertices. Moreover, if y does have two unmarked sons x for which $S(x) \cup S^2(x)$ contains more than 2 active vertices, then y has only two active unmarked sons.

Assume that at the moment that the last of the two vertices y and $f(y)$ is marked, y has two unmarked sons, say x_1 and x_2, such that $S(x_i) \cup S^2(x_i)$ contains more than 2 active vertices ($i = 1, 2$). By Lemma 6.4, $f^2(y)$ is marked.

Suppose the third son x_3 of y is activated. Since $f^2(y), f(y), y$ are all marked, by the strategy, one of x_1 and x_2, say x_1, will be marked. At the time x_3 is activated, $S(x_3) \cup S^2(x_3)$ contains at most two active vertices and at most one marked vertex. If one more vertex of $S(x_3) \cup S^2(x_3)$ is activated or marked, then Alice should have marked x_3. When the fourth son x_4 of y is activated, Alice should have marked x_2. Once both x_1 and x_2 are marked, then for any son x of y, if $S(x) \cup S^2(x)$ contains more

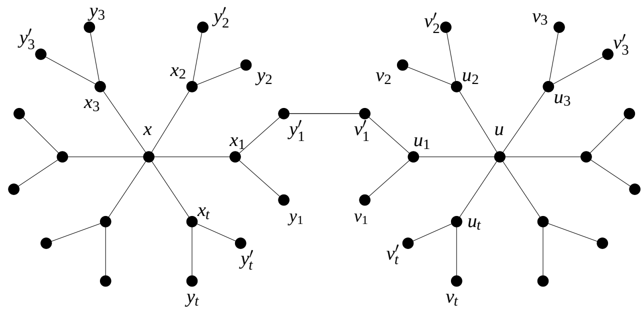

Figure 6.2: A tree T with $\text{col}_g(T^2) = \Delta + 3$.

than 2 active vertices or contains more than one marked vertex, Alice should have marked x. ∎

Lemma 6.8 *Assume $\Delta(G) \geq 9$. If Alice has just finished a move and x is an unmarked vertex, then there are at most $\Delta + 1$ marked vertices at distance at most 2 (in G) from x.*

Proof. By Lemma 6.5, $S(x) \cup S^2(x)$ contains at most 6 active vertices, and so at most 6 marked vertices since after any of Alice's moves all the marked vertices are active. The other marked vertices at distance at most 2 from x are $f(x)$ and the neighbors of $f(x)$. By Lemma 6.7, if $S(x) \cup S^2(x)$ contains at least 2 two marked vertices then $f(x)$ has at most 3 active sons (including x), hence the set $N[f(x)] - \{x\}$ contains at most 4 marked vertices : $f(x)$, $f^2(x)$, and two sons of $f(x)$. So in this case there are at most $4 + 6 = 10 \leq \Delta + 1$ marked vertices at distance at most 2 from x. If $S(x) \cup S^2(x)$ contains at most one marked vertex, then again there are at most $\Delta + 1$ marked vertices at distance at most 2 from x. ∎

After Bob's move, an unmarked vertex x has at most $\Delta + 2$ active vertices that are of distance at most 2 from x. This proves that the game coloring number of the square of a forest F is at most $\Delta + 3$.

The bound $\text{col}_g(G) \leq \Delta + 3$ is tight for trees. To see this, consider the graph depicted in Figure 6.2. By symmetry, we can assume that Alice does not mark x or x_i during her first move. Let $X = \{x_i, 1 \leq i \leq t\}$, $Y_i = \{y_i, y_i'\}$, and $Y = \bigcup_{1 \leq i \leq t} Y_i$. We say that Y_i has been marked if any of y_i and y_i' has been marked. Bob's strategy is the following : if there is an unmarked vertex x_i, such that Y_i is not marked, Bob marks y_i. Otherwise he just marks any u_j, v_j, or v_j'.

We now prove that if Bob follows this strategy, some unmarked vertex will be adjacent to at least $\Delta + 2$ marked vertices in T^2 at some point of the game.

After Bob's first move, the number of marked Y_i's is one more than the number of marked x_i's. If Alice marks an x_i whenever Bob marks Y_i, then eventually x will have too many marked neighbors in T^2. So before all the x_i's are marked, Alice needs to mark x at a certain move. Then before all the x_i's are marked, if Bob has just finished a move, the number of marked Y_i's is at least two more than the number of marked x_i's.

Let x_i and x_j be the last vertices of X to be marked. Before x_i, x_j are marked, Bob has already marked y_i and y_j. Without loss of generality, assume that Alice chooses to mark x_i first, then Bob marks y'_j and after his move, x_j is unmarked and has at least $\Delta + 2$ neighbors in T^2.

6.3 Outerplanar graphs

A graph G is an outerplanar graph if G can be embedded in the plane in such a way that all the vertices of G lie on the boundary of the infinite face. This section gives an upper bound for $\chi_g(G^2)$ for outerplanar graphs.

Theorem 6.9 *Let G be an outerplanar graph with maximum degree Δ, then $\chi_g(G^2) \leq col_g(G^2) \leq 2\Delta + 16$.*

Let $G = (V, E)$ be an outerplanar graph with maximum degree Δ, and let $H = (V, E')$ be a maximal outerplanar graph containing G. Since H is a 2-tree, there exists an orientation $\vec{H}$ of H such that:

- every vertex of $\vec{H}$ has out-degree at most two;
- the two out-neighbors of any vertex, if they exist, are adjacent.

If a vertex x of H has two out-neighbors y, z, and $\overrightarrow{yz}$ is an arc of H, then we say that z is the *major parent* of x, x is a *major son* of z, y is the *minor parent* of x, and x is a *minor son* of z. If x has only one out-neighbor z, then z is the major parent of x and x is a major son of z. For a vertex x, we denote by $f(x)$ (resp. $l(x)$) its major (resp. minor) parent, if it exists. We also define $S(x)$ as the set of in-neighbors of x and $S^2(x)$ as the set of in-neighbors of the vertices of $S(x)$.

Observation 6.10 *For every vertex $x \in \vec{H}$, at most two in-neighbors of x are minor sons of x. The minor sons of x, if any, are major sons of $f(x)$ or $l(x)$.*

This observation is an easy consequence of the definition of $\vec{H}$ (see Figure 6.3, where only x_1 and x_t may be minor sons of x).

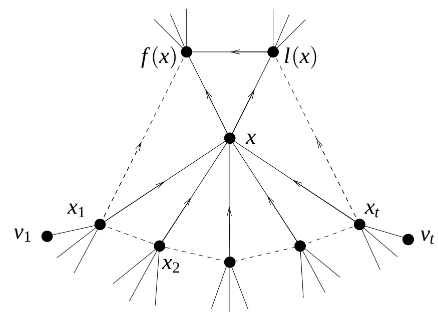

Figure 6.3: The neighborhood of a vertex x in $\vec{H}$. The dashed arcs may not be here in the graph.

Let $\overrightarrow{T}$ be the directed tree defined by the arcs $\{\overrightarrow{xf(x)}, x \in \vec{H}\}$. As in the previous section, Alice's strategy is a variation of the activation strategy and she will keep track of a set V_a of active vertices.

Alice's strategy

- At her first move, Alice will mark the root r of $\vec{T}$, and set $V_a = \{r\}$.
- Assume Bob just marked a vertex x. Let P_x be the path constructed as follows: At the beginning $P_x = \{x\}$. Let z be the last vertex of P_x. If z is inactive, then add $f(z)$ to P_x. Otherwise if $l(z)$ is inactive, add $l(z)$ to P_x. Eventually the procedure will stop and the last vertex y of P_x, as well as its parents, are all active (note that if z is active then $f(z)$ must be active). Alice adds all the vertices of P_x to V_a and marks the first unmarked vertex from the sequence $f(y), l(y), y, v$, where v is an unmarked vertex with no unmarked ancestors.

Lemma 6.11 *Let x be an unmarked vertex after a move of Alice, then x has at most $2\Delta + 14$ active vertices at distance one or two in G.*

Proof. Assume x is an unmarked vertex. We denote by $x_1, \ldots, x_t$ the sons of x (see Figure 6.3). Notice that by Observation 6.10 only x_1 and x_t may be minor sons of x. Let v_1 be the minor son of x_1 that is possibly a major son of $f(x)$, and v_t be the minor son of x_t that is possibly a major son of $l(x)$.

Assume that $f(x)$ and $l(x)$ both exist. Once they are both marked and x is activated, only two vertices of $S(x)$ (the two minor sons x_1 and x_t of x) and four vertices of $S(x) \cup S^2(x) - \{v_1, v_t\}$ can be activated. If some major son of x was activated, then Alice should have marked x. If a son of x_1 distinct from v_1 was activated, then x_1 would have been activated (x_2 could not be activated, since otherwise x would have been marked). If a second son of x_1 distinct from v_1 was activated, then x would have been marked by Alice's strategy. The same holds for x_t.

The first time a vertex y_1 of $S(x) \cup S^2(x) - \{v_1, v_t\}$ is activated, Alice activates x and $f(x)$. The second time, $l(x)$ is activated. The third and fourth times, $f(x)$ and $l(x)$ are marked. If x_1 or x_t are activated during these moves, the only change is the order of activation and marking of x, $f(x)$, and $l(x)$. In any case, at the time the last vertex of $f(x), l(x)$ is marked and x is activated (whichever is later), there are at most four moves in which some vertices in $S(x) \cup S^2(x) - \{v_1, v_t\}$ are activated. During these four moves, at most eight vertices of $S(x) \cup S^2(x)$ are activated.

Combining the two previous remarks, $S(x) \cup S^2(x)$ contains at most 14 active vertices: 8 vertices in $S(x) \cup S^2(x)$ activated before the moment that $f(x), l(x)$ are marked and x is activated, four vertices in $S(x) \cup S^2(x) - \{v_1, v_t\}$ activated after (including x_1 and x_t), and finally v_1 and v_t. If $l(x)$ does not exist, the same computation shows that $S(x) \cup S^2(x)$ contains at most 8 active vertices. If they are neighbors of x in G, the parents of x have at most $2\Delta - 2$ neighbors in G distinct from x. Hence, x has at most $2\Delta + 14$ active vertices at distance one or two in G. ∎

After Bob's move, an unmarked vertex has at most $2\Delta + 15$ active vertices at distance one or two in G. This proves that the game coloring number of the square of an outerplanar graph with maximum degree Δ is at most $2\Delta + 16$.

Observe that in the description and analyse of the strategy, we always use the graph H, which is a triangulated outerplanar graph obtained from G by adding some edges. But the degree of a vertex x refers to its degree in G, and Δ is the maximum degree of G.

6.4 Partial 2-trees and planar graphs

The two following lemmas are particular cases of an implicit lemma in the proof of Theorem 4 in [Zhu00] :

Lemma 6.12 [Zhu00] *In any partial 2-tree, Alice has a strategy such that at the end of each of her moves, any unmarked vertex has at most 6 marked neighbors.*

Lemma 6.13 [Zhu00] *In any planar graph, Alice has a strategy such that at the end of each of her moves, any unmarked vertex has at most 17 marked neighbors.*

We use these two results, combined with the same idea as in Observation 6.2 to obtain the following corollary.

Corollary 6.14 *Let G be a partial 2-tree with maximum degree $\Delta \geq 6$, then $col_g(G^2) \leq 12\Delta - 34$.*

Corollary 6.15 *Let G be a planar graph with maximum degree $\Delta \geq 17$, then $col_g(G^2) \leq 34\Delta - 287$.*

Proof. Let v be an unmarked vertex just after Alice's move, and let $N_M(v)$ (resp. $N_U(v)$) be the set of marked (resp. unmarked) neighbors of v. If every unmarked vertex is adjacent to at most $l \leq \Delta$ marked vertices at this moment, then using a similar counting as in Observation 6.2, v has at most $|N_M(v)|\Delta + |N_U(v)|l \leq |N_M(v)|(\Delta - l) + l\Delta \leq 2l\Delta - l^2$ marked vertices at distance one or two. Hence, after any of Bob's moves, no unmarked vertex has more than $2l\Delta - l^2 + 1$ marked vertices at distance one or two. These two facts prove that in this case, the game coloring number is bounded by $2l\Delta - l^2 + 2$. ∎

6.5 Conclusion

Using Observation 6.1, Theorem 6.9, as well as Corollaries 6.14 and 6.15 have immediate consequences on the acyclic game chromatic number of outerplanar graphs, partial 2-trees, and planar graphs.

However, we conjecture that in the case of acyclic games, less colors are necessary:

Conjecture 6.16 *For some constant C_1, any planar graph G with maximum degree Δ satisfies $\chi_{a,g}(G) \leq \frac{\Delta}{2} + C_1$.*

Based on what is known on the chromatic number of the square of partial 2-trees and planar graphs (see Chapters 2 and 3), we also conjecture the following:

Conjecture 6.17 *For some constant C_2, any outerplanar graph G with maximum degree Δ satisfies $col_g(G^2) \leq \Delta + C_2$.*

Conjecture 6.18 *For some constant C_3, any planar graph G with maximum degree Δ satisfies $col_g(G^2) \leq \frac{3}{2}\Delta + C_3$.*

Chapter 7

Boxicity

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In this final chapter, we define a specific coloring at distance two and use it to bound the boxicity of graphs with maximum degree Δ.

The boxicity of a graph $G = (V, E)$ is the smallest k for which there exist k interval graphs $G_i = (V, E_i)$, $1 \leq i \leq k$, such that $E = E_1 \cap \ldots \cap E_k$. Graphs with boxicity at most d are exactly the intersection graphs of (axis-parallel) boxes in $\mathbb{R}^d$. We prove that graphs with maximum degree Δ have boxicity at most $\Delta^2 + 2$, which improves the previous bound of $2\Delta^2$ obtained by Chandran *et al.* (J. Combin. Theory Ser. B **98** (2008) 443–445).

7.1 Introduction

For a family $\mathcal{F} = \{S_1, \ldots, S_n\}$ of subsets of a set Ω, the *intersection graph* of $\mathcal{F}$ is defined as the graph with vertex set $\mathcal{F}$, in which two sets are adjacent if and only if their intersection is non-empty. A *d-box* is the Cartesian product $[x_1, y_1] \times \ldots \times [x_d, y_d]$ of d closed intervals of the real line. For any graph G, the *boxicity* of G, denoted by $\mathrm{box}(G)$, is the smallest d such that G is the intersection graph of a family of d-boxes. $\mathrm{box}(G)$

For a family of graphs $\{G_i = (V, E_i),\ 1 \leq i \leq k\}$ defined on the same vertex set, we set $G_1 \cap \ldots \cap G_k$ to be the graph with vertex set V, and edge set $E_1 \cap \ldots \cap E_k$), and we naturally say that the graph $G_1 \cap \ldots \cap G_k$ is the intersection of the graphs $G_1, \ldots, G_k$. The boxicity of a graph G can be equivalently defined as the smallest k such that G is the intersection of k interval graphs. Graphs with boxicity one are exactly interval graphs, which can be recognized in linear time. On the other hand, Kratochvíl [Kra94] proved that determining whether $\mathrm{box}(G) \leq 2$ is NP-complete.

The concept of boxicity was introduced in 1969 by Roberts [Rob69]. It is used as a measure of the complexity of ecological [Rob76] and social [Fre83] networks, and has applications in fleet maintenance [OR81]. Boxicity has been investigated for various classes of graphs [CR83, Sch84, Tho86], and has been related with other parameters, such as treewidth [CS07]. Recently, Chandran *et al.* [CFS08] proved that every graph with maximum degree at most Δ has boxicity at most $2\Delta^2$. To prove this bound, Chandran *et al.* use the fact that if a graph G is the intersection of k graphs $G_1, \ldots, G_k$, we have $\mathrm{box}(G) \leq \sum_{1 \leq i \leq k} \mathrm{box}(G_i)$.

In the remaining of the chapter, we use the same idea to prove the following theorem:

Theorem 7.1 [Esp08] *Every graph with maximum degree Δ has boxicity at most $2\left\lfloor \Delta^2/2 \right\rfloor + 2$.*

7.2 Proof of Theorem 7.1

Let $G = (V, E)$ be a graph with maximum degree Δ, and let c be a (not necessarily proper) coloring of the vertices of G with colors from $\{1, \ldots, 2k\}$ such that:

(i) there is no path uvw with $c(u) = c(w)$;

(ii) for any $1 \leq j \leq k$, there is no edge between a vertex colored with $2j - 1$ and a vertex colored with $2j$.

Observe that condition (i) implies that the graph induced by each color class is a graph with maximum degree at most one (the disjoint union of a stable set and a matching). The first step of the proof is to find the smallest k such that a $2k$-coloring as defined above exists. Define the function f such that for every $j \geq 1$, $f(2j) = 2j-1$ and $f(2j-1) = 2j$. We color the vertices of G one by one with the following procedure: while coloring a vertex $u \in V$, we choose for u a color from $\{1, \ldots, 2k\} \backslash (N_1 \cup N_2)$, where $N_1 = \{f(c(v)) \mid v \text{ is a colored neighbor of } u\}$ and $N_2 = \{c(v) \mid u \text{ and } v \text{ have a common (not necessarily colored) neighbor}\}$.

If we follow this procedure, the partial coloring obtained at the end of each step has the desired properties : since $c(u) \notin N_1$, condition (ii) is still verified, and since $c(u) \notin N_2$, condition (i) is also still verified. At each step, N_1 has size at most Δ and N_2 has size at most $\Delta(\Delta - 1)$. Hence if $k = \left\lceil \frac{\Delta^2+1}{2} \right\rceil = \left\lfloor \frac{\Delta^2}{2} \right\rfloor + 1$, a $2k$-coloring of G as defined above exists.

From now on, we assume that $k = \lfloor \Delta^2/2 \rfloor + 1$. Hence, a $2k$-coloring c of G with the properties defined above exists. For any $1 \leq i \leq k$, let G_i be the graph obtained from G by adding an edge between any two non-adjacent vertices u, v such that $c(u), c(v) \notin \{2i-1, 2i\}$. Using conditions (i) and (ii), G_i can be decomposed into a clique K_i (induced by the vertices colored neither with $2i-1$, nor with $2i$), and two sets S_{2i-1} and S_{2i} corresponding to the vertices colored with $2i-1$ and $2i$ respectively (see Figure 7.2(a)). By condition (ii), there is no edge between S_{2i-1} and S_{2i}, and by condition (i), every vertex of K_i is adjacent to at most one vertex of S_{2i-1} and one vertex of S_{2i}. Moreover, S_{2i-1} and S_{2i} both induce a graph with maximum degree one by condition (i).

Now observe that $G = \cap_{1 \leq i \leq k} G_i$. If two vertices are adjacent in G they are also adjacent in any G_i, since $G \subseteq G_i$. On the other hand, if two vertices u and v are not adjacent in G, then they are not adjacent in $G_{\lceil c(u)/2 \rceil}$, and so they are not adjacent in the intersection of the G_i's.

As a consequence, $\text{box}(G) \leq \sum_{1 \leq i \leq k} \text{box}(G_i)$. We now show that every graph G_i has boxity at most two, which implies that $\text{box}(G) \leq 2(\lfloor \Delta^2/2 \rfloor + 1)$ and concludes the proof.

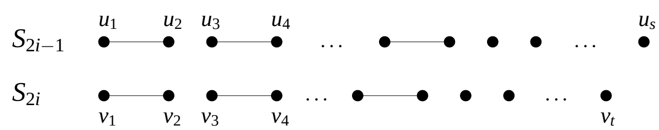


Figure 7.1: The ordering of the vertices of S_{2i-1} and S_{2i}.

For any $1 \leq i \leq k$, we represent G_i as the intersection graph of 2-dimensional boxes. We order the vertices $u_1, \ldots, u_s$ of S_{2i-1} and the vertices $v_1, \ldots, v_t$ of S_{2i} as depicted in Figure 7.1 (recall that S_{2i-1} and S_{2i} both induce a graph with maximum degree at most one). Let r be the maximum of s and t. For every j such that u_{2j-1} and u_{2j} are adjacent in S_{2i-1}, u_{2j-1} is represented by the box $\{-r+2j-1\} \times [-2j+2, -2j+1]$ and u_{2j} is represented by the box $[-r+2j-1, -r+2j] \times \{-2j+1\}$. If a vertex u_j is isolated in S_{2i-1}, it is represented by the point $(-r+j, -j+1)$.

Similarly, for every j such that v_{2j-1} and v_{2j} are adjacent in S_{2i}, v_{2j-1} is represented by the box $[2j-2, 2j-1] \times \{r-2j+1\}$ and v_{2j} is represented by the box $\{2j-1\} \times [r-2j, r-2j+1]$. If a vertex v_j is isolated in S_{2i}, it is represented by the point $(j-1, r-j)$ (see Figure 7.2(b) for an example).

Observe that :

(1) the boxes of two adjacent vertices u_{2j-1} and u_{2j} intersect in $(-r+2j-1, -2j+1)$;

(2) the boxes of two adjacent vertices v_{2j-1} and v_{2j} intersect in $(2j-1, r-2j+1)$;

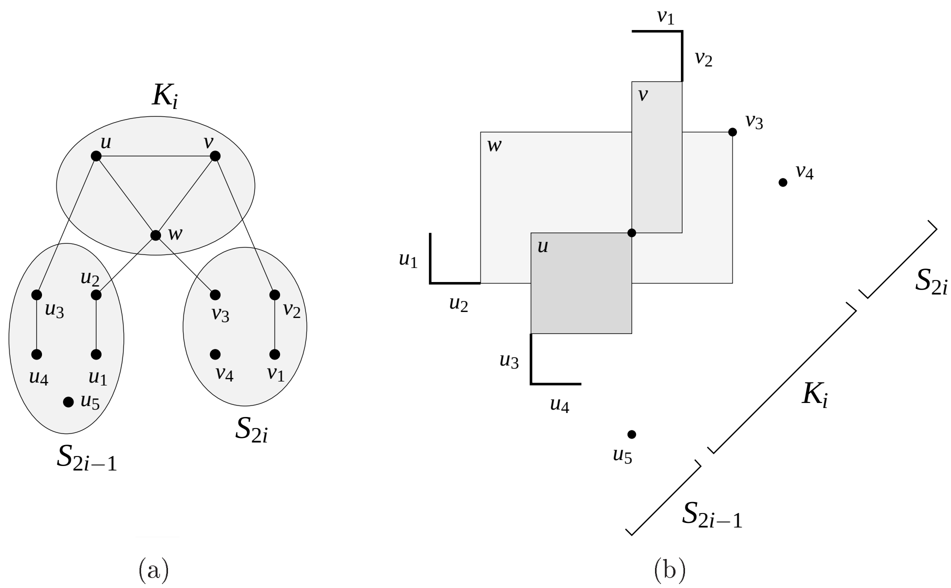


Figure 7.2: (a) A graph G_i and (b) a representation of G_i as the intersection graph of 2-dimensional boxes.

(3) the boxes of all the other pairs of vertices colored with $2i-1$ or $2i$ are not intersecting.

(4) the top-right corner of the box of u_j is the point $(-r+j, -j+1)$ and the bottom-left corner of the box of v_j is the point $(j-1, r-j)$

We now have to represent the vertices from K_i. We represent the vertices having no neighbor outside K_i by the point $(0,0)$. If a vertex u from K_i has only one neighbor outside K_i, say $u_j \in S_{2i-1}$, we represent u by the box $[-r+j, 0] \times [-j+1, 0]$. If a vertex v from K_i has only one neighbor outside K_i, say $v_j \in S_{2i}$, we represent v by the box $[0, j-1] \times [0, r-j]$. If a vertex w of K_i has one neighbor $u_j \in S_{2i-1}$ and one neighbor $v_\ell \in S_{2i}$, we represent w by the box $[-r+j, \ell-1] \times [-j+1, r-\ell]$ (see Figure 7.2(b) for an example).

The boxes representing the vertices from K_i are pairwise intersecting, since they all contain the point $(0,0)$. Moreover, using Observation (4) above, the box of every vertex v from K_i only intersects the boxes of the neighbors of v. Hence, G_i is the intersection graph corresponding to this representation, and so G_i has boxicity two, which concludes the proof.

7.3 Conclusion

The best known lower bound for the boxicity of graphs with maximum degree Δ was given by Roberts [Rob69]. Consider the graph H_{2n} obtained by removing a perfect matching from a clique of $2n$ vertices. If this graph has boxicity $k \leq n-1$, let $G_1, \ldots, G_k$ be interval graphs such that

$H_{2n} = G_1 \cap \ldots \cap G_k$. Since $k \leq n-1$ and H_{2n} have n non-edges, two non-edges of H_{2n} have to lie in the same interval graph, say G_i. This is impossible since otherwise G_i contains an induced cycle of length four and is not an interval graph. Hence, $\text{box}(H_{2n}) \geq n \geq \left\lceil \frac{1}{2}\Delta(H_{2n}) \right\rceil$.

Cozzens and Roberts [CR83] gave another construction of a graph with maximum degree Δ and boxicity at least $\lceil \Delta/2 \rceil$ based on a complete bipartite graph, but the proof is slightly more difficult.

Chandran *et al.* [CFS08] conjectured that for any graph G, $\text{box}(G) \leq O(\Delta)$. It is interesting to remark that this conjecture is true when the graphs $G_1, \ldots, G_k$ with $G = \cap_{1 \leq i \leq k} G_i$ are only required to be chordal. McKee and Scheinerman [MS93] defined the *chordality* of a graph G, denoted by $\text{chord}(G)$, as the smallest k such that G is the intersection of k chordal graphs. Since a graph is an interval graph if and only if it is chordal and its complement is a comparability graph, we clearly have $\text{chord}(G) \leq \text{box}(G)$ for any graph G. McKee and Scheinerman proved that the chordality of a graph is bounded by its chromatic number. As a corollary, it is easy to show that for any graph G with maximum degree Δ, $\text{chord}(G) \leq \Delta$. $\text{chord}(G)$

We conclude with general remarks. We denote by $a(G)$ the *arboricity* of G, that is the minimum number of induced forests into which the edges of G can be partitioned. For outerplanar graphs, planar graphs, graphs with bounded treewidth, and graphs with bounded degree, the boxicity seems to be bounded by the arboricity. Unfortunately it seems to be false in general: there exists trees with boxicity at least two, and graphs with arboricity two and boxicity at least three. This leads to two natural questions: $a(G)$

1. Is there a constant $\kappa \geq 1$, such that any graph G satisfies $\text{box}(G) \leq a(G) + \kappa$?
2. Is there a constant $\lambda > 1$, such that any graph G satisfies $\text{box}(G) \leq \lambda a(G)$?

