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A GAME THEORY APPROACH FOR THE GROUNDWATER POLLUTION CONTROL*

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Abstract. A differential game modeling the noncooperative outcome of pollution in groundwater is studied. Spatio-temporal objectives are constrained by a convection-diffusion-reaction equation ruling the spread of the pollution in the aquifer, and the velocity of the flow solves an elliptic partial differential equation. The existence of a Nash equilibrium is proved using a fixed point strategy. A uniqueness result for the Nash equilibrium is also proved under some additional assumptions. Some numerical illustrations are provided.

Key words. PDEs, differential game theory, Nash equilibrium, hydrogeological state equations

AMS subject classifications. 49A20, 49A25, 49K20, 91A10, 91B76

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1. Introduction.

The preservation of the groundwater is a leading concern in order to maintain the quality of water supply. Human activities and uses are the main causes for pollution. On the one hand, when related to economic activities such as, for instance, farming or manufacturing, the pollution often corresponds to a productivity gain, thus to benefits. On the other hand, the contamination requires depolluting actions in the captation wells and consequently entails cleaning costs. The present paper is a contribution to the study of the trade-off between the benefits of the polluting activities and the decontamination costs. Theoretical developments in the specific context of groundwater are all the more interesting because the flow dynamics are slow in the aquifers thus inducing a large delay before the possible evaluation of any political decision limiting the pollution. Here moreover, we focus on nonpoint source contaminations, such as agricultural nitrate pollution, for which the emissions of pollutants are not directly observable by the regulator, making any control difficult to apply, the regulator being able to measure pollution only at specific points ([25]).

The fact that aquifers are common goods contaminated by many actors is one of the reasons that may explain why their pollution level remains high in spite of the institutional protection attempts. Indeed, a negative externality comes into play for the individual decisions, leading to the suboptimality of the noncooperative solution compared to that which would have been obtained in a cooperative way. This type of behavior is now well documented for nonspatialized models: as stated in the seminal paper of Van Der Ploeg and De Zeeuw ([21]), the Nash open-loop equilibria lead to a higher pollutant stock and a higher fertilizer use than in the cooperative situation; similar results have been obtained in the framework of shallow lakes (Mäler, Xepapadeas, and De Zeeuw [17], Wagener [23], and Kossioris et al. [16]). However, most of

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the existing studies do not take into account the space dimension, whereas it is notoriously the main ingredient for a proper modeling of transport and diffusion processes in the underground (see any hydrogeological monograph, e.g., Scheidegger [20]). We thus only refer to [11] by de Frutos and Martin-Herran devoted to a time-space model of multiregional pollution. More precisely the authors consider a discrete space approximation and provide numerical illustrations of Nash equilibria. Here rather, we focus on the mathematical analysis.

The problem under consideration belongs to the class of infinite dimensional multiobjective control problems constrained by PDEs. There is a large literature devoted to the corresponding Pareto strategies. The Nash strategy remains seldom addressed. A general theory is solely developed for the case of linear PDEs. We refer to the fundamental work of Ramos, Glowinski, and Periaux [19]), based on the adjoint formulation and thus needing stronger assumptions than the one considered here for the existence result. See also Borzi and Kanzow [6] for a focus on the elliptic setting and Hintermüller and Surowiec [15] for the extension to the so-called generalized Nash equilibrium problems, where the set of admissible controls of each player depends on the other players' strategies. We also mention the recent results of Araruna et al. [1] for the Stackelberg–Nash strategy. In the particular context of water management, a related work is the article [13] of García-Chan, Munoz-Sola, and Vásquez-Méndez with pointwise pollution sources and control, constrained by a parabolic and once again linear PDE. A substantial difficulty of the present work relies on the nonlinearity of the parabolic PDE governing the state unknown.

In the present paper, for the sake of the simplicity, the formulation is limited to two players. Two polluters with two different spread policies are considered. The state equations are given by the time and space dependent model ruling the hydrogeological dynamics. The spread of the pollutants is modeled by a convection-diffusion-reaction equation and the velocity of the flow by an elliptic PDE. Generic reaction terms are assumed, including all the classical (nonlinear) isotherms. Two economic objectives are defined.

The paper is organized as follows. The problem is presented in section 2, introducing the hydrogeological state equations system and the objective functions. Assumptions are listed in section 3, and the main result of the paper, an existence result of a Nash equilibrium, is stated. Its proof is developed in section 4, the necessary compactness results being obtained by convexity arguments thus without unnatural assumptions on the controls. Finally, in section 5, we derive the corresponding optimality conditions using the Pontryagin's approach. This allows to prove a uniqueness result for the Nash equilibrium. More precisely, due to the nonlinearity of the state equation, the global result is obtained by propagation of a local in a time uniqueness result. Finally, some numerical illustrations are provided in section 6.

2. Description of the model.

We first set up the space-time domain of the study. Pollution spreading in areas, groundwater, and water collection wells are contained in a bounded domain $\Omega \subset \mathbb{R}^N$, where $N \geq 1$, with a boundary $\partial\Omega$ assumed $\mathcal{C}^2$. For practical applications $N = 3$. Two subdomains $\Omega_1 \subset \Omega$ and $\Omega_2 \subset \Omega$ are considered. They correspond to the areas of the soil polluted by two independent agents. The latter two are hereafter referred as Player 1 and Player 2. We assume that Ω_1 and Ω_2 are such that $\Omega_1 \cap \Omega_2 = \emptyset$. The contamination flow rate in Ω_i is denoted by $p_i(t, x)$, $i = 