A positive answer to the second question (and thus to the first), would imply that for any graph G with maximum degree Δ, $\text{box}(G) \leq \lambda \left\lceil \frac{\Delta+1}{2} \right\rceil$, proving the conjecture of Chandran *et al.* [CFS08].

Conclusion

In Chapter 2, we proved that the vertices of any planar graph can be colored with $(\frac{3}{2}+o(1))\beta$ colors, in such way that any two vertices that are adjacent or have a common neighbor of degree at most β, have distinct colors. It might be interesting to investigate a similar problem on surfaces of bounded genus:

Question 1 *Is there a function f such that the vertices of any graph embeddable on a surface of genus g can be colored with $f(g)\beta$ colors, in such way that any two vertices that are adjacent or have a common neighbor of degree at most β have distinct colors?*

A consequence of the main result of Chapter 2 is that Wegner's conjecture [Weg77] that the square of any planar graph of maximum degree $\Delta \geq 8$ can be properly colored with $\left\lfloor \frac{3}{2}\Delta(G) \right\rfloor + 1$ colors is asymptotically true. In Chapter 3, we investigated a generalization of this problem: recall that a p-frugal coloring of a graph G is a proper coloring of the vertices of G such that every color appears at most p times in the neighborhood of every vertex. We generalized Wegner's conjecture in the following way:

Conjecture 2 [AEH07] *For any integer $p \geq 1$ and planar graph G with maximum degree $\Delta \geq \max\{2p, 8\}$ we have*

$$\chi_p(G) \leq \begin{cases} \left\lfloor \frac{\Delta-1}{p} \right\rfloor + 2, & \text{if } p \text{ is even;} \\ \left\lfloor \frac{3\Delta-2}{3p-1} \right\rfloor + 2, & \text{if } p \text{ is odd.} \end{cases}$$

Using connections between frugal coloring and $L(p,q)$-labelling, we then proved that for fixed p, any planar graph G with maximum degree Δ satisfies $\chi_p(G) \leq \frac{3\Delta}{2p} + o(\Delta)$.

In [KW01], Kostochka and Woodall conjectured that for any graph G, the chromatic number and the list chromatic number of G^2 are the same. We generalize this conjecture in the following way:

Conjecture 3 [AEH07] *For any multigraph G and any integer $p \geq 1$, we have $\chi_p(G) = ch_p(G)$.*

The *List Coloring Conjecture* states that for any multigraph G the chromatic index and the list chromatic index of G are the same. Again, this can be seen as a special case of the following conjecture:

Conjecture 4 [AEH07] *For any multigraph G and any integer $p \geq 1$, we have $\chi'_p(G) = ch'_p(G)$.*

When $p = 2$, a p-frugal coloring of the vertices of a graph G corresponds to a coloring in which the (bipartite graph) induced by every two color classes has maximum degree two. In Chapter 4, we remarked that in this case, it does not cost too much to also require that the coloring be acyclic. Define a linear coloring as an acyclic 2-frugal coloring, then the union of any two color classes is a forest of paths.

In Chapter 4, we gave bounds on the linear chromatic number of various classes of graphs, such as graphs with small maximum degree, graph with small maximum average degree, outerplanar graphs, and planar graphs. We give here two nice conjectures about graphs with maximum degree at most three and planar graphs:

Conjecture 5 [EMR08] *If G has maximum degree three, and is different from $K_{3,3}$, then $\Lambda^l(G) \leq 4$.*

Conjecture 6 [RW06] *For some constant C, every planar graph G with maximum degree Δ satisfies $\Lambda^l(G) \leq \frac{\Delta}{2} + C$.*

In Chapter 5, we studied the $(p, 1)$-total number of graphs with bounded maximum degree. Our aim was to prove a weaker version of the following conjecture of Havet and Yu [HY08].

Conjecture 7 [HY08] *Let G be a graph with maximum degree Δ, then $\lambda_p^T(G) \leq \Delta + 2p$.*

Observe that any $(2, 1)$-total labelling of K_4 requires 7 colors. However, Havet and Yu conjectured the following:

Conjecture 8 [HY08] *Let G be a graph with maximum degree at most three, with $G \neq K_4$, then $\lambda_2^T(G) \leq 6$.*

In Chapter 6, we considered a two-player game in which Alice and Bob are properly coloring the square of a graph. If the coloring is completed, Alice wins, and otherwise Bob wins. We investigated winning strategies for Alice in forests, outerplanar graphs, partial 2-trees and planar graphs, and our results had direct consequences on the acyclic game chromatic number of these graphs. However most of our bounds are conjectured to be far from tight:

Conjecture 9 *There exist a constant C_1, such that if G is a planar graph with maximum degree Δ, then $\chi_{a,g}(G) \leq \frac{\Delta}{2} + C_1$.*

Conjecture 10 [EZ08] *For some constant C_2, any outerplanar graph G with maximum degree Δ satisfies $col_g(G^2) \leq \Delta + C_2$.*

Conjecture 11 [EZ08] *For some constant C_3, any planar graph G with maximum degree Δ satisfies $col_g(G^2) \leq \frac{3}{2}\Delta + C_3$.*

In Chapter 7, we investigated the boxicity of graphs with bounded maximum degree. We proved that any graph with maximum degree Δ could be seen as the intersection of Δ^2+2 interval graphs. The concept of boxicity seems to be related with the arboricity of graphs, so we asked the following:

Question 12

1. *Is there a constant $\kappa \geq 1$, such that any graph G satisfies box$(G) \leq a(G)+\kappa$?*
2. *Is there a constant $\lambda > 1$, such that any graph G satisfies box$(G) \leq \lambda a(G)$?*

*****

We conclude with a couple of questions and conjectures about distance-two colorings in general.

$L(p,q)$-labellings of oriented graphs have been investigated for graphs with maximum degree, trees, and Halin graphs [CL03, CW06, GRS06], but interesting questions remain. Define the *2-dipath chromatic number* $\vec{\chi}_2(\vec{G})$ of an oriented graph $\vec{G}$ as the minimum number of colors in a coloring of the vertices of $\vec{G}$, such that any two vertices joined by a directed path of length (number of arcs) at most two have distinct colors.

We saw in Chapter 2 that a coloring of the square of a non-oriented planar graph of maximum degree Δ might require at least $\frac{3}{2}\Delta$ colors. Surprisingly, a coloring of the square of an oriented planar graph only requires a constant number of colors. To see this, observe that for any oriented graph $\vec{G}$, $\vec{\chi}_2(\vec{G})$ is at most the oriented chromatic number of $\vec{G}$ (see Appendix A for more details about oriented coloring). Since the oriented chromatic number of planar graphs is at most 80, we obtain that for any oriented planar graph $\vec{G}$, $\vec{\chi}_2(\vec{G}) \leq 80$. On the other hand, there exists an oriented planar graph with 15 vertices, in which any two vertices are joined by a a directed path of length one or two. Hence, there exists an oriented planar graph $\vec{G}$, with $\vec{\chi}_2(\vec{G}) = 15$. Note that Klostermeyer and MacGillivray [KM04] proved that the order of an oriented planar graph in which all the vertices are joined by a a directed path of length one or two is at most 36.

The problem of improving the bound of 80 for oriented coloring of planar graphs is supposed to be quite difficult, but improving this bound for the 2-dipath chromatic number might be slightly easier. We propose the following optimistic conjecture:

Conjecture 13 *For any oriented planar graph $\vec{G}$, we have $\vec{\chi}_2(\vec{G}) \leq 15$.*

In this thesis, we mainly studied distance-two colorings of the vertices of graphs. The problem of coloring the edges of graphs with a condition at distance two is also very interesting. Erdös and Nešetřil (see [FGS$^+$89]) defined a *strong edge-coloring* of a graph G as a (proper) coloring of the edges of G in which every color class is an induced matching. This coloring can be seen as a proper vertex-coloring of the square of the line graph of G. If G has maximum degree Δ then $L(G)^2$ has maximum degree at most $2\Delta^2 - 2\Delta$, so it is easy to prove that any graph with maximum degree Δ has a strong edge-coloring using at most $2\Delta^2 - 2\Delta + 1$ colors. Erdös and Nešetřil conjectured the following:

Conjecture 14 *Every graph with maximum degree Δ has a strong edge-coloring with $\left\lfloor \frac{5}{4}\Delta^2 \right\rfloor$ colors.*

They also provided examples showing that this bound would be best possible. The closest result so far was given by Molloy and Reed [MR97], who proved that for some constant $\varepsilon > 0$, every graph with maximum degree Δ has a strong edge-coloring using at most $\lfloor (2-\varepsilon)\Delta^2 \rfloor$ colors.

An *incidence* in a graph G is a pair $(v, e) \in V(G) \times E(G)$ such that v and e are incident (it corresponds intuitively to a half-edge of G). Two incidences (u, e) and (v, f) are adjacent if one of the following holds: (i) $u = v$, (ii) $e = uv$ or (iii) $f = uv$.

An *incidence coloring* of a graph G, defined by Brualdi and Massey [BM93], is a coloring of the incidences of G such that any two adjacent incidences have distinct colors. Let $G^\star$ denote the graph obtained from G by subdividing every edge exactly once (see Figure 5.1 in Chapter 5 for an example). Then it is clear that an incidence coloring of G is exactly a strong edge-coloring of $G^\star$.

Guiduli [Gui97] proved that every graph with maximum degree Δ has an incidence coloring with $\Delta + O(\log\Delta)$ colors, which is best possible. Hosseini *et al.* [HSZ04] proved that any planar graph with maximum degree Δ has an incidence coloring with $\Delta+7$ colors. We ask the following question:

Question 15 *Is it true that any planar graph with maximum degree Δ has an incidence coloring with $\Delta + 2$ colors?*

Appendix A

[EO07a]

Oriented colorings of 2-outerplanar graphs

Abstract

A graph G is 2-outerplanar if it has a planar embedding such that the subgraph obtained by removing the vertices of the outer face is outerplanar. The oriented chromatic number of an oriented graph H is defined as the minimum order of an oriented graph H' such that H has a homomorphism to H'. In this paper, we prove that 2-outerplanar graphs are 4-degenerate. We also show that oriented 2-outerplanar graphs have a homomorphism to the Paley tournament QR_{67}, which implies that their (strong) oriented chromatic number is at most 67.

This article appeared in *Information Processing Letters*.

Oriented colorings of 2-outerplanar graphs

Louis Esperet*, and Pascal Ochem†
LaBRI UMR CNRS 5800, Université Bordeaux I,
33405 Talence Cedex
FRANCE.

March 21, 2008

Abstract

A graph G is 2-outerplanar if it has a planar embedding such that the subgraph obtained by removing the vertices of the outer face is outerplanar. The oriented chromatic number of an oriented graph H is defined as the minimum order of an oriented graph H' such that H has a homomorphism to H'. In this paper, we prove that 2-outerplanar graphs are 4-degenerate. We also show that oriented 2-outerplanar graphs have a homomorphism to the Paley tournament QR_{67}, which implies that their (strong) oriented chromatic number is at most 67.

Keywords: combinatorial problems, oriented coloring, 2-outerplanar graphs.

1 Introduction

Oriented graphs are directed graphs without loops nor opposite arcs. In other words an oriented graph is an orientation of an undirected simple graph, obtained by assigning to every edge one of the two possible orientations. If G is a graph, $V(G)$ denotes its vertex set, $E(G)$ denotes its set of edges. A homomorphism from an oriented graph G to an oriented graph H is a mapping φ from $V(G)$ to $V(H)$ which preserves the arcs, that is $(x, y) \in E(G) \Longrightarrow (\varphi(x), \varphi(y)) \in E(H)$. We say that H is a *target graph* of G if there exists a homomorphism from G to H. The oriented chromatic number $\chi_o(G)$ of an oriented graph G is defined as the minimum order of a target graph of G. The oriented chromatic number $\chi_o(G)$ of an undirected graph G is then defined as the maximum oriented chromatic number taken over all orientations of G. Nešetřil and Raspaud introduced in [5] the *strong oriented chromatic number* of an oriented graph G (denoted by $\chi_s(G)$), which definition differs from that of $\chi_o(G)$ by requiring that the target graph is an oriented Cayley graph. They show in particular that the strong oriented chromatic number of a planar graph G corresponds to the antisymmetric flow of the dual of G. Upper bounds on the (strong) oriented chromatic number have been found for various subclasses of planar graphs. In particular:

1. if G is a planar graph, then $\chi_o(G) \leq 80$ [8].
2. if G is an outerplanar graph, then $\chi_s(G) \leq 7$ [9].

* *esperet@labri.fr*
† *ochem@labri.fr*

A graph G is 2-outerplanar if it has a planar embedding such that the subgraph obtained by removing the vertices of the outer face is outerplanar. The second author proved that 2-outerplanar graphs have an acyclic partition into three independent sets and an outerplanar graph [7]. By Theorem 1 in [1], the oriented chromatic number of a 2-outerplanar graph is thus at most $2^{4-1} \times (1+1+1+7) = 80$. The same result follows from the bound of Raspaud and Sopena [8] holding for planar graphs.

In Section 2, we prove among other results that 2-outerplanar graphs G are 4-degenerate, that is, every subgraph H of G has minimum degree at most 4. In Section 3, we use these results to show that 2-outerplanar graphs have a homomorphism to QR_{67}, which improves the previous bounds of 80.

In the following, we call a k-vertex (resp. $^{\geq}k$-vertex, $^{\leq}k$-vertex) a vertex of degree k (resp. at least k, at most k). Figures are drawn with the following convention : the star symbol indicates the outer face, *white* vertices correspond to vertices which neighbors are all depicted in the figure, whereas *black* vertices may have other neighbors in the graph.

2 Structural properties of 2-outerplanar graphs

Definition 1 *A 2-outerplanar graph embedded in the plane is said to be a* block *if its outer face is an induced cycle.*

Theorem 2 *If G is a 2-outerplanar graph, then it contains a $^{\leq}4$-vertex.*

Proof. Let G be a 2-outerplanar graph embedded in the plane. We consider the subgraph H induced by the outer face of G. H is an outerplanar graph, so it contains an internal face F incident to at most one other internal face of H (see Proof of Lemma 2 in [4]). Let B be the subgraph of G induced by the vertices of F and the vertices inside F. By construction, the graph B obtained is a block. Moreover, B contains only two vertices x and x' such that the degree of x and x' in G may be higher than their degree in B. By construction, x and x' are two adjacent vertices belonging to the outer face of B (see Figure 1).

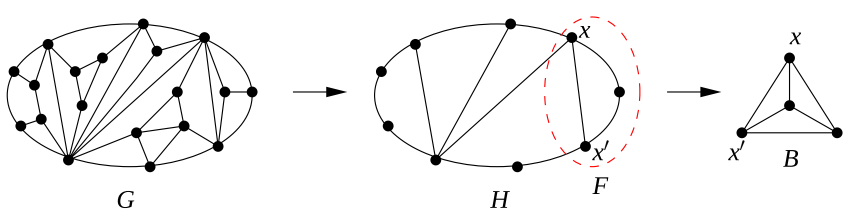


Figure 1: The decomposition of a 2-outerplanar graph into blocks.

Let B_c be the graph induced by the outer face of B, and B_o be the graph obtained from B by removing the vertices of B_c. By definition of 2-outerplanar graphs, B_o is outerplanar. So it contains two non-adjacent 2-vertices u and v (see Figure 2).

As mentioned above, vertices of B_o have the same degree in B and in G, so $d_B(u) = d_G(u)$ and $d_B(v) = d_G(v)$. Let us find a $^{\leq}4$-vertex in B. If B_o contains a $^{\leq}4$-vertex, it is done. Otherwise, it means that B_o contains only $^{\geq}5$-vertices; in particular u (resp. v) is adjacent to

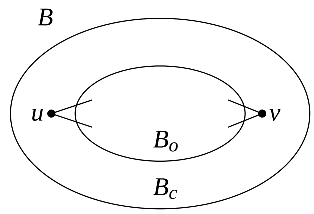


Figure 2: The decomposition of B into B_c and B_o.

three vertices u_1, u_2, u_3 (resp. v_1, v_2, v_3), where $u_1u_2u_3$ (resp. $v_1v_2v_3$) is an induced path of B_c (see Figure 3).

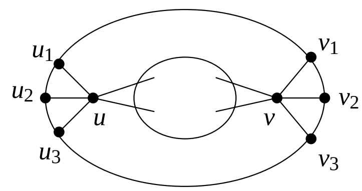


Figure 3: u and v have three neighbors in B_c.

We now use the fact that B contains only two vertices x and x' having a degree in G possibly higher than their degree in B. As xx' is an edge of B_c, this means that u_2 or v_2 have the same degree in B and in G, i.e. $d_G(u_2) = d_B(u_2) = 3$ or $d_G(v_2) = d_B(v_2) = 3$. Hence B always contains a vertex with degree at most 4 in G. □

We now prove that outerplanar graphs have properties stronger than 2-degeneration, in order to find more precise configurations in 2-outerplanar graphs.

Lemma 3 *Let G be an outerplanar graph. G contains either a 1-vertex, two adjacent 2-vertices, a 2-vertex adjacent to a 3-vertex as depicted in Figure 4.a, or two 2-vertices adjacent to a 4-vertex as depicted in Figure 4.b.*

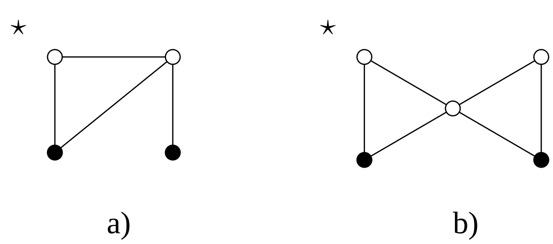


Figure 4: Unavoidable configurations in an outerplanar graph without two adjacent 2-vertices.

Proof. We prove this lemma by induction. Let G be an outerplanar graph, and let v be a 2-vertex of G (v exists, see [4] for details). The graph $H = G \setminus v$ is outerplanar, and smaller than G. By induction, H contains either two adjacent 2-vertices, or the configurations of

Figure 4. If v is not adjacent to such a configuration of H, then it is a configuration of G, and the induction is finished. Otherwise v is adjacent to a configuration, and we have to make the distinction between various cases. Notice that the neighbors of v must be adjacent in H in order to obtain an outerplanar graph.

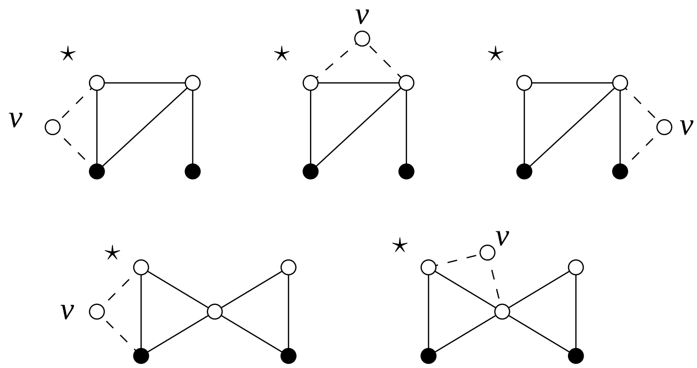


Figure 5: Induction step in the proof of Lemma 3.

- If H contains two adjacent 2-vertices, we obtain the configuration of Figure 4.a.
- If H contains a configuration of Figure 4, we obtain either the configuration of Figure 4.a, or the configuration of Figure 4.b (see Figure 5).

In any case, G contains one of the three configurations described earlier. □

We now use Lemma 3 to prove a key structural theorem on 2-outerplanar graphs admitting a block embedding in the plane. The following result can be extended to the whole class of 2-outerplanar graphs by using the same kind of proof as in Theorem 2.

Theorem 4 *Let G be a 2-outerplanar graph admitting a* block *embedding in the plane. G contains either a* $^{\leq}3$*-vertex, two adjacent 4-vertices, or the configuration depicted in Figure 6.*

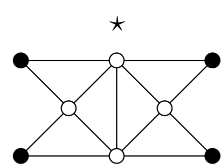


Figure 6: Unavoidable configuration in a 2-outerplanar block containing neither a $^{\leq}3$-vertex nor two adjacent 4-vertices.

Proof. We consider a block embedding of G in the plane. Then the subgraph induced by the outer face is a cycle. Let G_c be this cycle and let G_o be the graph obtained from G by removing the vertices of G_c. By definition of G and G_c, the graph G_o is outerplanar. We then know by Lemma 3 that G_o contains either two adjacent 2-vertices, a 2-vertex adjacent to a 3-vertex as depicted in Figure 4.a, or two 2-vertices having a common neighbor of degree 4 as depicted in Figure 4.b.

- If G_o contains a 1-vertex or two adjacent 2-vertices, we easily find a ${}^{\leq}3$-vertex or two adjacent 4-vertices in G.

- If G_o contains a 2-vertex v adjacent to a 3-vertex u, we can prove that either $d_G(v) = 4$ or there is a vertex of degree 3 in G (which is a neighbor of v belonging to the outer face). This is done by applying the same method as in the previous proof. Thus G must contain the configuration depicted in Figure 7. Notice that u and w are adjacent, since otherwise one of them would be a ${}^{\leq}3$-vertex. For reasons of planarity, if u is adjacent to another vertex of G_c, w cannot be adjacent to another vertex of G_o. Conversely, if w is adjacent to another vertex of G_o, u cannot be adjacent to a vertex of G_c. This proves that either u or w has degree 4 in G, say u. If there is no 3-vertex in G, we found two adjacent 4-vertices: u and v.

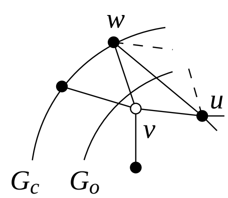


Figure 7: G_o contains a 2-vertex v adjacent to a 3-vertex u.

- If G_o contains two 2-vertices v and v' both adjacent to a 4-vertex u as depicted in Figure 4.b, we first prove that either v and v' have degree 4 in G or G contains a 3-vertex (in which case the proof is finished). Let v_1 and v_2 (resp. v'_1 and v'_2) be the neighbors of v (resp. v') belonging to the outer face. As depicted in Figure 8, we have to make a distinction between two cases : $\{v_1, v_2\}$ and $\{v'_1, v'_2\}$ are disjoint (case 1), or they have a vertex in common, say $v_2 = v'_1$ (case 2).

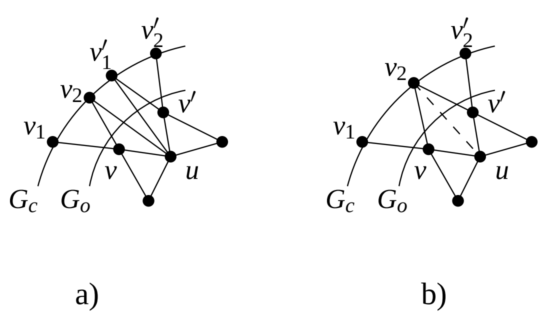


Figure 8: G_o contains two 2-vertices v and v' adjacent to a common 4-vertex u.

case 1 (see Figure 8.a) If v_2 and v_1' have degree at least 4 in G, they both have to be adjacent to u, in which case $d_G(v_2) = d_G(v_1') = 4$, and we found two adjacent 4-vertices in G.

case 2 (see Figure 8.b) If u is adjacent to $v_2 = v_1'$, we obtain exactly the configuration depicted in Figure 6. Otherwise, we simply have two adjacent 4-vertices (v and v_2).

□

3 Strong oriented coloring of 2-outerplanar graphs

Theorem 5 *If G is a 2-outerplanar graph, then $\chi_s(G) \leq 67$.*

Let q be prime power and let $\mathbb{F}_q$ denote the unique finite field with q elements. For a prime power $q \equiv 3 \pmod 4$, the vertices of the Paley tournament QR_q are the elements of $\mathbb{F}_q$ and (i,j) is an arc in QR_q if and only if $j-i$ is a non-zero quadratic residue of $\mathbb{F}_q$. Since $q \equiv 3 \pmod 4$, we have that for $i,j \in \mathbb{F}_q$, $i \neq j$, $j-i$ is a quadratic residue if and only if $i-j$ is not a quadratic residue. This means that QR_q is an oriented Cayley graph whose set of generators are the non-zero quadratic residue of $\mathbb{F}_q$. It can be proven [3] that Payley tournaments are *arc-transitive*, that is, for every arcs uv and tw, there is an automorphism φ of QR_q such that $tw = \varphi(uv)$). As a consequence, each QR_q is also a *circular tournament*, that is, a tournament admitting an automorphism which is a circular permutation.

An *orientation vector* of size k is a sequence $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ in $\{0,1\}^k$. Let G be an oriented graph and $X = (x_1, x_2, \ldots, x_k)$ be a sequence of distinct vertices of G. A vertex y of G is said to be an *α-successor of X* if for every i, $1 \leq i \leq k$, we have $\alpha_i = 1 \Rightarrow (x_i, y) \in E(G)$ and $\alpha_i = 0 \Rightarrow (y, x_i) \in E(G)$. The graph G satisfies property $S_{k,n}$ if for every sequence $X = (s_1, s_2, \ldots, s_k)$ of k *distinct* vertices of G, and for every orientation vector α of size k, there exist at least n vertices in $V(G)$ which are α-successors of X.
Notice that property $S_{k,n}$ implies $S_{k',n'}$ for every $k' \leq k$ and $n' \leq n$.

A computer check (similar to the one described in [6]) proves the following lemma:

Lemma 6 *The tournament QR_{67} satisfies properties $S_{3,6}$ and $S_{4,1}$.*

We use the method of reducible configurations to show that every 2-outerplanar graph is QR_{67}-colorable. Let $w(G) = |V(G)| + |E(G)|$. We consider a 2-outerplanar graph G having no homomorphism to QR_{67} such that $w(G)$ is minimum.

Lemma 7 *G is 2-connected and does not contain a cut consisting in two adjacent vertices.*

Proof. If G is not 2-connected, then we can obtain a QR_{67}-coloring of G from the coloring of its 2-connected components, since QR_{67} is a circular tournament. Moreover G cannot contain a cut set consisting of two adjacent vertices, since QR_{67} is arc-transitive. □

Notice that Lemma 7 implies that every 2-outerplanar embedding of G is a block.

Lemma 8

1. *The graph G does not contain any ${}^{\leq}3$-vertex.*
2. *The graph G does not contain two adjacent 4-vertices.*
3. *The graph G does not contain the configuration depicted in Figure 6.*

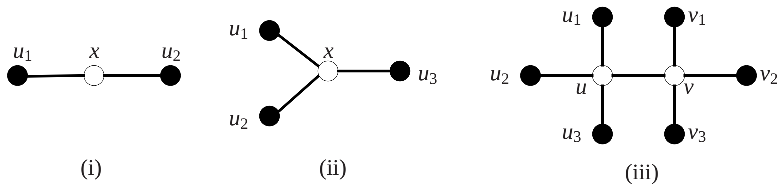


Figure 9: Forbidden configurations for Lemma 8.

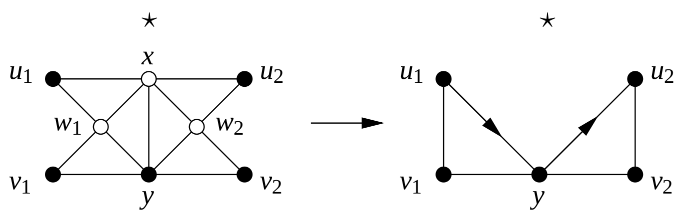


Figure 10: Construction of G' in the proof of Lemma 8.3.

Proof.

1. Notice that G does not contain ${}^{\leq}1$-vertices by Lemma 7. Suppose that G contains a 2-vertex x adjacent to vertices u_1 and u_2 (see configuration (i) in Figure 9). Let G' be the graph obtained from $G \setminus \{x\}$ by adding the arc $\overrightarrow{u_1u_2}$ if u_1 and u_2 are not already adjacent in G. Notice that G' is 2-outerplanar and $w(G') < w(G)$. Any QR_{67}-coloring f of G' induces a coloring of $G \setminus \{x\}$ such that $f(u_1) \neq f(u_2)$, which can be extended to G by property $S_{2,1}$.

   Suppose that G contains a 3-vertex x adjacent to vertices u_1, u_2, and u_3 (see configuration (ii) in Figure 9). Since QR_{67} is self-reverse, we assume w.l.o.g. that $d^-(x) \leq d^+(x)$ by considering either G or G^R. We have $d^-(x) \neq 0$, since otherwise we could extend any QR_{67}-coloring of $G \setminus \{x\}$ to G. Suppose now $d^-(x) = 1$, which is the only remaining case. Let us set $N^-(x) = \{u_1\}$, $N^+(x) = \{u_2, u_3\}$. Let G' be the graph obtained from $G\setminus\{x\}$ by adding the arc $\overrightarrow{u_1u_2}$ (resp. $\overrightarrow{u_1u_3}$) if u_1 and u_2 (resp. u_1 and u_3) are not already adjacent in G. Notice that G' is 2-outerplanar and $w(G') < w(G)$. Any QR_{67}-coloring f of G' induces a coloring of $G \setminus \{x\}$ such that $f(u_1) \neq f(u_2)$ and $f(u_1) \neq f(u_3)$, which can be extended to G by property $S_{3,1}$.

2. Suppose that G contains configuration (iii) in Figure 9. Let G' be the graph obtained from G by removing the arc connecting u and v. Notice that G' is 2-outerplanar and $w(G') < w(G)$. Let f be any QR_{67}-coloring of G'. By property $S_{3,6}$, we can choose f such that $f(u) \notin \{f(v_1), f(v_2), f(v_3)\}$. Now by property $S_{4,1}$, we can choose f such that $f(v) \notin \{f(u), f(u_1), f(u_2), f(u_3)\}$ and extend this coloring to G.

3. Suppose that G contains the configuration depicted in Figure 6. Let G' be the graph obtained from $G \setminus \{w_1, w_2, x\}$ by adding the arcs $\overrightarrow{u_1 y}$ and $\overrightarrow{y u_2}$, and the arc $\overrightarrow{u_1 v_1}$ (resp. $\overrightarrow{u_2 v_2}$) if u_1 and v_1 (resp. u_2 and v_2) are not adjacent in G. This construction is depicted in Figure 10. Notice that G' is 2-outerplanar and $w(G') < w(G)$. Any QR_{67}-coloring f of G' induces a coloring of $G \setminus \{w_1, w_2, x\}$ such that $f(u_1)$, $f(v_1)$, $f(y)$ (resp. $f(u_2)$, $f(v_2)$, $f(y)$; resp. $f(u_1)$, $f(u_2)$, $f(y)$) are pairwise distinct. By Property $S_{3,6}$, we can assign x a color $f(x) \notin \{f(v_1), f(v_2)\}$. By Property $S_{4,1}$, we can assign w_1 a color $f(w_1) \notin \{f(u_1), f(v_1), f(y), f(x)\}$ and assign w_2 a color $f(w_2) \notin \{f(u_2), f(v_2), f(y), f(x)\}$. We thus obtain a QR_{67}-coloring of G, which is a contradiction.

□

By Lemma 7, G is a block. Using Theorem 4, G must contain one of the configurations that are forbidden by Lemma 8. This contradiction completes the proof of Theorem 5.

References

[1] P. Boiron, E. Sopena, and L. Vignal. Acyclic improper colourings of graphs, *J. Graph Theory* **32** (1999), 97–107.

[2] O.V. Borodin, A.V. Kostochka, J. Nešetřil, A. Raspaud, and E. Sopena. On the maximum average degree and the oriented chromatic number of a graph, *Discrete Math.* **206** (1999), 77–89.

[3] E. Fried. On homogeneous tournaments, *Combinatorial theory and its applications* **II** (1970), 467–476.

[4] A. Hackmann and A. Kemnitz. List edge colorings of outerplanar graphs, *Ars Combinatoria* **60** (2001), 181–185.

[5] J. Nešetřil and A. Raspaud. Antisymmetric flows and strong colorings of oriented planar graphs, *Ann. Inst. Fourier* **49(3)** (1999), 1037–1056.

[6] P. Ochem. Oriented colorings of triangle-free planar graphs, *Inform. Process. Lett.* **92(2)** (2004), 71–76.

[7] P. Ochem. Ph.D. thesis, Université Bordeaux 1, 2005.
`http://dept-info.labri.fr/~ochem/these.ps`

[8] A. Raspaud and E. Sopena. Good and semi-strong colorings of oriented planar graphs, *Inform. Process. Lett.* **51(4)** (1994), 171–174.

[9] E. Sopena. Oriented graph coloring, *Discrete Math* **229(1–3)** (2001), 359–369.

Appendix B

[EO07b]

On circle graphs with girth at least five

Abstract

Circle graphs with girth at least five are known to be 2-degenerate (Ageev, 1999). In this paper, we prove that circle graphs with girth at least $g \geq 5$ contain a vertex of degree at most one or a chain of $g - 4$ vertices of degree two, which implies Ageev's result in the case $g = 5$. We then use this structural property to give an upper bound on the circular chromatic number of circle graphs with girth at least $g \geq 5$ as well as a precise estimate of their maximum average degree.

On circle graphs with girth at least five

Louis Esperet * and Pascal Ochem *

* *LaBRI, Université Bordeaux 1, Talence, France*

May 2007

Abstract

Circle graphs with girth at least five are known to be 2-degenerate (Ageev, 1999). In this paper, we prove that circle graphs with girth at least $g \geq 5$ contain a vertex of degree at most one or a chain of $g-4$ vertices of degree two, which implies Ageev's result in the case $g = 5$. We then use this structural property to give an upper bound on the circular chromatic number of circle graphs with girth at least $g \geq 5$ as well as a precise estimate of their maximum average degree.

1 Introduction

Let C denote the unit circle, and let us take the clockwise orientation as the positive orientation of C. Let $\{x_0, \ldots, x_{k-1}\} \subset C$, we say that $(x_0, \ldots, x_{k-1})$ are in *cyclic order* if the minimum between the sum of the length of the arcs $\overrightarrow{x_i x_{i+1}}$, $0 \leq i \leq k-1$, and the sum of the length of the arcs $\overrightarrow{x_{i+1} x_i}$, $0 \leq i \leq k-1$, is equal to one, where i is taken modulo k. A pair $\{x, y\}$ of elements of C is called a *chord* of C with *endpoints* x and y. Two chords $\{x_1, y_1\}$ and $\{x_2, y_2\}$ *intersect* if $(x_1 x_2 y_1 y_2)$ are in cyclic order, otherwise they are said to be *parallel*.

All graphs considered in this paper are simple: they do not have any loop nor parallel edges. The *girth* of a graph G is the size of a shortest cycle in G. We call a k-vertex (resp. ${}^{\leq}k$-vertex, ${}^{\geq}k$-vertex) a vertex of degree k (resp. at most k, at least k).

By definition, every circle graph G with set of vertices $V(G) = \{v_1, \ldots, v_n\}$ admits a *representation* $\mathcal{C} = \{\{x_1, y_1\}, \ldots, \{x_n, y_n\}\}$ such that for all i, j, v_i and v_j are adjacent in G if and only if the chords $\{x_i, y_i\}$ and $\{x_j, y_j\}$ intersect in $\mathcal{C}$. We only consider representations in which endpoints and intersection points of chords are all distinct. Observe that in general, circle graphs do not have a unique representation. A representation $\mathcal{C}'$ obtained from $\mathcal{C}$ only by removing chords is called a *sub-representation* of $\mathcal{C}$. Observe that if $\mathcal{C}$ is a representation of G, a sub-representation of $\mathcal{C}$ corresponds to an induced subgraph of G.

Observation 1 *Let G be a circle graph with representation $\mathcal{C}$, and let $v_1, \ldots, v_k$ be an independent set in G. The chords of $\mathcal{C}$ corresponding to $v_1, \ldots, v_k$ are pairwise parallel.*

In order to prove that circle graphs with girth at least five are 2-degenerate, Ageev [1] does not consider their circle representation, but an equivalent representation on the real axis,

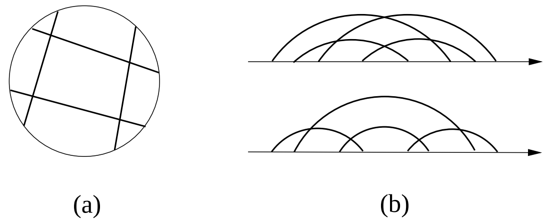


Figure 1: (a) The unique circle representation of C_4. (b) The two non-equivalent representations of C_4 on the real axis.

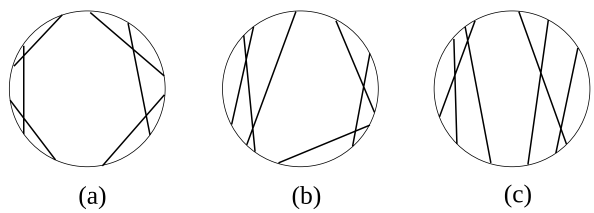


Figure 2: Three non-equivalent circle representations of the union of two paths of length two.

usually called *interval-overlap*. The major difference is that some graphs, for example cycles, have a unique circle representation whereas they have several non-equivalent representations on the real axis (see Figure 1). Hence, even if considering a real axis representation can be very convenient to define an order on the endpoint of the chords, the case study is then much harder. Unfortunately, even in the circle representation, some very simple graphs such as the union of two disjoint paths do not have a unique representation (see Figure 2). Observe that in Figure 2(a), the representation of the two paths is a sub-representation of the representation of a cycle. In this case we make a slight abuse of notation and say that the two paths are *in cyclic order*.

In Section 2, we prove the following extension of Ageev's result:

Theorem 1 *Every circle graph with girth $g \geq 5$ contains a $^{\leq}1$-vertex or a chain of $(g-4)$ 2-vertices.*

In [1], Ageev uses his structural result to prove that circle graphs with girth at least five have chromatic number at most three. We can use Theorem 1 to obtain a refinement of this result for circle graphs with larger girth. Instead of considering the chromatic number of these graphs, we consider their circular chromatic number. For two integers $1 \leq q \leq p$, a (p,q)-coloring of a graph G is a coloring c of the vertices of G with colors $\{0, \ldots, p-1\}$ such that for any pair of adjacent vertices x and y, we have $q \leq |c(x) - c(y)| \leq p - q$. The circular chromatic number of G is

$$\chi_c(G) = \inf\left(\frac{p}{q} \mid \text{ there exists a } (p,q)\text{-coloring of } G\right).$$

It is known that $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$, and so $\chi(G) = \lceil \chi_c(G) \rceil$. The chromatic number can thus be considered as an approximation of the circular chromatic number.

Class	PLANAR	OUTERPLANAR	PARTIAL 2-TREE	SEG	1-STRING
μ_g	$2+\frac{4}{g-2}$	$2+\frac{2}{g-2}$	$2+\frac{2}{\lfloor\frac{g-1}{2}\rfloor}$	$2+\frac{4}{g-4}$	$2+\frac{4}{g-4}$

Table 1: Values of μ_g for some classes of graphs.

Using a well-known observation on circular coloring (see e.g. Corollary 2.2 in [2]), the existence of a chain of $(g-4)$ 2-vertices implies the following result:

Corollary 1 *Every circle graph G with girth $g \geq 5$ has circular chromatic number*

$$\chi_c(G) \leq 2 + \frac{1}{\left\lfloor\frac{g-3}{2}\right\rfloor}.$$

In Section 3, we study an invariant giving a very precise idea of the local structure of graphs. The *maximum average degree* of a graph G is defined as

$$\text{mad}(G) = \max\left\{\text{ad}(H), H \subseteq G\right\}, \text{ where } \text{ad}(H) = \frac{2|E(H)|}{|V(H)|}.$$

For planar graphs, there is a simple relation between girth and maximum average degree: any planar graph G with girth g is such that $\text{mad}(G) < 2g/(g-2)$. On the other hand, there exists a family $(G_n)_{n\geq 0}$ of planar graphs with girth g, such that $\text{mad}(G_n) \to 2g/(g-2)$ when $n \to \infty$. We would like to obtain the same kind of link between the girth and the maximum average degree of circle graphs. The following corollary is a straightforward consequence of Theorem 1:

Corollary 2 *Any circle graph G with girth $g \geq 5$ is such that $mad(G) < 2 + 2/(g-4)$.*

note that Corollary 2 has some implications on the circular choosability of circle graphs. Using Proposition 32(i) in Section 5.4 of [3], we can prove :

Corollary 3 *Every circle graph G with girth $g \geq 5$ has circular choice number $cch(G) \leq 2 + \frac{4}{g-2}$.*

To improve Corollary 2, we consider

$$\mu_g(\mathcal{F}) = \sup\left\{\text{mad}(G) \mid G \in \mathcal{F} \text{ and } G \text{ has girth at least } g\right\}.$$

Let SEG denote the class of graphs defined as intersection of segments in the plane, and 1-STRING denote the class of graphs defined as intersection of jordan curves in the plane, such that any two curves intersect at most once. Table 1 gives an idea of the function μ_g for some classes of graphs. Note that for SEG and 1-STRING, g has to be at least five, since otherwise μ_g is not bounded.

We can remark that for all these classes, μ_g is a rational number. The following theorem shows that this is not the case for the class of circle graphs. It is proved in Section 3.

Theorem 2 *For every $g \geq 5$, $\mu_g(\text{CIRCLE}) = 2\sqrt{\frac{g-2}{g-4}}$*

2 Proof of Theorem 1

Let $G = (V, E)$ be a circle graph with girth $g \geq 5$ and minimum degree two, and let $\mathcal{C} = \{\{x_1, x'_1\}, \ldots, \{x_n, x'_n\}\}$ be a circle representation of G. We first decompose the chords of $\mathcal{C}$ into two sets, using the following rules:

(1) for every set of 3 distincts chords $\{x, x'\}$, $\{y, y'\}$, and $\{z, z'\}$, such that $\{y, y'\}$ is uncolored and $(xyzz'y'x')$ are in cyclic order, colour the chord $\{y, y'\}$ in blue,

(2) colour all the uncolored chords in red.

By construction, the red chords are exactly the chords $\{x, y\}$ such that at least one of the arcs $\overrightarrow{xy}$ and $\overrightarrow{yx}$ does not contain both endpoints of a chord distinct from $\{x, y\}$. Let $\mathcal{C}^R$ (resp. $\mathcal{C}^B$) be the representation induced by the red (resp. blue) chords and G^R (resp. G^B) be the corresponding graph. We first prove the following lemma.

Lemma 1 *$\mathcal{C}^R$ is a sub-representation of the representation of a cycle.*

Proof. Assume that G^R contains a $^{\geq}3$-vertex v, adjacent to x, y, and z in G^R. Since $g \geq 5$, the graph G does not contain any triangle, and so $\{x, y, z\}$ is an independent set. Using Observation 1, this implies that the three corresponding red chords are parallel in any representation, which contradicts Rule (1).

Hence, G^R has maximum degree two. Suppose now that G^R contains a cycle. Then if there exists a vertex which is not in the cycle, the corresponding chord, as well the chords corresponding to two non-adjacent vertices of the cycle, are parallel (recall that the cycle has length at least five, since $g \geq 5$). This contradicts Rule (1). So G^R is either a cycle or a union of disjoint paths.

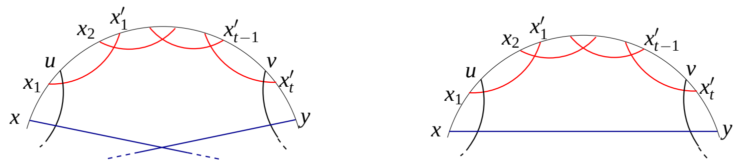
Suppose now that $\mathcal{C}^R$ is not a sub-representation of a cycle. Then G^R is necessarily a union of disjoint paths, and two of them are not in cyclic order in $\mathcal{C}^R$. This also contradicts Rule (1), so $\mathcal{C}^R$ is a sub-representation of the representation of a cycle. □

Observe that each blue chord $\{x, y\}$ induces two complementary arcs $\overrightarrow{xy}$ and $\overrightarrow{yx}$ on the circle. We denote by $\mathcal{A}_1$ the set of such arcs. Similarly, two intersecting blue chords $\{u, v\}$ and $\{x, y\}$ induce four consecutive arcs whose lengths add up to one, say without loss of generality $\overrightarrow{ux}$, $\overrightarrow{xv}$, $\overrightarrow{vy}$, and $\overrightarrow{yu}$. We denote by $\mathcal{A}_2$ the set of all such arcs.

For any arc $\overrightarrow{xy}$ of the circle, we define $\rho(\overrightarrow{xy})$ as the number of red chords having both endpoints in $\overrightarrow{xy}$. We consider the integer $t = \min\{\rho(\overrightarrow{xy}), \overrightarrow{xy} \in \mathcal{A}_1 \cup \mathcal{A}_2, \rho(\overrightarrow{xy}) > 0\}$.

If there is no blue chord in our decomposition, then G is either a cycle or a union of paths, and thus contains a $^{\leq}1$-vertex or g adjacent 2-vertices. So we can assume from now on that G^B is non empty. Observe that for any blue chord $\{x, y\}$, we have $\rho(\overrightarrow{xy}) > 0$ and $\rho(\overrightarrow{yx}) > 0$ since otherwise $\{x, y\}$ would be red. Hence, the integer t exists. We now consider two cases, depending on whether the minimum is reached by two intersecting chords or by a single chord.

Case 1: The minimum $t > 0$ is reached by two intersecting blue chords, say $\{x, x'\}$ and $\{y, y'\}$, and for every blue chord $\{u, v\}$, we have $\rho(\overrightarrow{uv}) \neq t$. Let us assume without loss of generality that $t = \rho(\overrightarrow{xy})$. According to the clockwise order, we denote by $\{x_1, x'_1\}, \ldots \{x_t, x'_t\}$ the red chords having both endpoints in $\overrightarrow{xy}$ (see Figure 3(a)). Observe that every blue chord



(a) (b)

Figure 3: A chain of $t \geq g-4$ vertices of degree two in G.

has at most one endpoint in $\overrightarrow{xy}$, since otherwise we would have a blue chord $\{u,v\}$ with $1 \leq \rho(\overrightarrow{uv}) \leq t$, which would contradict the hypothesis.

We first prove that the graph induced by the chords $\{x_i, x'_i\}$ $(1 \leq i \leq t)$ is a path. If this is not the case, then for some i the chords $\{x_i, x'_i\}$ and $\{x_{i+1}, x'_{i+1}\}$ do not intersect. Then either one of them corresponds to a $^{\leq}$1-vertex, or each of them intersects a blue chord. Such a blue chord also intersects $\{x, x'\}$ or $\{y, y'\}$, since it has only one endpoint in $\overrightarrow{xy}$. This contradicts the minimality of t.

We now prove that the arc $\overrightarrow{x_2 x'_{t-1}}$ does not contain any endpoint of a blue chord. Observe that if the arc contains the endpoint u of a blue chord, then there exists $1 \leq i \leq t-2$ such that $u \in \overrightarrow{x'_i x_{i+2}}$, since otherwise this would create a triangle. If such an endpoint u exists, the related blue chord along with $\{x, x'\}$ or $\{y, y'\}$ contradicts the minimality of t.

Hence, the vertices corresponding to $\{x_i, x'_i\}$ $(2 \leq i \leq t-1)$ are a chain of $(t-2)$ 2-vertices in G. Since G does not contain any 1-vertex, the chord $\{x_1, x'_1\}$ intersects a chord $\{u, u'\}$ distinct from $\{x_2, x'_2\}$. Such a chord may be blue or red, but by the minimality of t it cannot intersect $\{y, y'\}$. So the chord $\{u, u'\}$ has to intersect $\{x, x'\}$ and since $g \geq 4$, exactly one such $\{u, u'\}$ exists. Similarly, $\{x_t, x'_t\}$ intersects exactly one chord distinct from $\{x_{t-1}, x'_{t-1}\}$, say $\{v, v'\}$, and $\{v, v'\}$ also intersects $\{y, y'\}$. Thus the vertices corresponding to $\{x_i, x'_i\}$ $(1 \leq i \leq t)$ form a chain of t 2-vertices in G. Since the chords $\{x, x'\}, \{u, u'\}, \{x_1, x'_1\}, \ldots, \{x_t, x'_t\}, \{v, v'\}, \{y, y'\}$ correspond to a cycle in G, we have $t \geq g-4$.

Case 2: The minimum $t > 0$ is reached by a blue chord $\{x, y\}$. The proof is the same as the previous one, except that we obtain a chain of $(g-3)$ 2-vertices instead of $(g-4)$ 2-vertices (see Figure 3(b)).

3 Proof of Theorem 2

Let us first give a construction to prove the lower bound. For every $g \geq 5$, we construct a family $(Q_{g,t})_{t \geq 0}$ of circle graphs with girth g such that $Q_{g,0} = C_g$ (the cycle on n vertices) and $Q_{g,t+1}$ is obtained by adding chords to the representation of $Q_{g,t}$.

These new chords (represented as thin chords in Figure 4) induce a cycle. Every old chord (i.e. that belongs to $Q_{g,t}$, represented as thick chords in Figure 4) intersects one new chord at each of its endpoints. A *k-region* is a region inside the circle, which is incident to the circle and to exactly k chords. Note that in any $Q_{g,t}$, every k-region is either a 2- or a 3-region. Any 2-region in $Q_{g,t}$ produces in $Q_{g,t+1}$ a face $\mathcal{F}$ of size g, $(g-3)$ vertices $(2(g-3)$ half-chords), $(g-2)$ edges, $(g-3)$ 2-regions, and $(g-2)$ 3-regions. Any 3-region in $Q_{g,t}$ produces in $Q_{g,t+1}$ a face $\mathcal{F}$ of size g, $(g-4)$ vertices, $(g-3)$ edges, $(g-4)$ 2-regions, and $(g-3)$ 3-regions.

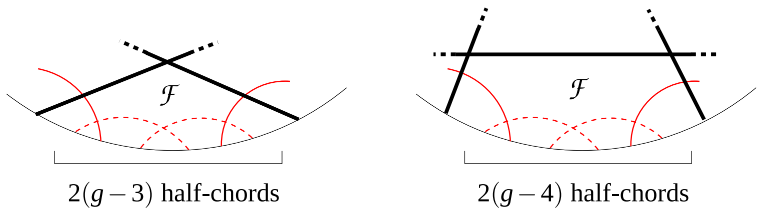


Figure 4: From $Q_{g,t}$ to $Q_{g,t+1}$

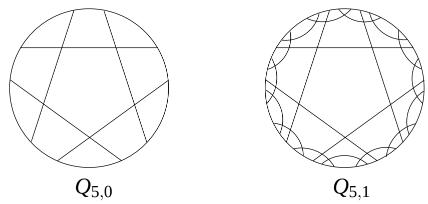


Figure 5: Examples

We now consider the vector $V_{g,t} = {}^t(n, m, R_2, R_3)$ whose components are respectively the number of vertices, edges, 2-regions, and 3-regions of $Q_{g,t}$. By construction, we have that $V_{g,t+1} = M_g V_{g,t}$, where

$$M_g = \begin{pmatrix} 1 & 0 & g-3 & g-4 \\ 0 & 1 & g-2 & g-3 \\ 0 & 0 & g-3 & g-4 \\ 0 & 0 & g-2 & g-3 \end{pmatrix}$$

The limit of the average degree $\mathrm{ad}(Q_{g,t})$ of $Q_{g,t}$ when $t \to \infty$ can be obtained from the unique eigenvector

$$V = \begin{pmatrix} g-3+\sqrt{(g-2)(g-4)} \\ g-2+(g-3)\sqrt{(g-2)/(g-4)} \\ g-4+\sqrt{(g-2)(g-4)} \\ g-2+\sqrt{(g-2)(g-4)} \end{pmatrix}$$

associated to the largest eigenvalue $g-3+\sqrt{(g-2)(g-4)}$ of M_g. We thus obtain:

$$\mu_g \geq \lim_{t\to\infty} \mathrm{ad}(Q_{g,t}) = 2 \cdot \frac{g-2+(g-3)\sqrt{(g-2)/(g-4)}}{g-3+\sqrt{(g-2)(g-4)}} = 2\sqrt{\frac{g-2}{g-4}}$$

Before proving the upper bound, we make some remarks on structure of the graphs $Q_{g,t}$. Observe that the graphs $Q_{g,t}$ with $t \geq 1$ are circle graphs with girth $g \geq 5$ that contain neither $^{\leq}1$-vertices nor chains of $(g-3)$ 2-vertices (see Figure 5 for an example with $g = 5$), which proves that Theorem 1 is optimal in a certain way. Another interesting property of these graphs is that for any $g \geq 5$, $Q_{g,t}$ contains K_{t+3}, the complete graph with $t+3$ vertices, as a minor (that is, K_{t+3} can be obtained from $Q_{g,t}$ by contracting edges and removing edges and vertices). To see this, contract $Q_{g,0}$ in order to obtain a triangle, and at each step contract the set of new vertices into a single vertex, which is universal by construction. The size of the clique we construct will increase by one at each step, and we will eventually obtain K_{t+3} as a minor of $Q_{g,t}$. This implies that for any integer $g \geq 5$ and any graph H, there exists a circle graph G with girth g such that G contains H as a minor.

We now prove the upper bound by contradiction. Since circle graphs of girth at least g are closed under taking induced subgraphs, it is sufficient to prove that every circle graph G with girth at least $g \geq 5$ has average degree $\mathrm{ad}(G) < 2\sqrt{\frac{g-2}{g-4}}$.

Let G be a circle graph and $\mathcal{C}$ be a circle representation of G. We denote by $R(\mathcal{C})$ the planar graph constructed as follows:

- the vertex set of $R(\mathcal{C})$ is the set of crossings of chords in $\mathcal{C}$,
- two distinct vertices are adjacent in $R(\mathcal{C})$ if and only if they correspond to consecutive crossings of a same chord in $\mathcal{C}$.

Observe that the construction above clearly gives a natural planar embedding of $R(\mathcal{C})$. In the following, we only consider this precise planar embedding. For example, the outerface of $R(\mathcal{C})$ will be well-defined. Note that $R(\mathcal{C})$ has maximum degree four.

Let us consider a fixed integer $g \geq 5$ and a circle graph G_1 with girth at least g, such that $\mathrm{ad}(G_1) > 2\sqrt{\frac{g-2}{g-4}}$, and such that G_1 is minimal with this property. That is, for any circle graph H with girth at least g and such that $|V(H)| < |V(G_1)|$, we have $\mathrm{ad}(H) < 2\sqrt{\frac{g-2}{g-4}}$. Observe that by minimality, G_1 does not contain any $^{\leq}1$-vertex, since otherwise by removing it we would obtain a smaller graph with larger average degree.

Let $\mathcal{C}_1$ be a representation of G_1. If the outerface of the planar embedding of $R(\mathcal{C}_1)$ contains a 4-vertex, we apply the following operation on $\mathcal{C}_1$, which gives a new representation $\mathcal{C}_2$ and a new circle graph G_2 with girth g. Let u denote a 4-vertex on the outerface of $R(\mathcal{C}_1)$. It corresponds to an edge between to vertices v_1 and v_2 of G_1, represented by two crossing chords c_1 and c_2 in $\mathcal{C}_1$. Since u is a 4-vertex in $R(\mathcal{C}_1)$, the chords c_1 and c_2 respectively cross two chords c'_1 and c'_2 as depicted in Figure 6. Let v'_1 and v'_2 be the vertices of G_1 associated to c'_1 and c'_2. Since u is on the outerface of $R(\mathcal{C}_1)$, v'_1 and v'_2 are not adjacent in G_1. Hence, we can add a path of $g-4$ chords between c'_1 and c'_2, as depicted in Figure 6. Let $\mathcal{C}_2$ denote the new representation, and G_2 be the associated circle graph. The $g-4$ vertices added to G_1 to obtain G_2 form a cycle of length exactly g in G_2 containing v_1, v_2, v'_1, and v'_2. Note that the number of 4-vertices on the outerface of the plane graph associated to the representation decreases by one after at most two iterations of this process.

Let n_1 and m_1 denote respectively the number of vertices and edges of G_1. By Corollary 2, we have that $\mathrm{ad}(G_1) < 2 \cdot \frac{g-3}{g-4}$. This implies that $\mathrm{ad}(G_2) = 2 \cdot \frac{m_1+g-3}{n_1+g-4} > 2 \cdot \frac{m_1}{n_1} = \mathrm{ad}(G_1)$. Thus the average degree increases during this operation.

We repeat this operation until we obtain a circle graph G with girth g having a representation $\mathcal{C}$ such that the outerface of the planar embedding of $R(\mathcal{C})$ does not contain any 4-vertex. The consequence of the previous observation is that $\mathrm{ad}(G) > \mathrm{ad}(G_1) > 2\sqrt{\frac{g-2}{g-4}}$. Let n and m be the number of vertices and edges of G. This implies in particular that:

$$\sqrt{\frac{g-2}{g-4}}n < m \tag{1}$$

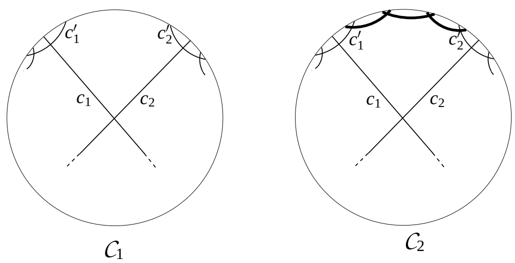


Figure 6: From $\mathcal{C}_1$ to $\mathcal{C}_2$

Let N, M, and F denote respectively the number of vertices, edges, and faces of $R(\mathcal{C})$. Since a crossing in $\mathcal{C}$ corresponds to both an edge in G and a vertex in $R(\mathcal{C})$, we have:

$$N = m \tag{2}$$

We can write Euler's formula for the planar embedding of $R(\mathcal{C})$ as follows:

$$M + 2 = F + N \tag{3}$$

Let N_d denote the number of d-vertices in $R(\mathcal{C})$. Since G_1 does not contain any $^{\leq}1$-vertex, and no new $^{\leq}1$-vertex is created during the transformation, the graph G does not contain any $^{\leq}1$-vertex either. This implies in particular that $R(\mathcal{C})$ does not contain $^{\leq}1$-vertices. Thus, the degree of a vertex in $R(\mathcal{C})$ is at least 2 and at most 4 and we have:

$$N = N_2 + N_3 + N_4 \tag{4}$$

The sum of vertex degrees is equal to twice the number of edges in $R(\mathcal{C})$:

$$2N_2 + 3N_3 + 4N_4 = 2M \tag{5}$$

Any chord in a representation of G corresponding to some vertex $v \in G$ contains $(deg(v) - 1)$ edges of $R(\mathcal{C})$. Since $\sum_{v \in G}(deg(v) - 1) = 2m - n$, we have:

$$2m - n = M \tag{6}$$

Note that the outerface of $R(\mathcal{C})$ contains every 2-vertex, every 3-vertex, and no 4-vertex of $R(\mathcal{C})$. Moreover, $R(\mathcal{C})$ cannot contain a face of degree strictly less than g, since otherwise G would contain a cycle of length strictly less than g. We thus obtain a lower bound on the sum of degrees of the faces of $R(\mathcal{C})$, which is equal to twice the number of edges in $R(\mathcal{C})$:

$$g(F - 1) + N_2 + N_3 \leq 2M \tag{7}$$

Let us decompose the chords of $\mathcal{C}$ into blue and red chords as done in the proof of Theorem 1. Using previous notation, $\mathcal{C}^B$ is the sub-representation of $\mathcal{C}$ induced by the blue chords and G^B is the corresponding circle graph. Note that G^B is a proper induced subgraph of G_1 and G. We thus have:

$$\mathrm{ad}\left(G^B\right) = \frac{2(m - N_2 - N_3)}{n - N_2} < 2\sqrt{\frac{g-2}{g-4}} < \frac{2m}{n} = \mathrm{ad}\left(G\right)$$

This implies that $\frac{2(N_2+N_3)}{N_2} > \frac{2m}{n} > 2\sqrt{\frac{g-2}{g-4}}$, which gives:

$$\left(\sqrt{\frac{g-2}{g-4}} - 1\right) N_2 < N_3 \tag{8}$$

The combination $(g-4)\times\mathbf{(1)}+(g-4)\left(2\sqrt{\frac{g-2}{g-4}}-1\right)\times\mathbf{(2)}+g\times\mathbf{(3)}+2(g-2)\left(1-\sqrt{\frac{g-4}{g-2}}\right)\times\mathbf{(4)}+\frac{1}{2}(g-2)\left(1-\sqrt{\frac{g-4}{g-2}}\right)\times\mathbf{(5)}+\sqrt{(g-2)(g-4)}\times\mathbf{(6)}+\mathbf{(7)}+\frac{1}{2}(g-4)\left(\sqrt{\frac{g-2}{g-4}}-1\right)\times\mathbf{(8)}$ gives $g<0$, a contradiction.

4 Perspectives

In the present paper, we study the structure of sparse circle graphs. The opposite problem of studying the structure of dense circle graphs seems to be much harder. For example, the relation between the clique number of circle graphs and their chromatic number is not precisely established. Kostochka and Kratochvíl [4] proved that every circle graph with clique number ω has chromatic number at most $2^{\omega+6}$, but this is still far from the lower bound of $\Omega(\omega \log \omega)$.

Note that the upper bound of $2^{\omega+6}$ even holds for polygon-circle graphs, a superclass of circle graphs, defined as the intersection class of chords and convex polygons of the circle. The size of this class is known to be much larger, but we suspect that polygon-circle graphs with girth at least five behave like circle graphs with girth at least five. It would be interesting to see if the results of the present paper extend to the class of polygon-circle graphs.

Acknowledgement

The authors would like to thank Daniel Gonçalves and Arnaud Labourel for fruitful discussion.

References

[1] A.A. Ageev. Every circle graph with girth at least 5 is 3-colourable, *Discrete Math.*, **195** (1999) 229–233.

[2] A. Galluccio, L.A. Goddyn, and P. Hell. High-Girth Graphs Avoiding a Minor are Nearly Bipartite *J. Combin. Theory. Ser. B* **83(1)** (2001), 1–14.

RR-5957

[3] F. Havet, R.J. Kang, T. Müller, and J.-S. Sereni. Circular choosability, *INRIA Sophia-Antipolis* Technical Report RR-5957 (2006).

[4] A. Kostochka, J. Kratochvíl. Covering and coloring polygon-circle graphs, *Discrete Math.*, **163** (1997) 299–305.

Appendix C

[ELO07]

On induced-universal graphs for the class of bounded-degree graphs

Abstract
For a family $\mathcal{F}$ of graphs, a graph U is said to be $\mathcal{F}$-induced-universal if every graph of $\mathcal{F}$ is an induced subgraph of U. We give a construction for an induced-universal graph for the family of graphs on n vertices with degree at most k. For k even, our induced-universal graph has $O(n^{k/2})$ vertices and for k odd it has $O(n^{\lceil k/2 \rceil - 1/k} \log^{2+2/k} n)$ vertices. This construction improves the main result of [But06] by a multiplicative constant factor for even case and by almost a multiplicative $n^{1/k}$ factor for odd case. We also construct induced-universal graphs for the class of oriented graphs with bounded incoming and outgoing degree, slightly improving another result of [But06].

On induced-universal graphs for the class of bounded-degree graphs

Louis Esperet* Arnaud Labourel† Pascal Ochem‡

February 6, 2008

Abstract

For a family $\mathcal{F}$ of graphs, a graph U is said to be $\mathcal{F}$-induced-universal if every graph of $\mathcal{F}$ is an induced subgraph of U. We give a construction for an induced-universal graph for the family of graphs on n vertices with degree at most k. For k even, our induced-universal graph has $O(n^{k/2})$ vertices and for k odd it has $O(n^{\lceil k/2\rceil-1/k}\log^{2+2/k} n)$ vertices. This construction improves a result of Butler by a multiplicative constant factor for even case and by almost a multiplicative $n^{1/k}$ factor for odd case. We also construct induced-universal graphs for the class of oriented graphs with bounded incoming and outgoing degree, slightly improving another result of Butler.

1 Introduction

All graphs are assumed to be without loops or multiples edges. For a graph G we denote by $V(G)$ its vertex set and by $E(G)$ its edge or arc set. Our terminology is standard and any undefined term can be found in standard theory books [11].

For a finite family $\mathcal{F}$ of graphs, a graph U is said to be *$\mathcal{F}$-universal* if every graph in $\mathcal{F}$ is a subgraph of U. For instance, if we denote by $\mathcal{F}_n$ the family of all graphs with at most n vertices, then the complete graph K_n is $\mathcal{F}_n$-universal. The universal graph problem consists in finding a n-vertex universal graph with minimal number of edges for specific subfamilies of $\mathcal{F}_n$. This problem was originally motivated by circuit design for computer chips [4]. Several families of graphs have been studied for this problem, including forests [10], bounded-degree forests [2, 3], and bounded-degree graphs [1].

The notion of induced-universal graph can be similarly defined. For a family $\mathcal{F}$ of graphs, a graph U is *$\mathcal{F}$-induced-universal* if every graph in $\mathcal{F}$ is an induced subgraph of U. The induced-universal graph problem consists in finding an induced-universal graph of minimal number of vertices for specific subfamilies of $\mathcal{F}_n$. The family $\mathcal{F}_n$ itself was considered by Moon [13], while Chung considered trees, planar graphs, and graphs with bounded arboricity on n vertices [9].

The induced-universal problem is strongly related to a notion of distributed data structure known as *adjacency labeling scheme* or *implicit representation*. An implicit representation for

* *esperet@labri.fr*
† *labourel@labri.fr*
‡ *pascal.ochem@lri.fr*

a family $\mathcal{F}$ of graphs consists in two functions: a labeling function that assigns labels to the vertices of any graph of $\mathcal{F}$ and an adjacency function that determines the adjacency between two vertices only by looking at their labels. The problem of finding an implicit representation with small labels for specific families of graphs was first introduced by Breuer [6, 7]. Kannan, Naor and Rudich [12] established the strong relation between the two problems by proving that the existence of an implicit representation using $k(n)$ bits per vertex for a family $\mathcal{F}_n$ is equivalent to the existence of an $\mathcal{F}_n$-induced-universal graph with $2^{k(n)}$ vertices.

In this paper, we focus on induced-universal graphs for bounded-degree graphs. We construct an induced-universal graph for the family $\mathcal{F}_{k,n}$ of n-vertex graphs of degree at most k. For k even, our induced-universal graph has $O(n^{k/2})$ vertices and for k odd our induced-universal graph has $O(n^{\lceil k/2\rceil - 1/k}\log^{2+2/k} n)$ vertices. Our result for graphs with maximum degree $k \equiv 0 \pmod 2$ is deduced from a construction similar to that of [8] but with an improvement of the base graph of the construction (Section 3). Actually, our $\mathcal{F}_{2,n}$-induced-universal graph forming the basis of the construction has $5n/2 + O(1)$ vertices while the best lower bound known for the order of such graphs is $11n/6 + \Omega(1)$. Our result for graphs with maximum degree $k \equiv 1 \pmod 2$ is deduced from a recent result of Alon and Capalbo [1] on universal graphs for bounded-degree graphs, combined with a construction of [9] that gives an interesting connection between induced-universal graphs and universal graphs (Section 4). Given that the best known lower bound for the number of vertices of an $\mathcal{F}_{k,n}$-induced-universal graph is $\Omega(n^{k/2})$ [8], our result for k even is tight up to a multiplicative constant and our result for k odd is equal to $O(n^{1/2-1/k}\log^{2+2/k} n)$ times the lower bound. We also give a generalization of our result for oriented graphs of bounded degree (Section 5). In Section 6, we show how to construct of an induced-universal graph for all orientations of the graphs of a family $\mathcal{F}$, only using a specific $\mathcal{F}$-induced-universal graph. We conclude the paper with some open problems (Section 7).

2 A small induced-universal graph for graphs with degree at most two

Our main concern here is to find an $\mathcal{F}_{k,n}$-induced-universal graphs for every k. We first investigate the case $k = 2$.

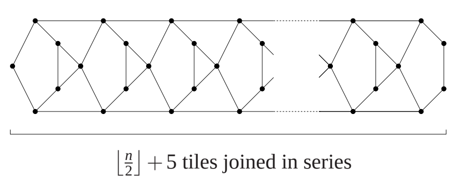


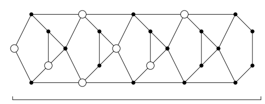
Figure 1: The $\mathcal{F}_{2,n}$-induced-universal graph U_n.

Lemma 1 *The graph U_n depicted in Figure 1 is an $\mathcal{F}_{2,n}$-induced-universal graph.*

Proof. It is sufficient to prove that any graph $G \in \mathcal{F}_{2,n}$ is an induced subgraph of the graph U_n depicted in Figure 1. For $1 \leq i \leq n$, let n_i be the number of connected components of G

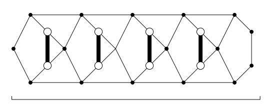
with i vertices. The degree of G is bounded by 2 so G contains n_1 isolated vertices, n_2 disjoint K_2's, and for $i \geq 3$, n_i cycles or paths of i vertices. We embed the connected components of G into U_n from left to right after having sort them by increasing size. The graph U_n is made of cycles of size 5 called *tiles* that are joined in series by 4 edges. Let us prove that we can embed all the connected components of G in an induced way using at most $\lfloor \frac{n}{2} \rfloor + 5$ tiles.

- The embedding of the stable set of size n_1, using $\lceil \frac{n_1}{2} \rceil + 1$ tiles.



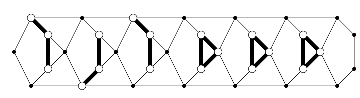
$\lceil \frac{n_1}{2} \rceil + 1$ tiles

- The embedding of n_2 K_2's, using $n_2 + 1$ tiles.



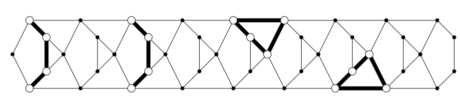
$n_2 + 1$ tiles

- The embedding of n_3 connected components of size 3, using $n_3 + 1$ tiles.



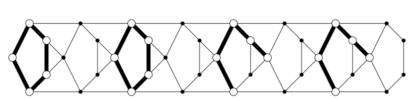
$n_3 + 1$ tiles

- The embedding of n_4 connected components of size 4, using $2n_4 + 1$ tiles.



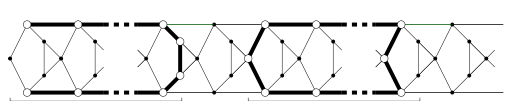
$2n_4 + 1$ tiles

- The embedding of n_5 connected components of size 5, using $2n_5$ tiles.



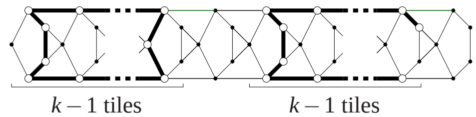
$2n_5$ tiles

- For $k \geq 3$, the embedding of n_{2k} connected components of size $2k$, using kn_{2k} tiles.



$k-1$ tiles $k-1$ tiles

- For $k \geq 3$, the embedding of n_{2k+1} connected components of size $2k+1$, using kn_{2k+1} tiles.



Observe that for each i the embedding of connected components of size i is induced. Moreover, at the end of the embedding of all connected components of size i, there is a tile in which no vertex of G is embedded. So, there are no edges of U_n between the embeddings of two connected components of different sizes. Hence, the embedding of G into U_n is induced. It remains to upper bound the number l of tiles used by such an embedding.

$$
\begin{aligned}
l &= \frac{n_1}{2} + 2 + n_2 + 1 + n_3 + 1 + 2n_4 + 1 + 2n_5 + \sum_{k=3}^{\lfloor n/2 \rfloor} 2kn_{2k} + \sum_{k=3}^{\lfloor n/2 \rfloor} 2kn_{2k+1} \\
&\leq 5 + \sum_{i=1}^{n} i\frac{n_i}{2} \\
&\leq 5 + \left\lfloor \frac{n}{2} \right\rfloor \text{, since } \sum_{i=1}^{n} in_i = n \text{ and the number of tiles is an integer.}
\end{aligned}
$$

□

A natural question is to investigate whether this construction is optimal. We now prove that it is optimal up to a constant multiplicative factor of approximately $\frac{3}{2}$.

Claim 1 *Every $\mathcal{F}_{2,n}$-induced-universal graph has at least $11\left\lfloor\frac{n}{6}\right\rfloor$ vertices.*

Proof. Let $n \in \mathbb{N}$ be a multiple of 6. Let $\mathcal{H}_n$ be the family containing the following three graphs:

- the stable set of n vertices,
- the disjoint union of $n/2$ K_2,
- the disjoint union of $n/3$ K_3.

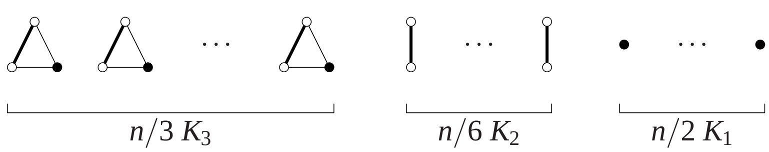


Figure 2: An induced subgraph of U_n.

Note that these three graphs have n vertices and degree at most two. Let U_n be an $\mathcal{H}_n$-induced-universal graph. Then U_n must contain $n/3$ triangles as induced subgraphs.

Since each of the triangles intersects at most one induced K_2, the graph U_n must contain an induced matching of size at least $n/2 - n/3 = n/6$ disjoint from the triangles. Since each K_2 and each triangle contains at most one isolated vertex as an induced subgraph, U_n must contain a stable set of size $n - n/3 - n/6 = n/2$ disjoint from the triangles and the induced matching (see Figure 2). Eventually, U_n has at least $3n/3 + 2n/6 + n/2 = 11n/6$ vertices and so any $\mathcal{F}_{2,n}$-induced-universal graph needs $11\lfloor n/6 \rfloor$ vertices because $\mathcal{H}_{6\lfloor n/6 \rfloor} \subseteq \mathcal{F}_{2,n}$. $\square$

We believe that the results in this section are not sharp. Indeed, we conjecture that there exists an $\mathcal{F}_{2,n}$-induced-universal graph with $2n + o(n)$ vertices, and that this is optimal.

3 Induced-universal graphs for graphs with even maximum degree

We now use our construction of an $\mathcal{F}_{2,n}$-induced-universal graph to construct an $\mathcal{F}_{k,n}$-induced-universal graph for k even (the same method was already used in [8]).

Theorem 1 *Let $k \geq 2$ be an even integer. There is an $\mathcal{F}_{k,n}$-induced-universal graph $U_{k,n}$ such that*

$$|V(U_{k,n})| = (1 + o(1)) \left(\frac{5n}{2}\right)^{k/2} \text{ and } |E(U_{k,n})| = \left(\frac{9k}{10} + o(1)\right) \left(\frac{5n}{2}\right)^{k-1}.$$

Proof. To prove this theorem, we first reduce the problem to the construction of an $\mathcal{F}_{2,n}$-induced-universal. Petersen [14] proved that any k-regular graph with k even can be decomposed into $k/2$ edge-disjoint graphs of degree at most 2. In [9], Chung proved that for two families of graphs $\mathcal{F}$ and $\mathcal{H}$ such that any graph of $\mathcal{F}$ can be decomposed into k graphs of $\mathcal{H}$, if we have an $\mathcal{H}$-induced-universal graph W, we can construct an $\mathcal{F}$-induced-universal graph U such that:

$$|V(U)| = |V(w)|^k \text{ and } |E(U)| = k|V(W)|^{2k-2}|E(W)|.$$

Using Lemma 1, we construct an $\mathcal{F}_{2,n}$-induced-universal graph U_n with $|V(U_n)| = \frac{5}{2}n + O(1)$ and $|E(U_n)| = \frac{9}{2}n + O(1)$. Eventually, using the fact that any graph of $\mathcal{F}_{k,n}$ can be decomposed into $k/2$ graphs of $\mathcal{F}_{2,n}$, we obtain an $\mathcal{F}_{k,n}$-induced-universal graph $U_{k,n}$ such that:

$$\begin{aligned} |V(U_{k,n})| &= |V(U)|^{k/2} = \left(\frac{5}{2}\right)^{k/2} n^{k/2} + o(n^{k/2}) \\ |E(U_{k,n})| &= \frac{k}{2}|V(U)|^{k-2}|E(U)| = \frac{k}{2} \cdot \frac{9}{2} \left(\frac{5}{2}\right)^{k-2} n^{k-1} + o(n^{k-1}). \end{aligned}$$

$\square$

4 Induced-universal graphs for graphs with odd maximum degree

To the best of your knowledge, there is no good result on edge decomposition for graphs belonging to $\mathcal{F}_{k,n}$ with k odd. Nevertheless, we can use $U_{k+1,n}$ as an $\mathcal{F}_{k,n}$-induced-universal graph since $\mathcal{F}_{k,n} \subset \mathcal{F}_{k+1,n}$. The graph obtained is from a multiplicative factor of $O(n^{1/2})$ of the best known lower bound for the number of vertices of $\mathcal{F}_{k,n}$-induced-universal graphs. We now show how to reduce the gap between lower and upper bounds with a construction deduced from universal graphs.

Theorem 2 *Let $k \geq 3$ be an odd integer. There is an $\mathcal{F}_{k,n}$-induced-universal graph $U_{k,n}$ such that*

$$|V(U_{k,n})| = c_1(k)n^{\lceil k/2\rceil - 1/k} \log^{2+2/k} n \text{ and } |E(U_{k,n})| = c_2(k)n^{k-2/k} \log^{4+4/k} n$$

Proof. The induced-universal graph is deduced from the $\mathcal{F}_{k,n}$-universal graph obtained by Alon and Capalbo [1], using a result of Chung [9] that gives a general construction of an induced-universal graph from an universal graph.

The construction of Chung [9] depends on the degree of the induced-universal graph and the arboricity of graphs of the family. Indeed, if we consider a family A_r of graphs with arboricity at most r and an A_r-universal graph G, then the construction produces an A_r-induced-universal graph H such that :

$$|V(H)| = \sum_{v \in V(G)} (d_G(v)+1)^r \text{ and } |E(H)| = \sum_{uv \in E(G)} (d_G(u)+1)^r d_G(v)^{r-1}.$$

The arboricity of graphs of the family $\mathcal{F}_{k,n}$ is at most $\lceil k/2 \rceil$. Moreover, the $\mathcal{F}_{k,n}$-universal graph described in [1] has degree at most $c(k)n^{2-2/k}\log^{4/k} n$. Hence, there is an induced-universal graph $U_{k,n}$ for the family $\mathcal{F}_{k,n} = \mathcal{A}_{\lceil k/2 \rceil}$ such that:

$$\begin{aligned}
|V(U_{k,n})| &= \sum_{v \in V(H_{k,n})} (d_{H_{k,n}}(v)+1)^{\lceil k/2\rceil} \\
&\leq |V(H_{k,n})|(2d_{H_{k,n}})^{\lceil k/2\rceil} \\
&\leq n(2c(k)n^{1-2/k}\log^{4/k} n)^{\lceil k/2\rceil} \\
&\leq c_1(k)n^{\lceil k/2\rceil - 1/k}\log^{2+2/k} n \text{ , where } c_1(k) = (2c(k))^{\lceil k/2\rceil}
\end{aligned}$$

$$\begin{aligned}
|E(U_{k,n})| &= \sum_{uv \in E(H_{k,n})} (d_{H_{k,n}}(u)+1)^{\lceil k/2\rceil} d_{H_{k,n}}(v)^{\lceil k/2\rceil - 1} \\
&\leq |E(H_{k,n})|(2d_{H_{k,n}})^{\lceil k/2\rceil}(d_{H_{k,n}})^{\lceil k/2\rceil - 1} \\
&\leq c(k)n^{2-2/k}\log^{4/k} n(2c(k)n^{1-2/k}\log^{4/k} n)^{\lceil k/2\rceil}(c(k)n^{1-2/k}\log^{4/k} n)^{\lceil k/2\rceil - 1} \\
&\leq c_2(k)n^{k-2/k}\log^{4+4/k} \text{ , where } c_2(k) = (2c(k))^{k+1}.
\end{aligned}$$

□

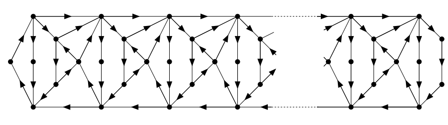
5 Induced-universal graphs for bounded-degree oriented graphs

An *orientation* $\overrightarrow{G}$ of a graph G consists in assigning to every edge of G one of its two possible orientations. $\overrightarrow{G}$ is called an oriented graph and by definition, it cannot have loops nor opposite arcs. The construction of Section 3 can be easily generalized to the family $\mathcal{O}_{k,n}$ of all the orientations of the graphs from $\mathcal{F}_{2k,n}$ having incoming and outgoing degree at most k. Indeed, any graph of $\mathcal{O}_{k,n}$ can be decomposed into k graphs of $\mathcal{O}_{1,n}$ [14] and the construction of induced universal graph using decomposition works in the oriented case.

Theorem 3 *There is an $\mathcal{O}_{k,n}$-induced-universal oriented graph $\overrightarrow{O_{k,n}}$ such that*

$$|V(\overrightarrow{O_{k,n}})| = (1+o(1))\,(3n)^k \text{ and } |E(\overrightarrow{O_{k,n}})| = (2+o(1))\,(3n)^{2k-1}.$$

Proof. The construction of an induced-universal graph for $\mathcal{O}_{k,n}$ is almost the same as the construction for $\mathcal{F}_{2k,n}$ presented in Section 3. Any graphs with outgoing and incoming degree at most k can be decomposed into k edge-disjoint graphs having outgoing and incoming degree at most 1 [14]. Let $\overrightarrow{O_n}$ be the graph depicted in Figure 3. If $\overrightarrow{O_n}$ is $\mathcal{O}_{1,n}$-induced-universal then, using the construction of Chung [9], we can construct an $\mathcal{O}_{k,n}$-induced-universal graph $\overrightarrow{O_{k,n}}$ having $|V(\overrightarrow{O_{k,n}})| = (1+o(1))\,(3n)^k$ vertices and $|E(\overrightarrow{O_{k,n}})| = (2+o(1))\,(3n)^{2k-1}$ edges. So, the only thing we need to prove is that $\overrightarrow{O_n}$ is $\mathcal{O}_{1,n}$-induced-universal.

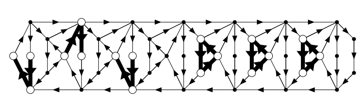


$\lfloor \frac{n}{2} \rfloor + 5$ tiles joined in series

Figure 3: The $\mathcal{O}_{1,n}$-induced-universal graph $\overrightarrow{O_n}$.

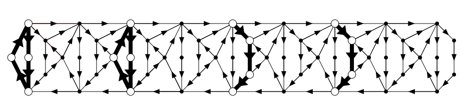
Let $\overrightarrow{G}$ be any graph of $\mathcal{O}_{1,n}$. The connected components of $\overrightarrow{G}$ are either directed paths (oriented paths with exactly one sink and one source) or directed cycles (oriented cycles with no source). We embed $\overrightarrow{G}$ in $\overrightarrow{O_n}$ almost the same way we embedded graphs of $\mathcal{F}_{2,n}$ in U_n in Section 2. The only differences are for the embeddings of connected components of size 3 or more that slightly differ from the non-oriented case.

- The embedding of n_3 connected components of size 3, using n_3+1 tiles.



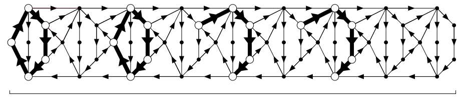
$n_3 + 1$ tiles

- The embedding of n_4 connected components of size 4, using $2n_4+1$ tiles.

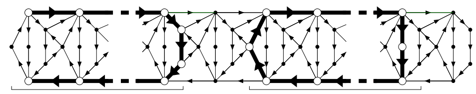


$2n_4 + 1$ tiles

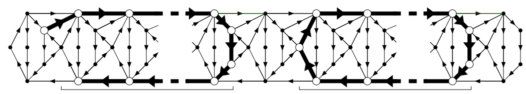
- The embedding of n_5 connected components of size 5, using $2n_5 + 1$ tiles.



- For $k \geq 3$, the embedding of n_{2k} connected components of size $2k$, using kn_{2k} tiles.



- For $k \geq 3$, the embedding of n_{2k+1} connected components of size $2k+1$, using kn_{2k+1} tiles.



We use for embeddings exactly the same number of tiles as for the non-oriented case, so the graph $\overrightarrow{O_n}$ has also $\lfloor \frac{n}{2} \rfloor + 5$ tiles. □

6 From induced-universal graphs to oriented induced-universal graphs

In Section 5, we constructed an induced-universal graph for a family of orientations of graphs in $\mathcal{F}_{2,n}$ by orienting the edges and adding some vertices to the non-oriented induced-universal graph. Let $\mathcal{F}$ be a family of graph and $\overrightarrow{\mathcal{F}}$ be a family of orientations of graphs from $\mathcal{F}$. One may ask if, taking an $\mathcal{F}$-induced-universal graph U, it is always possible to construct an $\overrightarrow{\mathcal{F}}$-induced-universal graph $\overrightarrow{U}$.

Given two graphs G and H, a *homomorphism* from G to H is a mapping $f : V(G) \rightarrow V(H)$ satisfying $[x, y] \in E(G) \Rightarrow [f(x), f(y)] \in E(H)$. In fact, the construction is possible if there is a graph $\overrightarrow{H}$ into which each graph of $\overrightarrow{\mathcal{F}}$ has a homomorphism. In this case, the graph $\overrightarrow{H}$ is said to be an $\overrightarrow{\mathcal{F}}$*-universal graph for homomorphism*. For instance, the directed cycle of length three is a universal graph for homomorphism for the family of orientation of trees. The graph $\overrightarrow{U}$ can be obtained by making a special product of the two graphs $\overrightarrow{H}$ and U. The *oriented tensor product* $G \times \overrightarrow{H}$ of a non-oriented graph G and an oriented graph $\overrightarrow{H}$ is defined to have vertex set $V(G \times \overrightarrow{H}) = V(G) \times V(\overrightarrow{H})$ and arc set $E(G \times \overrightarrow{H}) = \left\{[(x, u), (y, v)] \mid xy \in E(G) \text{ and } uv \in E(\overrightarrow{H})\right\}$.

Theorem 4 *Let U and $\overrightarrow{H}$ be two graphs. If U is $\mathcal{F}$-induced-universal and $\overrightarrow{H}$ is $\overrightarrow{\mathcal{F}}$-universal for homomorphism then $U \times \overrightarrow{H}$ is $\overrightarrow{\mathcal{F}}$-induced-universal.*

Proof. It suffices to show that we can embed an arbitrary graph $\overrightarrow{G} \in \overrightarrow{\mathcal{F}}$ as an induced subgraph of $U \times \overrightarrow{H}$. Let $v \in \overrightarrow{G}$. There is a homomorphism of $\overrightarrow{G}$ to $\overrightarrow{H}$ since $\overrightarrow{H}$ is $\overrightarrow{\mathcal{F}}$-universal for homomorphism. We denote by $h(v) \in V(\overrightarrow{H})$ the vertex into which v is mapped. If we forget about the orientation, we can embed $\overrightarrow{G}$ into U since U is $\mathcal{F}$-induced-universal. Let denote by $u(v) \in V(U)$ the vertex into which v is embedded. The embedding of $\overrightarrow{G}$ into $U \times \overrightarrow{H}$ consists in embedding each vertex v of G into the vertex $(u(v), h(v))$ of $U \times \overrightarrow{H}$. The embedding is correct in the sense that if there is an arc $[x, y]$ in $\overrightarrow{G}$ then there is an arc $[(u(x), h(x)), (u(y), h(y))]$ in $U \times \overrightarrow{H}$. Indeed, there is an edge $[u(x), u(y)]$ in U due to the non-oriented embedding of $\overrightarrow{G}$ into U and an arc $[h(x), h(y)]$ in $\overrightarrow{H}$ due to the mapping of $\overrightarrow{G}$ into $\overrightarrow{H}$. Moreover, the embedding is induced. Indeed, if two vertices x and y of G are not adjacent then $u(x)$ and $u(y)$ are not adjacent in U because the non-oriented embedding of $\overrightarrow{G}$ into U is induced. So, by construction, $(u(x), h(x))$ and $(u(y), h(y))$ are not adjacent in $U \times \overrightarrow{H}$. □

Families such as trees, planar graphs, partial 2-trees, outerplanar graphs, and subcubic graphs are known to have universal graphs for homomorphism with constant number of vertices [5, 15]. So for these families, induced-universal graphs and induced-universal oriented graphs have asymptotically the same order.

7 Concluding remarks and open problems

In Section 2, we proved that a minimal $\mathcal{F}_{2,n}$-induced-universal has at least $5n/2 + O(1)$, and and at most $11n/6 + O(1)$ vertices. The natural question that arises is whether it is possible to reduce the gap between 5/2 and 11/6 for the multiplicative constant. This question seems to be quite difficult, even though graphs of $\mathcal{F}_{2,n}$ have a very simple structure. For k odd, if we drop the polylogarithmic factor, there remains a multiplicative factor of $n^{1/2-1/k}$ between the lower and the upper bound for the number of vertices in a minimal $\mathcal{F}_{k,n}$-induced-universal graph. An interesting problem would be to lower this factor, especially for large values of k. In our construction, for k even, our $\mathcal{F}_{k,n}$-induced-universal graph have maximum degree $4^{k/2}$ depending only on k whereas for k odd, it has maximum degree $c_2(k)n^{k-1-2/k}\log^{4+4/k} n$. Considering that for k even our construction is almost tight whereas for k odd it is not, we conjecture that $\mathcal{F}_{k,n}$-induced-universal graphs with minimal number of vertices and edges have degree only depending on k. In other words, we conjecture that there is a function $f(k)$ such that the existence of a $\mathcal{F}_{k,n}$-induced-universal graph $U_{k,n}$ implies that there exists another one with at most the same number of vertices, but with degree at most $f(k)$.

A more general problem concerning induced-universal graphs should be to solve the induced-universal version of the implicit graph conjecture of Kannan, Naor and Rudich [12]:

Conjecture 1 (Implicit Graph Conjecture (induced-universal version)) *Every hereditary class of graphs which contains $2^{O(n \log n)}$ graphs on n vertices admits an induced-universal graph with $n^{O(1)}$ vertices.*

Solving this conjecture seems rather difficult even if it is known that families of graphs closed by taking minor fulfill the conjecture since they admit induced-universal graph of $n^{O(1)}$ vertices.

References

[1] N. Alon and M. Capalbo, *Sparse universal graphs for bounded degree graphs*, Random Structures and Algorithms **31** (2007), 123–133.

[2] S.N. Bhatt, F.R.K. Chung, F.T. Leighton, and A.L. Rosenberg, *Optimal simulations of tree machines*, In *27th IEEE Foundations of Computer Science* (1986), 274–282.

[3] S.N. Bhatt, F.R.K. Chung, F.T. Leighton, and A.L. Rosenberg, *Universal graphs for bounded-degree trees and planar graphs*, SIAM J. Discrete Math. **2** (1989), 145–155.

[4] S.N. Bhatt and C.E. Leiserson, *How to assemble tree machines*, F. Preparata, 1984.

[5] O.V. Borodin, A.V. Kostochka, J. Nešetřil, A. Raspaud, and E. Sopena *On universal graphs for planar oriented graphs of a given girth*, Discrete Math. **188** (1998), 73–85.

[6] M.A. Breuer, *Coding the vertexes of a graph*, IEEE Transactions on Information Theory **12** (1966), 148–153.

[7] M.A. Breuer and J. Folkman, *An unexpected result on coding the vertices of a graph*, J. Math. Analysis and Applications **20** (1967), 583–600.

[8] S. Butler, *Induced-universal graphs for graphs with bounded maximum degree*, preprint.

[9] F.R.K. Chung, *Universal graphs and induced-universal graphs*, J. Graph Theory **14** (1990) 443–454.

[10] F.R.K. Chung and R.L. Graham, *On universal graphs for spanning trees*, J. London Math. Soc. **27** (1983) 203–211.

[11] R. Diestel, *Graph Theory*, 3rd edition. Springer-Verlag, Berlin and Heidelberg, 2005.

[12] S. Kannan, M. Naor, and S. Rudich, *Implicit representation of graphs*, SIAM J. Discrete Math. **5** (1992), 596–603.

[13] J.W. Moon, *On minimal n-universal graphs*, In *Proc. Glasgow Math. Soc.* **7** (1965), 32–33.

[14] J. Petersen, *Die Theorie der regulären Graphs*, Acta Math. **15** (1891) 193–220.

[15] E. Sopena, *The chromatic number of oriented graphs*, J. Graph Theory **25** (1997) 191–205.

Appendix D

[AEK$^+$07]

Acyclic improper colourings of graphs with bounded maximum degree

Abstract

For graphs of bounded maximum degree, we consider acyclic t-improper colourings, that is, colourings in which each bipartite subgraph consisting of the edges between two colour classes is acyclic and each colour class induces a graph with maximum degree at most t.

We consider the supremum, over all graphs of maximum degree at most d, of the acyclic t-improper chromatic number and provide t-improper analogues of results by Alon, McDiarmid and Reed (1991, RSA 2(3), 277–288) and Fertin, Raspaud and Reed (2004, JGT 47(3), 163–182).

Acyclic improper colourings of graphs with bounded maximum degree

Louigi Addario-Berry [†§] Louis Esperet [‡¶]

Ross J. Kang [†‖] Colin J. H. McDiarmid [†]

Alexandre Pinlou [‡¶]

April 7, 2008

Abstract

For graphs of bounded maximum degree, we consider acyclic t-improper colourings, that is, colourings in which each bipartite subgraph consisting of the edges between two colour classes is acyclic and each colour class induces a graph with maximum degree at most t.

We consider the supremum, over all graphs of maximum degree at most d, of the acyclic t-improper chromatic number and provide t-improper analogues of results by Alon, McDiarmid and Reed (1991, RSA 2(3), 277–288) and Fertin, Raspaud and Reed (2004, JGT 47(3), 163–182).

[†]Department of Statistics, 1 South Parks Road, Oxford OX1 3TG, UK

[‡]LaBRI, Université Bordeaux 1, 351 Cours de la Libération, 33405 Talence Cedex, France

[§]This author is supported by the Fonds québécois de la recherche sur la nature et les technologies.

[¶]Part of this work was supported by the European training network COMBSTRU (Combinatorial Structure of Intractable Problems), contract grant n° HPRN CT 2002 00278.

[‖]This author is partially supported by NSERC of Canada and the Commonwealth Scholarship Commission (UK).

[**]laddar@cs.mcgill.ca, esperet@labri.fr, kang@stats.ox.ac.uk, cmcd@stats.ox.ac.uk, pinlou@labri.fr

1 Introduction

Given a graph $G = (V, E)$, a proper colouring $\mathcal{V} = (V_1, \ldots, V_k)$ of V is *acyclic* if for all $1 \leq i < j \leq k$, the subgraph of G induced by $V_i \cup V_j$, which we denote $G[V_i \cup V_j]$, contains no cycles (i.e., is a forest). The *acyclic chromatic number* $\chi_a(G)$ is the smallest value k for which there exists a proper acyclic k-colouring of G. It is easily seen that $\chi_a(G) \leq \Delta(G)(\Delta(G) - 1) + 1$, as any proper colouring of the square G^2 of G is *de facto* a proper acyclic colouring of G, and G^2 has maximum degree at most $\Delta(G)(\Delta(G) - 1)$. In 1976, Erdős (see (cf. [1])) conjectured that $\chi_a(G) = o(\Delta(G)^2)$; this conjecture was proved by Alon et. al. [2], who showed the existence of a fixed constant $c < 50$ such that for all G, $\chi_a(G) \leq c\Delta(G)^{4/3}$. Alon et. al. also showed that their bound was close to optimal by proving via probabilistic arguments that

$$\max\{\chi_a(G) : \Delta(G) \leq \Delta\} = \Omega\left(\frac{\Delta^{4/3}}{(\log \Delta)^{1/3}}\right).$$

When studying the asymptotics of $\chi_a(G)$ in terms of $\Delta(G)$, the restriction that the colouring be *proper* is not of great importance. Indeed, suppose we define the *laid-back acyclic chromatic number* $\chi_\ell(G)$ to be the smallest value k for which there exists a colouring $\mathcal{V} = (V_1, \ldots, V_k)$ of G such that, for all $1 \leq i < j \leq k$, $G[V_i \cup V_j]$ is a forest (placing no further restriction on edges within a given block $G[V_i]$). Clearly, $\chi_\ell(G) \leq \chi_a(G)$. On the other hand, given such a colouring, it follows in particular that for all $1 \leq i \leq k$, $G[V_i]$ is a forest, so $\chi(G[V_i]) \leq 2$. By splitting V_i into stable sets $V_i^{(1)}$ and $V_i^{(2)}$ (for each $1 \leq i \leq k$), we may then obtain an acyclic *proper* colouring of G with at most $2k$ colours. It follows that $\chi_a(G)$ and $\chi_\ell(G)$ are within a factor of two of each other.

In this paper we investigate another relaxation of the acyclic chromatic number; in order to define it we first note that we may reformulate the definition of $\chi_a(G)$ by observing that if V_i and V_j are distinct stable sets in G, then $G[V_i \cup V_j]$ is exactly the bipartite graph $G[V_i, V_j]$ containing all edges with one endpoint in V_i and one endpoint in V_j. We may then equivalently define $\chi_a(G)$ as the smallest value k for which there exists a proper colouring $\mathcal{V} = (V_1, \ldots, V_k)$ of V such that for all $1 \leq i < j \leq k$, $G[V_i, V_j]$ is a forest (i.e. such that with this colouring, G contains no *alternating cycle*).

Starting from this definition, we may now relax the requirement that $\mathcal{V}$ be a proper colouring while continuing to impose that G contain no alternating cycle. To wit: given an integer $t \geq 0$, we say that a colouring $\mathcal{V} = (V_1, \ldots, V_k)$ is *t-improper* if for all $1 \leq i \leq k$, $G[V_i]$ has maximum degree at most t (in this case we say that V_i is *t-dependent*, for each $1 \leq i \leq t$). The *t-improper acyclic chromatic number* $\chi_a^t(G)$ is the smallest k for which there exists a t-improper colouring $\mathcal{V} = (V_1, \ldots, V_k)$ such that with this colouring, G contains no alternating cycle.

For an integer $d \geq 0$, we let

$$\chi_a^t(d) = \max\{\chi_a^t(G) : \Delta(G) \leq d\}.$$

The object of this paper is to study how $\chi_a^t(d)$ varies as a function of t and of d. Clearly, for any d, $\chi_a^0(d) \geq \chi_a^1(d) \geq \ldots \geq \chi_a^d(d) = 1$.

It is easily seen that $\chi_a^t(d) = \Omega\left((d/t)^{4/3}/(\ln d)^{1/3}\right)$; given an acyclic t-improper colouring, by applying the first of the results from [2] mentioned above, we can acyclically colour each colour class with at most $ct^{4/3}$ new colours (where c is some fixed constant which is less than 50) to obtain an acyclic colouring of the entire graph. Our first result is to show that this straightforward lower bound on $\chi_a^t(d)$ can be much improved upon asymptotically, as long as $t \leq d - 10\sqrt{d \ln d}$. More fully,

Theorem 1. *If $t \leq d - 10\sqrt{d \ln d}$, then $\chi_a^t(d) = \Omega\left((d-t)^{4/3}/(\ln d)^{1/3}\right)$.*

In particular, if $t = (1-\varepsilon)d$ for any fixed constant ε, $0 < \varepsilon \leq 1$, then we obtain the same asymptotic lower bound as Alon *et al.* Comparing this lower bound with the upper bound $\chi_a^t(d) = O(d^{4/3})$, we see the surprising fact that even allowing $t = \Omega(d)$ does not greatly reduce the number of colours needed for improper acyclic colourings of graphs with large maximum degree.

At some point, $\chi_a^t(d)$ must drop significantly as t increases, because $\chi_a^d(d) = 1$. Although we are unable to pin down the behaviour of $\chi_a^t(d)$ viewed as a function of t, we can improve upon the upper bound of Alon *et al.* when t is very close to d (more precisely, when $d - t = o(d^{1/3})$). We prove:

Theorem 2. $\chi_a^t(d) = O(d \ln d + (d-t)d)$.

As for lower bounds on $\chi_a^t(d)$ when $d - t = o(d)$, we first note that [3] showed $\chi_a^{d-2}(d) \geq 3$; we can straightforwardly generalise this result by showing that $\chi_a^t(d) \geq d - t + 1$. This is done as follows: if K_{d+1} is the complete graph on $d + 1$ vertices, then $\chi_a^t(K_{d+1}) \geq d - t + 1$, since, in any acyclic t-improper colouring of K_{d+1}, at most one colour class has more than one vertex and no colour class has more than $t + 1$ vertices. We can, however, improve upon this further and, in the final section, we exhibit a set of examples showing the following lower bound.

Theorem 3. $\chi_a^{d-1}(d) = \Omega(d^{2/3})$.

We would like to reduce the gaps between the lower and upper bounds on $\chi_a^t(d)$. For $t = d - 1$, the problem is particularly tantalising, and, in this case, the lower bound of Theorem 3 and the upper bound of Theorem 2 differ by a factor of $d^{1/3} \ln d$. For this choice of t, the problem also includes the conjecture from [3] that every subcubic graph is acyclically 2-improperly 2-colourable.

In the rest of the paper, we use the following notation. The degree of a given vertex v is denoted by $d(v)$. We denote by $N(v)$ the set of the neighbours of v. A k-cycle (resp. a ${}^{\geq}k$*-cycle*) is a cycle containing k vertices (resp. at least k vertices). For a graph G and a vertex $v \in V(G)$, we denote by $G \setminus \{v\}$ the graph obtained from G by removing v and its incident edges; for an edge uv of $E(G)$, $G \setminus \{uv\}$ denotes the graph obtained from G by removing the edge uv. These notions are extended to sets of vertices and edges in an obvious way. Let G be a graph and f be a colouring of G. For a given vertex v of G, we denote by $\mathrm{im}_f(v)$, or simply $\mathrm{im}(v)$ when the colouring is clear from the context, the number of neighbours of v having the same colour as v and call this quantity the *impropriety* of the vertex v. For notation not defined here, we refer the reader to [9].

2 A probabilistic lower bound for $\chi_a^t(d)$

In this section, we prove Proposition 6 below, a more explicit version of Theorem 1. Our argument mirrors that of Alon *et al.* but uses upper bounds on the *t-dependence number* α^t, the size of a largest t-dependent set, in the random graph $G_{n,p}$. For more precise upper bounds on $\alpha^t(G_{n,p})$, consult [7].

Lemma 4. *Fix an integer $n \geq 1$ and $p \in \mathbb{R}$ with $4(\ln n/n)^{1/4} \leq p \leq 1$. Let $m = \lfloor n - 128 \ln n/p^4 \rfloor$. Then asymptotically almost surely and uniformly over*

p in the above range, any colouring of $G_{n,p}$ with $k \le (n-m)/4$ colours and in which each colour class contains at most m vertices contains an alternating 4-cycle.

Proof. As there are at most $k^n \le n^n$ possible k-colourings of $G_{n,p}$, to prove the lemma it suffices to show that for any fixed k-colouring of the vertices of $G_{n,p}$ (which we denote $\{v_1, \ldots, v_n\}$) with colour classes $C_1, \ldots, C_k$ in which $|C_i| \le m$ for all $1 \le i \le k$, the probability that $G_{n,p}$ does not contain an alternating 4-cycle is $o(n^{-n})$.

Fix a colouring as above, and let q be minimal such that $|C_1 \cup \ldots \cup C_q| \ge (n-m)/2$. Let $A = C_1 \cup \ldots \cup C_q$ and let $B = C_{q+1} \cup \ldots \cup C_k$. As no colour class has size greater than m, $|A| \le (n+m)/2$ and so $|B| \ge (n-m)/2$. By symmetry, we may also assume that $|A| \ge n/2$.

Next, let $P = \{\{x_1, x_1'\}, \ldots, \{x_r, x_r'\}\}$ be a maximal collection of pairs of elements of A such that for $1 \le i \le r$, x_i and x_i' have the same colour, and for $1 \le i < j \le r$, $\{x_i, x_i'\}$ and $\{x_j, x_j'\}$ are disjoint. As we may place all but perhaps one vertex from each colour class C_i in some such pair (with one vertex left over precisely if $|C_i|$ is odd), it follows that

$$r \ge \frac{1}{2}\left(|A| - q\right) \ge \frac{1}{2}\left(\frac{n}{2} - k\right) \ge \frac{n}{8}.$$

Similarly, let $Q = \{\{y_1, y_1'\}, \ldots, \{y_s, y_s'\}\}$ be a maximal collection of pairs of elements of B satisfying identical conditions; by an identical argument to that above, it follows that $s \ge (n-m)/8$.

Let E be the event that for all $1 \le i \le r$, $1 \le j \le s$, $\{x_i, y_j, x_i', y_j'\}$ is not an alternating 4-cycle, and let E' be the event that $G_{n,p}$ contains no alternating 4-cycle; clearly $E' \subseteq E$. For *fixed* $1 \le i \le r$ and $1 \le j \le s$, the probability that $\{x_i, y_j, x_i', y_j'\}$ is not an alternating 4-cycle is $(1-p^4)$ and this event is independent from all other such events. As $(n-m) \ge 128 \ln n / p^4$ it follows that

$$\begin{aligned}\mathbf{Pr}\left(E'\right) &\le \mathbf{Pr}\left(E\right) \le (1-p^4)^{rs} \le e^{-p^4 rs} \\ &\le \exp\left\{-\frac{p^4 n(n-m)}{64}\right\} \le e^{-2n\ln n} = o(n^{-n}),\end{aligned}$$

as required. □

Using this lemma, we next bound the acyclic t-improper chromatic number of $G_{n,p}$ for p in the range allowed in Lemma 4.

Lemma 5. *Fix an integer $n \geq 1$ and $p \in \mathbb{R}$ with $4(\ln n/n)^{1/4} \leq p \leq 1$. Let $m = \lfloor n - 128 \ln n/p^4 \rfloor$ and let $t(n,p) = p(m-1) - 2\sqrt{np}$. Then asymptotically almost surely, for all integers $t \leq t(n,p)$, $\chi_a^t(G_{n,p}) \geq 32 \ln n/p^4$, uniformly over p and t in the above ranges.*

Proof. Fix n and p as above, and choose $t \leq t(n,p)$. We will show that asymptotically almost surely $G_{n,p}$ contains no t-dependent set of size greater than m, from which the claim follows immediately by applying Lemma 4 as $(n-m)/4 \geq 32 \ln n/p^4$. Let $G[m]$ represent the subgraph of $G_{n,p}$ induced by $\{v_1, \ldots, v_m\}$. By a union bound and symmetry, we have

$$\mathbf{Pr}\left(\alpha^t(G_{n,p}) \geq m\right) \leq \binom{n}{m} \mathbf{Pr}\left(\Delta(G[m]) \leq t\right) \leq 2^n \mathbf{Pr}\left(\Delta(G[m]) \leq t\right).$$

Since, if $\Delta(G[m]) \leq t$ then $G[m]$ has at most $tm/2$ edges, it follows that

$$\begin{aligned}\mathbf{Pr}\left(\alpha^t(G_{n,p}) \geq m\right) &\leq 2^n \mathbf{Pr}\left(E(G[m]) \leq \frac{tm}{2}\right) \\ &\leq 2^n \mathbf{Pr}\left(E(G[m]) - p\binom{m}{2} \leq \frac{tm}{2} - p\binom{m}{2}\right)\end{aligned}$$

Finally, by a Chernoff bound and by the definition of $t(n,p)$, we conclude that

$$\begin{aligned}\mathbf{Pr}\left(\alpha^t(G_{n,p}) \geq m\right) &\leq 2^n \exp\left\{-\left(\frac{tm}{2} - p\binom{m}{2}\right)^2 \cdot \left(2p\binom{m}{2}\right)^{-1}\right\} \\ &\leq 2^n \exp\left\{-\frac{(t-p(m-1))^2}{4p}\right\} \leq (2/e)^n = o(1),\end{aligned}$$

as claimed. □

Using Lemma 5, it is a straightforward calculation to bound $\chi_a^t(d)$ for d sufficiently large and t sufficiently far from d.

Proposition 6. *For all sufficiently large integers d and all non-negative integers $t \leq d - 10\sqrt{d \ln d}$,*

$$\chi_a^t(d) \geq \frac{(d-t)^{4/3}}{2^{14}(\ln d)^{1/3}}.$$

Proof. Choose n so that

$$2^{13}n^3 \ln n \leq d^3(d-t) \leq 2^{14}n^3 \ln n; \tag{1}$$

such a choice of n clearly exists as long as d is large enough. Let $p = (d - 4\sqrt{d \ln d})/n$; we first check that p and t satisfy the requirements of Lemma 5. Presuming d is large enough that $np \geq d/2$, by the lower bound in (1) and the fact that $d(d-t) \leq d^2$ we have

$$p \geq \frac{d}{2n} \geq \frac{(d^3(d-t))^{1/4}}{2n} \geq \frac{8n^{3/4}(\ln n)^{1/4}}{2n} = 4\left(\frac{\ln n}{n}\right)^{1/4}. \tag{2}$$

Furthermore, letting $m = \lfloor n - 128 \ln n / p^4 \rfloor$, we have

$$\begin{aligned} p(m-1) - 2\sqrt{np} &\geq np - \frac{128 \ln n}{p^3} - 2\sqrt{np} - 2 = d - 4\sqrt{d \ln d} - 2\sqrt{np} - 2 - \frac{128 \ln n}{p^3} \\ &\geq d - 8\sqrt{d \ln d} - \frac{128 \ln n}{p^3}. \end{aligned} \tag{3}$$

Since $p \geq d/2n$ and by the lower bound in (1),

$$\frac{128 \ln n}{p^3} \leq \frac{2^{10} n^3 \ln n}{d^3} \leq \frac{d-t}{8},$$

which combined with (3) yields

$$\begin{aligned} p(m-1) - 2\sqrt{np} &> d - 8\sqrt{d \ln d} - \frac{(d-t)}{8} \\ &= t + \frac{7(d-t)}{8} - 8\sqrt{d \ln d} > t, \end{aligned} \tag{4}$$

the last inequality holding since $t \leq d - 10\sqrt{d \ln d}$. As (2) and (4) hold we may apply Lemma 5 to bound $\chi_a^t(G_{n,p})$ with this choice of t and p; as $n > d$, it follows that as long as d is sufficiently large,

$$\mathbf{Pr}\left(\chi_a^t(G_{n,p}) \geq \frac{32 \ln n}{p^4}\right) \geq \frac{3}{4}, \tag{5}$$

say. Furthermore, by a union bound and a Chernoff bound,

$$\begin{aligned} \mathbf{Pr}\left(\Delta(G_{n,p}) > d\right) &\leq n\mathbf{Pr}\left(\mathrm{BIN}\left(n, \frac{d - 4\sqrt{d \ln d}}{n}\right) > d\right) \\ &\leq ne^{-16 \ln d/3} \leq \frac{1}{n}, \end{aligned} \tag{6}$$

the last inequality holding as $\ln d \geq \ln n/2$ (which is an easy consequence of (1)). Combining (5) and (6), we obtain that

$$\mathbf{Pr}\left(\chi_a^t(G_{n,p}) \geq \frac{32\ln n}{p^4}, \Delta(G_{n,p}) \leq d\right) \geq \frac{3}{4} - \frac{1}{n} \geq \frac{1}{2}$$

as long as $n \geq 4$, so there is some graph G with maximum degree at most d and with $\chi_a^t(G) \geq 32\ln n/p^4$. Since χ_a^t is monotonically increasing in d, it follows that

$$\chi_a^t(d) \geq \frac{32\ln n}{p^4} > \frac{32n^4\ln n}{d^4}. \tag{7}$$

An easy calculation using the upper bound in (1) and the fact that $\ln n < 2\ln d$ gives the bound

$$d^4 < \frac{2^{19}n^4(\ln d)^{4/3}}{(d-t)^{4/3}},$$

so $32n^4\ln n/d^4 > (d-t)^{4/3}/2^{14}(\ln d)^{1/3}$. By (7), it follows that

$$\chi_a^t(d) \geq \frac{(d-t)^{4/3}}{2^{14}(\ln d)^{1/3}},$$

as claimed. $\square$

3 A probabilistic upper bound for $\chi_a^t(d)$

In this section, we study the situation when $d - t = o(d^{1/2})$. Theorem 2, which improves the upper bound of [2] when $d - t = o(d^{1/3}$, is a corollary of our main result here.

We analyse a different parameter from, but one that is closely related to, the acyclic t-improper chromatic number. A *star colouring* of G is a colouring such that no path of length three (i.e. with four vertices) is alternating; in other words, each bipartite subgraph consisting of the edges between two colour classes is a disjoint union of stars. The *star chromatic number* $\chi_s(G)$ is the least number of colours needed in a proper star colouring of G. We analogously define the parameters $\chi_s^t(G)$ and $\chi_s^t(d)$ in the natural way. The star chromatic number was one of the main motivations for the original study of acyclic colourings [6]. Clearly, any star colouring is acyclic; thus, $\chi_a^t(d) \leq \chi_s^t(d)$. Fertin, Raspaud and Reed [5] showed that $\chi_s(d) = O(d^{3/2})$ and that $\chi_s(d) = \Omega\left(d^{3/2}/(\ln d)^{1/2}\right)$. We note that a natural adaptation to star colouring of the argument given in the last section gives the following:

Theorem 7. *There exists a fixed constant $C > 0$ such that, if $t \leq d - C\sqrt{d \ln d}$, then $\chi_s^t(d) = \Omega\left((d-t)^{3/2}/(\ln d)^{1/2}\right)$.*

Given a graph G of maximum degree d, the idea behind our method for improved upper bounds is to find a dominating set $\mathcal{D}$ and a function $g = g(d) = o(d^{3/2})$ such that $|(N(v) \cup N^2(v)) \cap \mathcal{D}| \leq g$ for all $v \in V(G)$. Given such a set $\mathcal{D}$ in G, we assign colours to the vertices in $\mathcal{D}$ by greedily colouring $\mathcal{D}$ in the square of G (i.e. vertices in $\mathcal{D}$ at distance at most two in G receive different colours) with at most $g+1$ colours; then we give the vertices of $G \setminus \mathcal{D}$ the colour $g+2$. It can be verified that this colouring prevents any alternating paths of length three (and so prevents alternating cycles) and ensures that every vertex has at least one neighbour of a different colour. Furthermore, we can generalise this idea by prescribing that our set $\mathcal{D}$ is *k-dominating* — each vertex outside of $\mathcal{D}$ has at least k neighbours in $\mathcal{D}$ — to give a bound on $\chi_s^{d-k}(d)$.

Theorem 8. $\chi_s^t(d) = O(d \ln d + (d-t)d)$.

This result provides an asymptotically better upper bound than $\chi_s^t(d) = O(d^{3/2})$ when $d - t = o(d^{1/2})$. It also provides a better bound than $\chi_a^t(d) = O(d^{4/3})$ when $d - t = o(d^{1/3})$. Theorem 8 is an easy consequence of the following lemma:

Lemma 9. *Given a d-regular graph G and an integer $k \geq 1$, let $\psi(G,k)$ be the least integer $k' \geq k$ such that there exists a k-dominating set $\mathcal{D}$ for which, for all $v \in V(G)$, $|N(v) \cap \mathcal{D}| \leq k'$. Let $\psi(d,k)$ be the maximum over all d-regular graphs G of $\psi(G,k)$. Then, for all d sufficiently large, $\psi(d,k) \leq \max\{3k, 31 \ln d\}$.*

We postpone the proof of this lemma, first using it to prove Theorem 8:

Proof of Theorem 8. We first remark that the function χ_s^t is monotonic with respect to graph inclusion in the following sense: if G and G' are graphs with $V(G) \subseteq V(G')$, and $E(G) \subset E(G')$, then $\chi_s^t(G) \leq \chi_s^t(G')$. As any graph G of maximum degree d is a subgraph of a d-regular graph (possibly with a greater number of vertices), to prove that $\chi_s^t(d) = O(d \ln d + (d-t)d)$ it therefore suffices to show that $\chi_s^t(G) = O(d \ln d + (d-t)d)$ for d-regular graphs G. We hereafter assume G is d-regular and d is large enough to apply Lemma 9. Let $k = d - t$. We will show that $\chi_s^t(G) \leq d\psi(d,k) + 2$, which proves the theorem.

By Lemma 9, there is a k-dominating set $\mathcal{D}$ such that $|N(v) \cap \mathcal{D}| \leq \psi(d,k)$ for all $v \in V(G)$. Fix such a dominating set $\mathcal{D}$ and form the auxiliary graph H as follows: let H have vertex set $\mathcal{D}$ and let uv be an edge of H precisely if u and v have graph distance at most two in G. As $|N(v) \cap \mathcal{D}| \leq \psi(d,k)$ for all $v \in V(G)$, H has maximum degree at most $d\psi(d,k)$.

To colour G, we first greedily colour H using at most $d\psi(d,k)+1$ colours, and assign each vertex v of $\mathcal{D}$ the colour it received in H. We next choose a new colour not used on the vertices of $\mathcal{D}$, and assign this colour to all vertices of $V(G) \setminus \mathcal{D}$. We remind the reader that $\mathrm{im}(v)$ denotes the number of neighbours of v of the same colour as v. If $v \in \mathcal{D}$ then $\mathrm{im}(v) = 0$, and if $v \in V \setminus \mathcal{D}$ then $\mathrm{im}(v) \leq d - |N(v) \cap \mathcal{D}| \leq d - k = t$, so the resulting colouring is t-improper.

Furthermore, given any path $P = v_1 v_2 v_3 v_4$ of length three in G, either two consecutive vertices v_i, v_{i+1} of P are not in $\mathcal{D}$ (in which case $c(v_i) = c(v_{i+1})$ and P is not alternating), or two vertices v_i, v_{i+2} are in $\mathcal{D}$ (in which case $c(v_i) \neq c(v_{i+2})$ and P is not alternating). Thus, the above colouring is a star colouring G of impropriety at most t and using at most $d(3k + 31 \ln d) + 2$ colours; as G was an arbitrary d-regular graph, it follows that $\chi_s^t(d) \leq d\psi(d,k) + 2$, as claimed. □

We next prove Lemma 9 with the aid of the following symmetric version of the Lovász Local Lemma:

Lemma 10 ([4], cf. [8], page 40)**.** *Let $\mathcal{A}$ be a set of bad events such that for each $A \in \mathcal{A}$*

1. $\mathbf{Pr}(A) \leq p < 1$, *and*
2. *A is mutually independent of a set of all but at most δ of the other events.*

If $4p\delta \leq 1$, then with positive probability, none of the events in $\mathcal{A}$ occur.

Proof of Lemma 9. We may clearly assume that k is at least $(31/3)\ln d$, since, if the claim of the lemma holds for such k, then it also holds for smaller k. Let $p = 2k/d$ and let $\mathcal{D}$ be a random set obtained by independently choosing each vertex v with probability p. We claim that, with positive probability, $\mathcal{D}$ is a k-dominating set such that $|N(v) \cap \mathcal{D}| \leq 3k$ for all $v \in V(G)$; we will prove our claim using the local lemma.

For $v \in V(G)$, let A_v be the event that either $|N(v) \cap \mathcal{D}| < k$ or $|N(v) \cap \mathcal{D}| > 3k$. By the mutual independence principle, cf. [8], page 41, A_v is mutually independent of all but at most d^2 events A_w (with $w \neq v$). Furthermore, since $|N(v) \cap \mathcal{D}|$ has a binomial distribution with parameters d and p, we have by a Chernoff bound that

$$\mathbf{Pr}\,(A_v) = \mathbf{Pr}\,(||N(v) \cap \mathcal{D}| - \mathbf{E}(|N(v) \cap \mathcal{D}|)| > k) \leq 2e^{-k/5} = o(d^{-2})$$

so $4\mathbf{Pr}\,(A_v)\,d^2 < 1$ for d large enough. By applying Lemma 10 with $\mathcal{A} = \{A_v \mid v \in V\}$, it follows that with positive probability none of the events A_v occur, i.e. $\mathcal{D}$ has the desired properties. $\square$

4 A deterministic lower bound for $\chi_a^{d-1}(d)$

In this section, we concentrate on the case $t = d - 1$ and exhibit an example G_n which gives the asymptotic lower bound of Theorem 3. Given a positive integer n, we construct the graph G_n as follows: G_n has vertex set $\{v_{ij} : i, j \in \{1, \dots n\}\} \cup \{w_{ij} : i, j \in \{1, \dots, n\}\}$. For $i, j \in \{1, \dots, n\}$ we let $\mathcal{V}_{ij} = \{v_{ij}, w_{ij}\}$. We can think of the set of vertices as an n-by-n matrix, each entry of which has been "doubled". Within each column $\mathcal{C}_i = \bigcup_{j=1}^{n} \mathcal{V}_{ij}$ and within each row $\mathcal{R}_j = \bigcup_{i=1}^{n} \mathcal{V}_{ij}$ we add all possible edges. The graph G_n has $2n^2$ vertices and is regular with degree $d = 4n - 3$. We will prove the following proposition, which directly implies Theorem 3:

Proposition 11. $\chi_a^{d-1}(G_n) \geq \frac{n}{n^{1/3}+1} + 1$.

Proof. Let $f : G_n \to \{1, \dots, k\}$ be an acyclic $(d-1)$-improper colouring of G_n; we will show that necessarily $k \geq \frac{n}{n^{1/3}+1}$. Since $n \geq 1$ it follows that $n/2 \geq \frac{n}{n^{1/3}+1}$ and thus we may assume that $k < n/2$. Clearly, some colour – say a_1 – appears on two vertices x, x' of $\mathcal{C}_1$. We call the colour a_1 "black" and refer to vertices receiving colour a_1 as black vertices. If $y, y' \in \mathcal{C}_1$ both receive colour $i \neq a_1$, then $xyx'y'$ forms an alternating cycle, so a_1 is the only colour appearing twice in $\mathcal{C}_1$. It follows that at most $k-1$ vertices in $\mathcal{C}_1$ are not black.

Applying the same logic to any column $\mathcal{C}_i$, we see that all but $k-1$ vertices in $\mathcal{C}_i$ must receive the same colour, say a_i. Since $k < n/2$, it is easily seen, then, that there must be a row $\mathcal{R}_m$ such that v_{m1} and w_{m1} are both black, and v_{mi} and w_{mi} are both coloured a_i. This implies that $a_i = a_1$, since otherwise $v_{m1}v_{mi}w_{m1}w_{mj}$ would be an alternating cycle. It follows that in all columns, at most $k-1$ vertices receive a colour other than a_1. Symmetrically, there is a colour b such that in all rows, at most $k-1$ vertices receive a colour other than b; clearly, it must the case that $b = a_1$.

If there are $i, j \in \{1, \ldots, n\}$ such that both $\mathcal{R}_i$ and $\mathcal{C}_j$ are entirely coloured black, then all the neighbours of v_{ij}, w_{ij} are coloured with a_1 and the colouring is not $(d-1)$-improper; therefore, it must be the case that either all rows, or all columns, contain a non-black vertex. Without loss of generality, we may assume that all rows contain a non-black vertex.

Let $x_1, \ldots, x_r$ be non-black vertices receiving the same colour, say a, and let $x_i \in \mathcal{V}_{\ell_i, m_i}$, for $1 \leq i \leq r$. As previously noted, no two of $x_1, \ldots, x_r$ may lie in the same row or column; i.e., for $i \neq j$, $\ell_i \neq \ell_j$ and $m_i \neq m_j$.

Claim 1. *At least $3\binom{r}{2}$ vertices of $\bigcup_{1 \leq i \neq j \leq r} \mathcal{V}_{\ell_i, m_j}$ receive a non-black colour other than a.*

Proof. No vertices in $\bigcup_{1 \leq i \neq j \leq r} \mathcal{V}_{\ell_i, m_j}$ receive colour a as each such vertex is in the same row as one of $x_1, \ldots, x_r$. On the other hand, for each pair i, j with $1 \leq i < j \leq r$, at least three of the vertices in $\mathcal{V}_{\ell_i, m_j} \cup \mathcal{V}_{\ell_j, m_i}$ must receive a colour other than a_1. For if $y, y' \in \mathcal{V}_{\ell_i, m_j} \cup \mathcal{V}_{\ell_j, m_i}$ both receive colour a_1, then $x_i y x_j y'$ forms an alternating cycle. The result follows as there are $\binom{r}{2}$ pairs i, j with $1 \leq i < j \leq r$. $\square$

Claim 2. *At least r distinct non-black colours appear on $\bigcup_{1 \leq i < j \leq r} \mathcal{V}_{\ell_i, m_j}$.*

Proof. By an argument just as above, each of $\mathcal{V}_{\ell_1, m_2}, \ldots, \mathcal{V}_{\ell_1, m_r}$ must contain a vertex receiving a colour other than a_1 or a. These colours must all be distinct as $\mathcal{V}_{\ell_1, m_2}, \ldots, \mathcal{V}_{\ell_1, m_r}$ are all contained within $\mathcal{R}_{\ell_1}$. $\square$

Let $\{a_2, a_3, \ldots, a_k\}$ be the set of non-black colours. Let $x_1^2, \ldots, x_{r_2}^2$ be the vertices receiving colour a_2, and for $i = 3, \ldots, k$ let $x_1^i, \ldots, x_{r_i}^i$ be the vertices receiving colour a_i *which are in a different row from all vertices in* $\bigcup_{j<i}\bigcup_{s\le r_j} x^j_s$. As every row contains a non-black vertex, $\sum_{i=2}^k r_i = n$; it is possible that $r_i = 0$ for certain i, if there is a vertex coloured with one of $a_2, \dots, a_i$ in every row.

For $i \in \{2, \dots, k\}$ and $s \in \{1, \dots, r_i\}$, say vertex $x^i_s \in \mathcal{V}_{\ell^i_s, m^i_s}$, and let

$$A_i = \bigcup_{1\le s<t\le r_i} \mathcal{V}_{\ell^i_s, m^i_t} \cup \mathcal{V}_{\ell^i_t, m^i_s}.$$

By Claim 1, at least $3\binom{r_i}{2}$ vertices of A_i are non-black. Furthermore, if $i \neq i'$ then for any $s \in \{1, \dots, r_i\}$, $s' \in \{1, \dots, r_{i'}\}$, x^i_s and $x^{i'}_{s'}$ are in different rows – so A_i and $A_{i'}$ are disjoint. It follows that in $\bigcup_{i=2}^k A_i \cup \{x^i_1, \dots, x^i_{r_i}\}$, at least

$$\sum_{i=2}^{k} \left(3\binom{r_i}{2} + r_i\right) \ge \sum_{i=2}^{k} r_i^2 \tag{8}$$

vertices are non-black. As $\sum_{i=2}^k r_i = n$, it is easily seen that

$$\sum_{i=2}^{k} r_i^2 \ge (k-1)\left(\left\lfloor \frac{n}{k-1} \right\rfloor\right)^2.$$

As there are only $k-1$ non-black colours, it follows that some non-black colour – say a_2 – appears at least $(\lfloor n/(k-1)\rfloor)^2$ times. If $(\lfloor n/(k-1)\rfloor)^2 \ge n^{2/3}$, then by Claim 2, at least $n^{2/3} + 1 > \frac{n}{n^{1/3}+1} + 1$ colours appear on G_n, so we may assume that $n^{2/3} > (\lfloor n/(k-1)\rfloor)^2 \ge (n/(k-1) - 1)^2$. But then $k > \frac{n}{n^{1/3}+1} + 1$, as claimed. □

It is worth noting that the correct asymptotic order of $\chi^{d-1}_a(G_n)$ is unknown; it is even conceivable that $\chi^{d-1}_a(G_n) = \Theta(d)$.

5 Conclusion

In our view, the most surprising result of this paper is that the same asymptotic lower bound for ordinary acyclic chromatic number by Alon *et al.* also holds for the acyclic t-improper chromatic number for any $t = t(d)$ satisfying $d - t = \Theta(d)$. As $\chi_a(G) \ge \chi^t_a(G)$ for any $t \ge 0$, this means that, for $d - t = \Theta(d)$, Theorem 1 is asymptotically tight up to a factor of $(\ln d)^{1/3}$.

In the case that t is very close to d, Theorem 8 improves upon upper bounds for $\chi_a^t(d)$ and $\chi_s^t(d)$ implied by the results of Alon *et al.* and Fertin *et al.*, respectively, giving for instance that $\chi_s^t(d) = O(d \ln d)$ for $d-t = O(\ln d)$. On the other hand, we showed that $\chi_a^{d-1}(d) = \Omega(d^{2/3})$ by a deterministic construction.

<table>
<tr><th rowspan="2">$d-t$</th><th colspan="2">$\chi_a^t(d)$</th><th colspan="2">$\chi_s^t(d)$</th></tr>
<tr><th>lower</th><th>upper</th><th>lower</th><th>upper</th></tr>
<tr><td>$\Theta(d)$</td><td>$\Omega\left(\frac{d^{4/3}}{(\ln d)^{1/3}}\right)$</td><td rowspan="3">$O(d^{4/3})$</td><td>$\Omega\left(\frac{d^{3/2}}{(\ln d)^{1/2}}\right)$</td><td rowspan="2">$O(d^{3/2})$</td></tr>
<tr><td>$\omega(\sqrt{d \ln d})$</td><td>$\Omega\left(\frac{(d-t)^{4/3}}{(\ln d)^{1/3}}\right)$</td><td>$\Omega\left(\frac{(d-t)^{3/2}}{(\ln d)^{1/2}}\right)$</td></tr>
<tr><td>$O(d^{1/2})$</td><td rowspan="3">$\Omega\left(d^{2/3}\right)$</td><td rowspan="3">$\Omega\left(d^{2/3}\right)$</td><td rowspan="2">$O((d-t)d)$</td></tr>
<tr><td>$O(d^{1/3})$</td><td>$O((d-t)d)$</td></tr>
<tr><td>$O(\ln d)$</td><td>$O(d \ln d)$</td><td>$O(d \ln d)$</td></tr>
<tr><td>0</td><td>1</td><td>1</td><td>1</td><td>1</td></tr>
</table>

Table 1: Asymptotic bounds for $\chi_a^t(d)$ and $\chi_s^t(d)$.

There is much remaining work in the case $d - t = o(d)$. Table 1 is a rough summary of the current bounds on $\chi_a^t(d)$ and $\chi_s^t(d)$ when d is large. A case of particular interest to the authors is when $d - t = 1$; in this case, it is unknown if $\chi_a^{d-1}(d)$ is $\Theta(d^{2/3})$, $\Theta(d \ln d)$ or lies somewhere strictly between these extremes.

References

[1] M. O. Albertson and D. M. Berman. The acyclic chromatic number. In *Proceedings of the Seventh Southeastern Conference on Combinatorics, Graph Theory, and Computing (Louisiana State Univ., Baton Rouge, La., 1976)*, pages 51–69. Congressus Numerantium, No. XVII, Winnipeg, Man., 1976. Utilitas Math.

[2] N. Alon, C. J. H. McDiarmid, and B. Reed. Acyclic coloring of graphs. *Random Structures and Algorithms*, 2(3):277–288, 1991.

[3] P. Boiron, É. Sopena, and L. Vignal. Acyclic improper colorings of graphs. *J. Graph Theory*, 32(1):97–107, 1999.

[4] P. Erdős and L. Lovász. Problems and results on 3-chromatic hypergraphs and some related questions. In *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II*, pages 609–627. Colloq. Math. Soc. János Bolyai, Vol. 10. North-Holland, Amsterdam, 1975.

[5] G. Fertin, A. Raspaud, and B. Reed. Star coloring of graphs. *J. Graph Theory*, 47(3):163–182, 2004.

[6] Branko Grünbaum. Acyclic colorings of planar graphs. *Israel J. Math.*, 14:390–408, 1973.

[7] R. J. Kang and C. J. H. McDiarmid. The t-improper chromatic number of random graphs. 2007. In preparation.

[8] M. Molloy and B. Reed. *Graph colouring and the probabilistic method*, volume 23 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, 2002.

[9] Douglas B. West. *Introduction to graph theory, 2nd ed.* Prentice Hall Inc., Upper Saddle River, NJ, 2001.

Appendix E

[EMZ08]

Adapted list colouring of planar graphs

Abstract

Given a (possibly improper) edge-coloring F of a graph G, a vertex coloring of G is *adapted to* F if no color appears at the same time on an edge and on its two endpoints. If for some integer k, a graph G is such that given any list assignment L to the vertices of G, with $|L(v)| \geq k$ for all v, and any edge-coloring F of G, G admits a coloring c adapted to F where $c(v) \in L(v)$ for all v, then G is said to be *adaptably k-choosable*. In this note, we prove that K_5-minor-free graphs are adaptably 4-choosable, which implies that planar graphs are adaptably 4-colorable and answers a question of Hell and Zhu. We also prove that triangle-free planar graphs are adaptably 3-choosable and give negative results on planar graphs without 4-cycle, planar graphs without 5-cycle, and planar graphs without triangles at distance t, for any $t \geq 0$.

Adapted list colouring of planar graphs

Louis Esperet* Mickaël Montassier† Xuding Zhu‡

Abstract

Given a (possibly improper) edge-colouring F of a graph G, a vertex colouring of G is *adapted to* F if no colour appears at the same time on an edge and on its two endpoints. If for some integer k, a graph G is such that given any list assignment L to the vertices of G, with $|L(v)| \geq k$ for all v, and any edge-colouring F of G, G admits a colouring c adapted to F where $c(v) \in L(v)$ for all v, then G is said to be *adaptably k-choosable*. In this note, we prove that K_5-minor-free graphs are adaptably 4-choosable, which implies that planar graphs are adaptably 4-colourable and answers a question of Hell and Zhu. We also prove that triangle-free planar graphs are adaptably 3-choosable and give negative results on planar graphs without 4-cycle, planar graphs without 5-cycle, and planar graphs without triangles at distance t, for any $t \geq 0$.

Keywords: Adapted colouring, list colouring, planar graphs.

Mathematical Subject Classification: 05C15

1 Introduction

The concept of adapted colouring of a graph was introduced by Hell and Zhu in [9], and has strong connections with matrix partition of graphs, graph homomorphisms, and full constraint satisfaction problems [4, 6, 7, 10]. The more general problem of adapted list colouring of hypergraphs was then considered by Kostochka and Zhu in [11], where an application to job assignment problems was also given.

*LaBRI, Université Bordeaux 1, France, E-mail address: esperet@labri.fr

†LaBRI, Université Bordeaux 1, France, E-mail address: montassi@labri.fr

‡Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, Taiwan and National Center for Theoretical Sciences. E-mail address: zhu@math.nsysu.edu.tw. Partially supported by the National Science Council under grant NSC95-2115-M-110-013-MY3.

In this note, we study adapted list colourings of simple graphs. Let G be a simple graph (that is, without loops nor multiple edges), and let $F : E(G) \to \mathbb{N}$ be a (possibly improper) colouring of the edges of G. A k-colouring $c : V(G) \to \{1, \ldots, k\}$ of the vertices of G is *adapted* to F if for every $uv \in E(G)$, $c(u) \neq c(v)$ or $c(v) \neq F(uv)$. In other words, the same colour never appears on an edge and both its endpoints. If there is an integer k such that for any edge colouring F of G, there exists a vertex k-colouring of G adapted to F, we say that G is *adaptably k-colourable.* The smallest k such that G is adaptably k-colourable is called the *adaptable chromatic number* of G, denoted by $\chi_{ad}(G)$.

Note that in [9] and [11], the authors require that the edge colouring F is a k-colouring. Even though we enable F to take any integer value, it is easy to see that our definition is equivalent to the original definition (whereas its extension to adapted list colouring is more natural). Let $L : V(G) \to 2^{\mathbb{N}}$ be a list assignment to the vertices of a graph G, and F be a (possibly improper) edge colouring of G. We say that a colouring c of G adapted to F is an *L-colouring adapted to F* if for any vertex $v \in V(G)$, we have $c(v) \in L(v)$. If for any edge colouring F of G and any list assignment L with $|L(v)| \geq k$ for all $v \in V(G)$ there exists an L-colouring of G adapted to F, we say that G is *adaptably k-choosable.* The smallest k such that G is adaptably k-choosable is called the *adaptable choice number* of G, denoted by $\mathrm{ch}_{ad}(G)$.

Since a proper vertex k-colouring of a graph G is adapted to any edge colouring of G, we clearly have $\chi_{ad}(G) \leq \chi(G)$ and $\mathrm{ch}_{ad}(G) \leq \mathrm{ch}(G)$ for any graph G, where $\chi(G)$ is the usual chromatic number of G, and $\mathrm{ch}(G)$ is the usual choice number of G. Using the Four-Colour Theorem and a theorem of Thomassen [13], this proves that for any planar graph G, $\chi_{ad}(G) \leq 4$ and $\mathrm{ch}_{ad}(G) \leq 5$. In [9], Hell and Zhu proved that there exist planar graphs that are not adaptably 3-colourable, and asked whether it would be possible to prove that every planar graph is adaptably 4-colourable without using the Four-Colour Theorem.

A graph H is called a *minor* of G if a copy of H can be obtained by contracting edges and/or deleting vertices and edges of G. A graph is said to be *H-minor-free* if it does not have H as a minor. Planar graphs are known to be a proper subclass of K_5-minor-free graphs. In this note, we answer to the question of Hell and Zhu by proving the following stronger statement:

Theorem 1 *Every K_5-minor-free graph is adaptably 4-choosable.*

Observe that this does not hold for the usual list colouring, since Voigt [15] proved that there exist planar graphs which are not 4-choosable.

Triangle-free planar graphs are known to be 3-colourable [5, 14] and 4-choosable (it is easy to prove that they are 3-degenerate using Euler Formula). On the other hand Voigt [16] proved that there exist triangle-free planar graphs that are not 3-choosable. In Section 3, we prove the following theorem:

Theorem 2 *Every triangle-free planar graph is adaptably* 3*-choosable.*

In Section 4, we investigate a problem related to a question of Havel [8]. We prove that for all t, there exist planar graph without triangles at distance less than t, which are not adaptably 3-choosable. In Sections 5 and 6, we prove that there exist planar graphs without 4-cycles, and planar graph without 5-cycles, which are not adaptably 3-colourable. These negative results seem to indicate that it may be hard to have a weaker hypothesis in Theorem 2.

2 K_5-minor-free graphs

Theorem 1 is a consequence of Lemma 2.3 in this section. Note that the adaptable 4-choosability of planar graphs can be deduced directly from Lemma 2.1.

Lemma 2.1 *Let G be an edge-coloured plane graph, and let $C = (v_1, \dots, v_k)$ be its outer face. Let ϕ be an adapted colouring of v_1 and v_2. Suppose finally that any vertex $v \in C$ distinct from v_1 and v_2 has a colour list $L(v)$ of size at least three and every vertex $v \in V(G) \setminus C$ has a colour list $L(v)$ of size at least four. Then the colouring ϕ can be extended to an adapted L-colouring of G.*

Proof. We prove this lemma by induction on $|V(G)|$. If $|V(G)| = 3$, the assertion is trivial. Suppose now that $|V(G)| \geq 4$ and assume that the assertion is true for any smaller graphs.

Since the subgraph G_C of G induced by C is an outerplanar graph, it contains two vertices v_i and v_j of degree at most two which are not adjacent in G_C and which are not cut-vertices of G_C. These vertices v_i and v_j are neither cut-vertices of G nor incident to a chord of C, and one of them (say v_i), is distinct from v_1 and v_2. Let $\alpha \in L(v_i)$ be a colour distinct from the colours of the edges v_iv_{i+1}, v_iv_{i-1}. For each neighbour x of v_i not in C, we remove the colour α from the colour list of x. Applying the induction hypothesis to $G \setminus v_i$ and then colouring v_i with α yields an adapted list colouring of G. ∎

Lemma 2.2 *Let G be an edge-coloured plane graph. Suppose that every vertex v of G has a list $L(v)$ of size at least four. Let H be a subgraph of G isomorphic to K_2 or*

K_3, and let ϕ be an adapted L-colouring of H. Then ϕ can be extended to an adapted L-colouring of G.

Proof. Let G be a counterexample with minimum order. If H is isomorphic to K_2, then consider a face incident to H as the outer face and apply Lemma 2.1 to this planar embedding of G.

Assume now that H is isomorphic to K_3 and $V(H) = \{u, v, w\}$. If H is a separating 3-cycle, then let G_1 (resp. G_2) be the graph induced by the vertices of H and the vertices inside (resp. outside) of H. By the minimality of G, extending ϕ to G_1 and to G_2 yields an adapted L-colouring of G. Suppose now that H is not a separating 3-cycle, and assume that H bounds the outer face of G. Let $G' = G \setminus w$ and let L' be the list assignment defined by $L'(x) = L(x) \setminus \{\phi(w)\}$ for every vertex x adjacent to w (and distinct from u, v) and by $L'(x) = L(x)$ for any other vertex distinct from u and v. Lemma 2.1 applied to G' allows to extend ϕ to G. ∎

Lemma 2.3 *Let G be an edge maximal K_5-minor-free graph. Suppose that every vertex v of G has a list $L(v)$ of size at least four. Let H be a subgraph of G isomorphic to K_2 or K_3, and let ϕ be an adapted L-colouring of H. Then ϕ can be extended to an adapted L-colouring of G.*

Proof. Let G be a counterexample with minimum order. Then G is not isomorphic to the Wagner graph (which is 3-regular, and hence adaptably L-colourable given a precolouring of H), and by Lemma 2.2, G is not a planar triangulation. It follows from Wagner's theorem [17], that $G = G_1 \cup G_2$ where G_1, G_2 are proper subgraphs of G such that $G_1 \cap G_2$ is isomorphic to K_2 or K_3. Clearly, $H \subseteq G_1$ or $H \subseteq G_2$. Without loss of generality, assume that $H \subseteq G_1$. By minimality of G, we can extend ϕ to G_1. This gives an adapted colouring to $G_1 \cap G_2$ which can be extended to G_2, by the minimality of G. This yields an extension of ϕ to an adapted L-colouring of G. ∎

3 Triangle-free planar graphs

Theorem 2 is a consequence of the following theorem:

Theorem 3 *Suppose G is an edge-coloured simple triangle-free plane graph, $C = (v_1, v_2, \cdots, v_k)$ is the outer face. Suppose L is a list assignment that assigns to each vertex x a set $L(x)$ of 3 permissible colours, except that some vertices on C have only 2 permissible colours. However, each edge of G has at least one end vertex x which has 3 permissible colours. Then G is adaptably L-colourable.*

Proof. We may assume G is connected and prove the theorem by induction on the number of vertices. If $|V(G)| \leq 4$, then the theorem is obviously true.

Assume $|V(G)| \geq 5$. A path $P = (v_i, x, v_j)$ is called a *long chord* of C connecting v_i and v_j, if $v_i, v_j \in C$, $x \notin C$ and $|L(v_i)| + |L(v_j)| = 5$. Let $\mathcal{P}$ be the set of chords, long chords, and cut-vertices of C. Suppose $P \in \mathcal{P}$ is a chord (v_i, v_j) or a long chord (v_i, x, v_j) connecting v_i and v_j. We denote by A_P and B_P the two components of $C - \{v_i, v_j\}$, and assume that $|A_P| \leq |B_P|$. If $P \in \mathcal{P}$ is a cut-vertex of C, we denote by A_P the smallest component of $C - P$. Let $P^* \in \mathcal{P}$ be a chord, long chord, or cut-vertex, for which $|A_{P^*}|$ is minimum.

Claim A_{P^*} contains a vertex v_t which is not a cut-vertex, such that $|L(v_t)| = 3$ and v_t is not contained in any chord or long chord of C.

First observe that A_{P^*} does not contain any cut-vertex, since otherwise this would contradict the minimality of P^*. Assume that P^* is a cut-vertex v. Then A_{P^*} contains at least two adjacent vertices v_i and v_{i+1}, and both of them are neither contained in a chord nor in a long chord of C by the minimality of P^*. By the hypothesis, there is a $t \in \{i, i+1\}$ such that $|L(v_t)| = 3$.

Assume $P^* = (v_i, x, v_j)$ is a long chord, $|L(v_j)| = 2$ and $A_{P^*} = (v_{i+1}, v_{i+2}, \cdots, v_{j-1})$. Then $|L(v_{j-1})| = 3$, for otherwise $v_j v_{j-1}$ is an edge of G connecting two vertices each with 2 permissible colours, in contrary to our assumption. Since G is triangle-free, v_{j-1} is not adjacent to x. If v_{j-1} is contained in a chord or a long chord P', then we would have $A_{P'} \subset A_{P^*}$ and hence $|A_{P'}| < |A_{P^*}|$, in contrary to our choice of P^*.

Assume $P^* = (v_i, v_j)$ is a chord, and $A_{P^*} = (v_{i+1}, v_{i+2}, \cdots, v_{j-1})$. Since G is triangle-free, $v_{i+1} \neq v_{j-1}$. Since each edge of G has at least one end vertex x which has 3 permissible colours, there exists $t \in \{i+1, i+2\}$ such that $|L(v_t)| = 3$. By the same argument as above, v_t is not contained in any chord or long chord of C. This completes the proof of the claim.

Let $v_t \in C$ be a vertex which is not a cut-vertex, such that $|L(v_t)| = 3$ and v_t is not contained in any chord or long chord of C. Let $\alpha \in L(v_t)$ be a colour distinct from the colours of the two edges $v_{t-1}v_t$ and $v_t v_{t+1}$. Let $G' = G - v_t$ and let L' be a list assignment of G' defined as $L'(x) = L(x) - \{\alpha\}$ if x is a neighbour of v_t distinct from v_{t-1}, v_{t+1}, and $L'(x) = L(x)$ otherwise. Then $L'(x)$ contains 3 colours for each interior vertex x of G' and $L'(x)$ contains at least 2 colours for each vertex x on the outer face of G', since v_t is not contained in any chord of C. Moreover, since v_t is not contained in any long chord of C, it follows that each edge of G' has at least one end vertex x which has 3 permissible colours. By induction hypothesis, G' is adaptably L'-colourable. Any L'-colouring of G' can be extended to an L-colouring of G by colouring v_t with colour α. So G is adaptably L-colourable. ∎

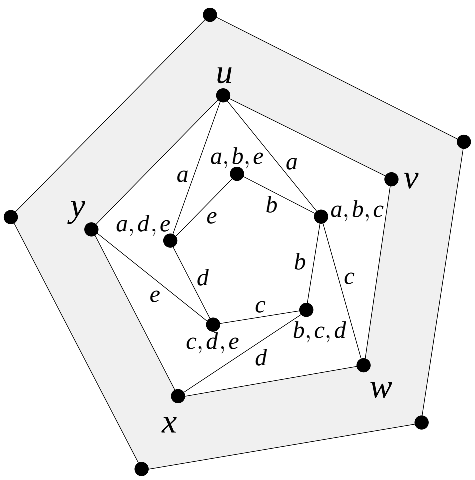


Figure 1: The construction of H_k.

4 Planar graphs without triangles at distance k

The distance between two triangles xyz and uvw is the minimum distance between a vertex of $\{x, y, z\}$ and a vertex of $\{u, v, w\}$. For any graph G, we denote by $d_t(G)$ the minimum distance between two triangles of G. If G contains at most one triangle, we take $d_t(G)$ to be infinite. Havel [8] asked the following question: is it true that for some k, every planar graph G with $d_t(G) \geq k$ is 3-colourable? Havel showed that such an integer k is at least 2, disproving a conjecture of Grűnbaum. In [1], Aksionov and L.S Mel'nikok proved that such a k is at least 4, and conjectured that the real value should be 5.

Since triangle-free planar graphs are adaptably 3-choosable, it is interesting to see if anything can be said about a relaxation similar to Havel's problem : is there an integer k, such that any planar graph G with $d_t(G) \geq k$ is adaptably 3-choosable? In the following, we prove that such a k does not exist: more precisely, for every k we construct a planar graph where every two triangles are at distance at least $2k$ apart, which is not adaptably 3-choosable.

Let us define the distance between two faces $\mathcal{F}_1$ and $\mathcal{F}_2$ of a graph as the minimum distance between a vertex of $\mathcal{F}_1$ and a vertex of $\mathcal{F}_2$. A face containing exactly k vertices is called a k-face. In the following, we construct inductively the plane graph H_i, such that the following is verified at each step:

(a) H_i is triangle-free.

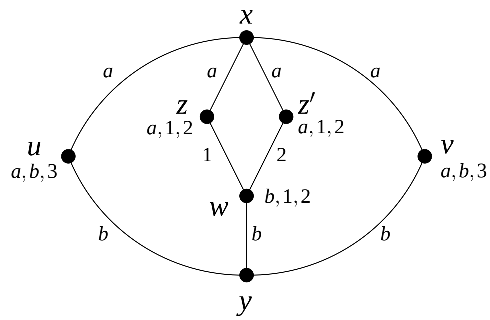


Figure 2: $H(a,b)$.

(b) H_i contains exactly two 5-faces (the outer face and another face, say $\mathcal{F}_i$). Moreover, the distance between these two faces is exactly i.

(c) Assume that the outer face is coloured with five distinct colours a, b, c, d and e in clockwise order. Then there exist an edge-colouring F_i of H_i and a list assignment L_i with $|L_i(v)| = 3$ for every vertex v which is not incident to the outer face, such that H_i has a unique L_i-colouring adapted to F_i. Moreover, this colouring is such that $\mathcal{F}_i$ is coloured with a, b, c, d and e in clockwise order.

Let H_0 be a 5-cycle. Then the three properties are trivially verified. Assume that for some $i \geq 1$, H_{i-1} also verifies these properties. Fix five different colours a, b, c, d, and e (in clockwise order) on the vertices of the outer face of H_{i-1}. By property (3), there exist an edge-colouring F_{i-1} of H_{i-1} and a list assignment L_{i-1} with lists of size three, such that H_{i-1} has a unique L_{i-1}-colouring adapted to F_{i-1}. In this colouring, the vertices u, v, w, x, and y of the 5-face $\mathcal{F}_{i-1}$ are coloured with a, b, c, d and e respectively. Let H_i be the graph obtained from H_{i-1} by adding five new vertices inside $\mathcal{F}_{i-1}$, as depicted in Figure 1. This figure also shows how to extend F_{i-1} and L_{i-1} to an edge-colouring F_i and a list-assignment L_i of H_i.

Since u and w are coloured with a and c respectively, the new vertex v' adjacent to u and w must be coloured with b. The new vertex w' adjacent to v' and x must be coloured with c; the new vertex x' adjacent to w' and y must be coloured with d; the new vertex y' adjacent to x' and y must be coloured with e, and the new vertex u' adjacent to y' and v' must be coloured with a. The graph H_i is still triangle-free, and only contains two 5-faces: the outer face and $\mathcal{F}_i = u'v'w'x'y'$. Moreover these two faces are at distance exactly $i-1+1 = i$. Hence, the graph H_i verifies properties (a), (b), and (c). We denote by G_i the graph obtained from H_i by adding inside the face $\mathcal{F}_i$ a 3-vertex z adjacent to u', w', and x'. We give the edges zu', zw' and zx' colours a, c, and d respectively, and we assign the list $\{a, c, d\}$ to z. Observe that the graph G_i contains only one triangle (which is at distance i from the outer face), and that the colouring of the outer face cannot be extended to an adapted list-colouring of G_i.

Let $H(a,b)$ be the edge-coloured graph depicted in Figure 2. Assume that x and y are coloured with a and b respectively. Then u and v must be coloured with 3, and w must be coloured either 1 or 2. If it is coloured with 1, the 5-face $xzwyu$ has its vertices coloured with a, 2, 1, b and 3. Otherwise, the 5-face $xvywz'$ has its vertices coloured with a, 3, b, 2, 1. Let $G(a,b)$ be the graph obtained from $H(a,b)$ by plugging the widget G_k in each of the two 5-faces (that is, each of these two faces becomes the outer face of a graph G_k). Using what has been done before, we know that with a suitable edge-colouring of the two widgets, there exists a list assignment with lists of size three, such that the colouring of $H(a,b)$ cannot be extended to a colouring of $G(a,b)$. Hence, if x and y are coloured with a and b respectively, this cannot be extended to an adapted list colouring of $G(a,b)$.

Consider 9 copies of $G(a,b)$, with $(a,b) \in \{4,5,6\} \times \{7,8,9\}$, and identify all the vertices x (resp. y) of these copies into a single vertex x^* (resp y^*). Assign the colour lists $\{4,5,6\}$ and $\{7,8,9\}$ to x^* and y^* respectively. Assume that there exists an adapted list colouring f of this graph, then there exist no adapted list colouring of the copy of $G(f(x^*), f(y^*))$, which is a contradiction. Hence, this planar graph is not adaptably 3-choosable, and any two triangles are at distance at least $2k$ apart.

5 Planar graphs without 4-cycles

In this section, we prove that there exist planar graphs without 4-cycles, which are not adaptably 3-colourable. Let $H(a,b,c)$ be the edge-coloured graph depicted in Figure 3. Consider that $\{a,b,c\} = \{1,2,3\}$, and assume that the vertices u and v of $H(a,b,c)$ are coloured with a and b respectively. Then at least one of the vertices w and w' is coloured with c. By symmetry, we can assume that w is coloured with c. Then x must be coloured with a, y must be coloured with c, and z and z' must be coloured with b. It is easy to check that in this situation, the remaining subgraph induced the vertices at distance one or two from z and z' cannot be adaptably coloured. Hence, if u and v are coloured with a and b, this colouring cannot be extended to an adapted 3-colouring of $H(a,b,c)$.

For every $1 \leq a \leq 3$, let b and c be the two colours from $\{1,2,3\}$ distinct from a. We denote by G_a the edge-coloured graph obtained from $H(a,b,c)$ and $H(a,c,b)$ by contracting the two vertices u (resp. v) into a single vertex u^* (resp.v^*). Observe that in any adapted 3-colouring of G_a, if u^* is coloured with a then v^* is also coloured with a.

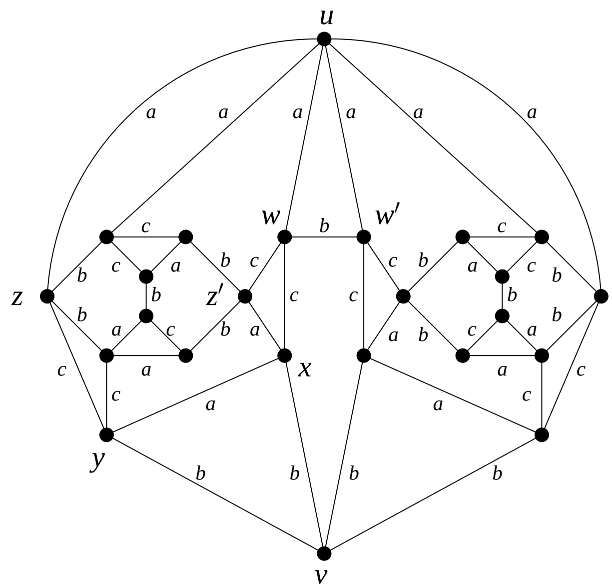


Figure 3: $H(a,b,c)$.

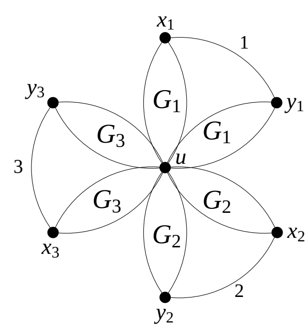


Figure 4: A planar graph without 4-cycle, which is not adaptably 3-colourable.

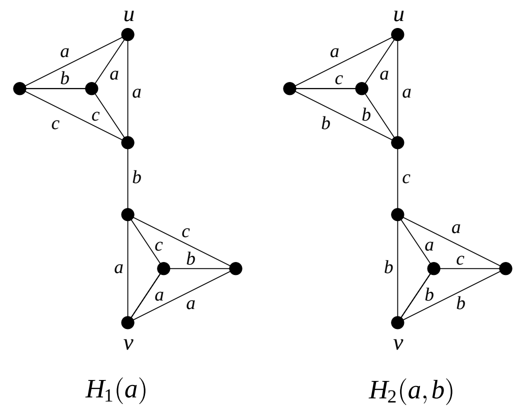


Figure 5: $H_1(a)$ and $H_2(a,b)$.

Consider now an adapted 3-colouring of the construction of Figure 4, which does not contain any 4-cycle. If the vertex u is coloured with $1 \le i \le 3$, then the two vertices x_i and y_i are both coloured with i, which is a contradiction since they are linked by an edge coloured with i. Hence, this graph is not adaptably 3-colourable.

6 Planar graphs without 5-cycles

In this section, we prove that there exist planar graphs without 5-cycles, which are not adaptably 3-colourable. For any $\{a,b,c\} = \{1,2,3\}$, let $H_1(a)$ and $H_2(a,b)$ be the two C_5-free planar graphs depicted in Figure 5. It is easy to check that in $H_1(a)$, if the vertices u and v are coloured with a, then this colouring cannot be extended to an adapted colouring of $H_1(a)$. Similarly in $H_2(a,b)$, if u and v are coloured respectively with a and b ($a \neq b$), then this colouring cannot be extended to an adapted colouring of $H_2(a,b)$.

Consider the three graphs $H_1(a)$ for $1 \le a \le 3$, and the six graphs $H_2(a,b)$ with $1 \le a \neq b \le 3$. Contract the nine vertices u (resp. v) of these graphs into a single vertex u^* (resp. v^*). Assume that there exists an adapted 3-colouring f of this graph. If $f(u^*) = f(v^*)$ then the copy of $H_1(f(u^*))$ is not adaptably 3-colourable, which is a contradiction. Otherwise $f(u^*) \neq f(v^*)$ and the copy of $H_2(f(u^*), f(v^*))$ is not adaptably 3-colourable, which is also a contradiction. Hence, this graph is planar and without 5-cycles, but is not adaptably 3-colourable.

7 Conclusion

In this note, we proved that triangle-free planar graphs are adaptably 3-choosable, whereas C_4-free planar graphs and C_5-free planar graphs are not even adaptably 3-colourable. We also showed that for any $k \geq 0$, there exist planar graphs without triangles at distance k which are not adaptably 3-choosable. However, the question remains open for adapted colouring:

Question 7.1 *Is there an integer k, such that every planar graph G with $d_t(G) \geq k$ is adaptably 3-colourable?*

If the answer to this question is negative, it implies that the answer to the original problem of Havel is also negative, whereas a positive answer to the original problem of Havel would imply a positive answer to Question 7.1.

In 1976, Steinberg conjectured that planar graphs without cycles of length 4 and 5 are 3-colourable (see [12] for a survey). We can ask the same for adapted 3-colouring and adapted 3-choosability :

Question 7.2 *Are planar graphs without 4-cycles and 5-cycles adaptably 3-colourable?*

Question 7.3 *Are planar graphs without 4-cycles and 5-cycles adaptably 3-choosable?*

A weaker version of the problem of Steinberg was proposed by Erdős in 1991: he asked what is the smallest i, such that every planar graph without cycles of length 4 to i is 3-colourable? The same can be asked for adapted 3-colouring and adapted 3-choosability:

Question 7.4 *What is the smallest i, such that every planar graph without cycles of length 4 to i is adaptably 3-colourable?*

Question 7.5 *What is the smallest i, such that every planar graph without cycles of length 4 to i is adaptably 3-choosable?*

Note that by [3], the answer of Question 7.4 is at most 7, and by [2, 18], the answer of Question 7.5 is at most 9.

References

[1] V.A. Aksionov and L.S Mel'nikok, *Some counterexamples associated with the Three Color Problem*, J. Combin. Theory Ser. B **28** (1980) 1–9.

[2] O.V. Borodin, *Structural properties of plane graphs without adjacent triangles and an application to 3-colorings*, J. Graph Theory **12** (1996) 183–186.

[3] O.V. Borodin, A.N. Glebov, A. Raspaud, and M.R. Salavatipour, *Planar Graphs without Cycles of Length from 4 to 7 are 3-colorable*, J. Combin. Theory Ser. B **93** (2005) 303–311.

[4] T. Feder and P. Hell, *Full constraint satisfaction problems*, SIAM J. Comput. **36** (2006) 230–246.

[5] H. Grőtzsch, *Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel*, Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe **8** (1959) 109–120.

[6] T. Feder, P. Hell, S. Klein, and R. Motwani, *Complexity of list partitions*, SIAM J. Discrete Math. **16** (2003) 449–478.

[7] T. Feder, P. Hell, D. Král, and J. Sgall, *Two algorithms for list matrix partition*, Proc. 16th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA) 2005, 870–876.

[8] I. Havel, *On a conjecture of B. Grűnbaum*, J. Combin. Theory **7** (1969) 184–186.

[9] P. Hell and X.Zhu, *Adaptable chromatic number of graphs*, European J. Combinatorics, to appear.

[10] P. Hell and J. Nešetřil, **Graphs and homomorphisms**, Oxford University Press, 2004.

[11] A. Kostochka and X. Zhu, *Adapted list coloring of graphs and hypergraphs*, SIAM J. Discrete Math., to appear.

[12] R. Steinberg, *The state of the three color problem* in. Quo Vadis, Graph Theory? Annals of Discrete Mathematics **55** (1993) 211–248.

[13] C. Thomassen, *Every planar graph is 5-choosable*, J. Combin. Theory Ser. B **62** (1994) 180–181.

[14] C. Thomassen, *A short list color proof of Grötzsch's theorem*, J. Combin. Theory Ser. B **88** (2003) 189–192.

[15] M. Voigt, *List colourings of planar graphs*, Discrete Math. **120** (1993), 215–219.

[16] M. Voigt, *A not 3-choosable planar graph without 3-cycles*, Discrete Math. **146** (1995) 325–328.

[17] K. Wagner, *Űber eine Eigenschaft der ebenen Komplexe*, Math. Ann. **144** (1937) 570–590.

[18] L. Zhang and B. Wu, *A note on 3-choosability of planar graphs without certain cycles*, Discrete Math. **297** (2005) 206–209.

Bibliography

[AEK$^+$07] L. Addario-Berry, L. Esperet, R.J. Kang, C.J.H. McDiarmid, and A. Pinlou, *Acyclic t-improper colourings of graphs with bounded maximum degree*, Discrete Math., in press.

[AEH07] O. Amini, L. Esperet, and J. van den Heuvel, *Frugal Colouring of Graphs*, Technical Report RR-1428-07, Université de Bordeaux, LaBRI, May 2007.

[AEH08] O. Amini, L. Esperet, and J. van den Heuvel, *A Unified Approach to Distance-Two Colouring of Planar Graphs*, manuscript.

[AH00] G. Agnarsson and M.M. Halldórsson, *Coloring powers of planar graphs*, SIAM J. Discrete Math. **16** (2003), 651–662.

[AMR91] N. Alon, C.J.H. McDiarmid, and B. Reed, *Acyclic coloring of graphs*, Random Structures and Algorithms **2** (1991), 277–288.

[Beh65] M. Behzad, *Graphs and Their Chromatic Numbers*, Ph.D. thesis, Michigan State University, 1965.

[Ber69] C. Berge, *Graphes et Hypergraphes*, Dunod, Paris, 1969.

[Bod91] H.L. Bodlaender, *On the complexity of some coloring games*, Internat. J. Found. Comput. Sci. **2** (1991), 133–147.

[BM76] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, Macmillan, London and Elsevier, New York, 1976.

[Bor79] O.V. Borodin, *On acyclic colorings of planar graphs*, Discrete Math. **25** (1979), 211–236.

[Bor84] O.V. Borodin, *Solution of the Ringel problem on vertex-face coloring of planar graphs and coloring of 1-planar graphs* (in Russian), Metody Diskret. Analyz. **41** (1984), 12–26.

[BBG+01] O.V. Borodin, H.J. Broersma, A. Glebov, and J. van den Heuvel, *Minimal degrees and chromatic numbers of squares of planar graphs* (in Russian), Diskretn. Anal. Issled. Oper. Ser. 1 **8**, no. 4 (2001), 9–33.

[BBG+07] O.V. Borodin, H.J. Broersma, A. Glebov, and J. van den Heuvel, *A new upper bound on the cyclic chromatic number*, J. Graph Theory **54** (2007), 58–72.

[BFK+02] O.V. Borodin, D.G. Fon-Der Flaass, A.V. Kostochka, A. Raspaud and E. Sopena, *Acyclic list 7-coloring of planar graphs*, J. Graph Theory **40(2)** (2002), 83–90.

[BKW97] O.V. Borodin, A.V. Kostochka, and D.R. Woodall, *List edge and list total colourings of multigraphs*, J. Combin. Theory Ser. B **71** (1997), 184–204.

[BKW99] O.V. Borodin, A.V. Kostochka and D.R. Woodall, *Acyclic Colourings of Planar Graphs with Large Girth*, J. London Math. Soc. **60(2)** (1999), 344–352.

[BSZ99] O.V. Borodin, D.P. Sanders, and Y. Zhao, *On cyclic colorings and their generalizations*, Discrete Math. **203** (1999), 23–40.

[BM93] R.A. Brualdi and J.J.Q. Massey, *Incidence and strong edge colorings of graphs*, Discrete Math. **122** (1993), 51–58.

[But06] S. Butler, *Induced-universal graphs for graphs with bounded maximum degree*, preprint.

[Cal06] T. Calamoneri, *The $L(h,k)$-Labelling Problem: A Survey and Annotated Bibliography*, The Computer Journal, **49(5)** (2006), 585–608.

[CS07] L.S. Chandran and N. Sivadasan, *Boxicity and treewidth*, J. Combin. Theory Ser. B **97(5)** (2007), 733–744.

[CFS08] L.S. Chandran, M.C. Francis, and N. Sivadasan, *Boxicity and maximum degree*, J. Combin. Theory Ser. B **98** (2008), 443–445.

[Cha07] H.-Y. Chang, *Game Colourings of Graphs*, Ph.D. thesis, National Sun Yat-sen University, Taiwan, 2007.

[CL03] G.J. Chang and S.-C. Liaw, *The $L(2,1)$-labeling problems on ditrees*, Ars Combin. **66** (2003), 23–31.

[CW06] M. Chen and W.-F. Wang, *The 2-dipath chromatic number of Halin graphs*, Inform. Process. Letters **99** (2006), 47–53.

[CR83] M.B. Cozzens and F.S. Roberts, *Computing the boxicity of a graph by covering its complement by cointerval graphs*, Discrete Applied Math. **6** (1983), 217–228.

[Die05] R. Diestel, *Graph Theory*, 3rd edition. Springer-Verlag, Berlin and Heidelberg, 2005.

[DZ99] T. Dinski and X. Zhu, *A bound for the game chromatic number of graphs*, Discrete Math. **196** (1999), 109–115.

[DKN+08] Z. Dvořák, D. Král', P. Nejedlý, and R. Škrekovski, *Coloring squares of planar graphs with no short cycles*, European J. Combin. **29** (2008), 838–849.

[EH99] N. Eaton and T. Hull, *Defective list colorings of planar graphs*, Bull. Inst. Combin. Appl. **25** (1999), 79–87.

[Edm65] J.R. Edmonds, *Matching and a polyhedron with 0-1 vertices*, J.Res, Nat. Bur. Standards **69B** (1965), 125–130.

[ERT79] P. Erdös, A.L. Rubin and H. Taylor, *Choosability in graphs*, Congr. Numer. **26** (1979), 125–157.

[Esp08] L. Esperet, *Boxicity of graphs with bounded degree*, manuscript.

[ELO07] L. Esperet, A. Labourel, and P. Ochem, *On induced-universal graphs for the class of bounded-degree graphs*, Inform. Process. Letters, in press.

[EMR06] L. Esperet, M. Montassier, and A. Raspaud, $(d, 1)$-*total labelling of sparse graphs*, Technical Report RR-1391-06, Université de Bordeaux, LaBRI, March 2006.

[EMR08] L. Esperet, M. Montassier, and A. Raspaud, *Linear choosability of graphs*, Discrete Math. **308(17)** (2008), 3938–3950.

[EMZ08] L. Esperet, M. Montassier, and X. Zhu, *Adapted list colouring of planar graphs*, Technical Report RR-1442-08, Université de Bordeaux, LaBRI, February 2008.

[EO07a] L. Esperet and P. Ochem, *Oriented colorings of 2-outerplanar graphs*, Inform. Process. Letters **101** (2007), 215–219.

[EO07b] L. Esperet and P. Ochem, *On circle graphs with girth at least five*, Discrete Math., in press.

[EP06] L. Esperet and A. Pinlou, *Acyclic improper choosability of graphs*, In: Proceedings of the Sixth Czech-Slovak International Symposium on Combinatorics, Graph Theory, Algorithms and Applications (CS'06), ENDM **28** (2007), 251–258.

[EZ08] L. Esperet and X. Zhu, *Game colouring of the square of graphs*, manuscript.

[FGS+89] R. Faudree, A. Gyárfás, R. Schelp, and Z. Tuza, *Induced matchings in bipartite graphs*, Discrete Math. **78** (1989), 83–87.

[FF83] J.-L. Fouquet and J.-L. Jolivet, *Strong edge-colorings of graphs and applications to multi-k-gons*, Ars Combin. **16** (1983), 141–150.

[Fre83] L.C. Freeman, *Spheres, cubes and boxes: graph dimensionality and network structure*, Social Networks **5** (1983), 139–156.

[Gal95] F. Galvin, *The list chromatic index of a bipartite multigraph*, J. Combin. Theory Ser. B **63** (1995), 153–158.

[GJS76] M.R. Garey, D.S. Johnson and L.J. Stockmeyer, *Some simplified NP-Complete graph problems*, Theor. Comput. Sci. **1(3)** (1976), 237–267.

[GMW94] J.P. Georges, D.W. Mauro, and M.A. Whittlesey, *Relating path coverings to vertex labelings with a condition at distance two*, Discrete Math. **135** (1994), 103–111.

[GMW95] J.P. Georges, D.W. Mauro, and M.A. Whittlesey, *On the λ-number of Q_n and related graphs*, SIAM, J. Discrete Math. **8** (1995), 449–506.

[GRS06] D. Gonçalves, A. Raspaud, and M.A. Shalu, *On oriented labelling parameters*, Formal Models, Languages and Applications, World Scientific, Singapore **66** (2006), 33–45.

[GY92] J.R. Griggs and R.K. Yeh, *Labelling graphs with a condition at distance two*, SIAM, J. Discrete Math **5** (1992), 586–595.

[Gru73] B. Grünbaum, *Acyclic colorings of planar graphs*, Israel J. Math. **14** (1973), 390–408.

[Gui97] B. Guiduli, *On incidence coloring and star arboricity of graphs*, Discrete Math. **163** (1997), 275–278.

[HC92] R. Häggkvist and A. Chetwynd, *Some upper bounds on the total and list chromatic numbers of multigraphs*, J. Graph Theory **16** (1992), 503–516.

[HHM+07] F. Havet, J. van den Heuvel, C.J.H. McDiarmid, and B. Reed, *List colouring squares of planar graphs*, In: Proceedings of the 2007 European Conference on Combinatorics, Graph Theory and Applications (EuroComb'07), ENDM **29** (2007), 515–519.

[HRS08] F. Havet,B. Reed and J.-S. Sereni, *L(2,1)-labelling of graphs*, In: Proceedings of the ACM-SIAM Symposium on Discrete Algorithm (SODA 2008), 621–630.

[HY08] F. Havet and M.-L. Yu, $(p, 1)$-*Total labelling of graphs*, Discrete Math. **308** (2008), 496–513.

[HW06] T.J. Hetherington and D.R. Woodall, *List-colouring the square of an outerplanar graph*, Ars Combin., in press.

[HM03] J. van den Heuvel and S. McGuinness, *Coloring the square of a planar graph*, J. Graph Theory **42** (2003), 110–124.

[HSS01] A.J.W. Hilton, T. Slivnik, and D.S.G. Stirlingilton, *Aspects of edge list-colourings*, Discrete Math. **231** (2001), 253–264.

[Hin90] H. Hind, *An improved bound for the total chromatic number of a graph*, Graphs and Combinatorics **6** (1990), 153–159.

[HMR97] H. Hind, M. Molloy, and B. Reed, *Colouring a graph frugally*, Combinatorica **17** (1997), 469–482.

[HMR99] H. Hind, M. Molloy, and B. Reed, *Total coloring with* $\Delta + \mathrm{poly}(\log \Delta)$ *colors*, SIAM J. Comput. **28** (1999), 816–821.

[HSZ04] M. Hosseini Dolama, É. Sopena, X. Zhu, *Incidence coloring of k-degenerated graphs*, Discrete Math. **283** (2004), 121–128.

[JT95] T.R. Jensen, B. Toft, *Graph Coloring Problems*, John-Wiley & Sons, New York, 1995.

[Jon93] T.K. Jonas, *Graph coloring analogues with a condition at distance two : $L(2,1)$-labelings and list λ-labelings*. Ph.D. thesis, University of South Carolina, 1993.

[Kah96a] J. Kahn, *Asymptotics of the Chromatic Index for Multigraphs*, J. Combin. Theory Ser. B **68** (1996), 233–254.

[Kah96b] J. Kahn, *Asymptotically Good List-Colorings*, J. Combin. Theory Ser. A **73** (1996), 1–59.

[Kah97] J. Kahn, *On the stochastic independence properties of hard-core distributions*, Combinatorica **17** (1997), 369–391.

[Kah00] J. Kahn, *Asymptotics of the list-chromatic index for multigraphs*, Random Structures and Algorithms **17** (2000), 117–156.

[KM07] R.J. Kang and T. Müller, *Acyclic and frugal colourings of graphs*, manuscript (2007).

[Kie00] H.A. Kierstead, *A simple competitive graph colouring algorithm*, J. Combin. Theory Ser. B **78** (2000), 57–68.

[KT94] H.A. Kierstead and W.T. Trotter, *Planar graph colouring with an uncooperative partner*, J. Graph Theory **18(6)** (1994), 569–584.

[KM04] W.F. Klostermeyer and G. MacGillivray, *Analogues of cliques for oriented coloring*, Discuss. Math. Graph Theory **24(3)** (2004), 373–387.

[Kos77] A.V. Kostochka, *An analogue of Shannon's estimate for complete colorings* (in Russian), Metody Diskret. Analyz. **30** (1977), 13–22.

[KW01] A.V. Kostochka and D.R. Woodall, *Choosability conjectures and multicircuits*, Discrete Math. **240** (2001), 123–143.

[Kra06] D. Král', *Channel assignment problem with variable weights*, SIAM J. Discrete Math. **20** (2006), 690–704

[Kra94] J. Kratochvíl, *A special planar satisfiability problem and a consequence of its NP-completeness*, Discrete Appl. Math. **52** (1994), 233–252.

[LW03] K.-W. Lih and W.-F. Wang, *Labeling planar graphs with conditions on girth and distance two*, SIAM J. Discrete Math. **17** (2003), 264–275.

[LW06] K.-W. Lih and W.-F. Wang, *Coloring the square of an outerplanar graph*, Taiwanese J. of Math. **10** (2006), 1015–1023.

[MS93] T.A. McKee and E.R. Scheinerman, *On the Chordality of a Graph*, J. Graph Theory **17(2)** (1993), 221–232.

[MR97] M. Molloy and B. Reed, *A bound on the strong chromatic index of a graph*, J. Combin. Theory Ser. B **69** (1997), 103–109.

[MR98] M. Molloy and B. Reed, *A bound on the total chromatic number*, Combinatorica **18** (1998), 241–280.

[MR98b] M. Molloy and B. Reed, *Further Algorithmic Aspects of the Local Lemma*, In: Proceedings of the 30th ACM Symposium on Theory of Computing, 1998.

[MR02] M. Molloy and B. Reed, *Graph coloring and the probabilistic method*, Vol. 23 of *Algorithms and combinatorics*, Springer-Verlag, 2002.

[MS05] M. Molloy and M.R. Salavatipour, *A bound on the chromatic number of the square of a planar graph*, J. Combin. Theory Ser. B **94** (2005), 189–213.

[MR03] M. Montassier and A. Raspaud, $(d,1)$-*total labelling of graphs with a given maximum average degree*, J. Graph Theory **51** (2006), 93–109.

[OR81] R.J. Opsut and F.S. Roberts, *On the fleet maintenance, mobile radio frequency, task assignment, and traffic phasing problems*, in G. Chartrand, et al (eds.), The Theory and Applications of Graphs, Wiley, New York, 1981, 479–492.

[OP69] O. Ore and M.D. Plummer, *Cyclic coloration of plane graphs*, In: Recent Progress in Combinatorics, Proceedings of the Third Waterloo Conference on Combinatorics, Academic Press, San Diego (1969), 287–293.

[Pet91] J. Petersen, *Die Theorie der regulären Graphs*, Acta Math. **15** (1891), 193–220.

[RW06] A. Raspaud and W.-F. Wang, *Linear coloring of planar graphs*, Technical Report RR-1384-06, University of Bordeaux 1, 2006.

[Rob69] F.S. Roberts, *On the boxicity and cubicity of a graph*, In: Recent Progresses in Combinatorics, Academic Press, New York, 1969, 301–310.

[Rob76] F.S. Roberts, *Discrete Mathematical Models, with Applications to Social, Biological, and Environmental Problems*, Prentice-Hall, Englewood Cliffs, New Jersey, 1976.

[SZ01] D.P. Sanders and Y. Zhao, *A new bound on the cyclic chromatic number*, J. Combin. Theory Ser. B **83** (2001), 102–111.

[Sha49] C.E. Shannon, *A theorem on colouring lines of a network*, J. Math. Physics **28** (1949), 148–151.

[Sch84] E.R. Scheinerman, *Intersection classes and multiple intersection parameters*, Ph.D. thesis, Princeton University, 1984.

[Sch03] A. Schrijver, *Combinatorial Optimization*, Algorithms and Combinatorics **24**, Springer, 2003.

[Skr99] R. Škrekovski, *List improper colorings of planar graphs*, Combin. Probab. and Comput. **8** (1999), 293–299.

[Tho86] C. Thomassen, *Interval representations of planar graphs*, J. Combin. Theory Ser. B **40** (1986), 9–20.

[Tuz97] Z. Tuza, *Graph colorings with local contraints - A survey*, Discussiones Mathematicae Graph Theory **17(2)** (1997), 161–228.

[Viz64] V.G. Vizing, *On an estimate of the chromatic class of a p-graph* (in Russian), Metody Diskret. Analiz. **3** (1964), 25–30.

[Viz68] V.G. Vizing, *Some Unsolved Problems in Graph Theory*, Russian Math. Surveys **23** (1968), 125–141.

[Viz76] V.G. Vizing, *Coloring the vertices of a graph in prescribed colors*, Metody Diskret. Anal. **19** (1976), 3–10.

[Weg77] G. Wegner, *Graphs with given diameter and a coloring problem*, Technical Report, University of Dortmund, 1977.

[Won96] S.A. Wong, *Colouring graphs with respect to distance*, M.Sc. Thesis, Department of Combinatorics and Optimization, University of Waterloo, 1996.

[WZ08] J. Wu and X. Zhu, *Lower bounds for the game colouring number of planar graphs and partial k-trees*, Discrete Math., in press.

[Yus98] R. Yuster, *Linear coloring of graphs*, Discrete Math. **185** (1998), 293–297.

[Zhu99] X. Zhu, *The Game Coloring Number of Planar Graphs*, J. Combin. Theory Ser. B **75** (1999), 245–258.

[Zhu00] X. Zhu, *The game coloring number of pseudo partial k-trees*, Discrete Math. **215** (2000), 245–262.

[Zhu08] X. Zhu, *Refined activation strategy for the marking game*, J. Combin. Theory Ser. B **98** (2008), 1–18.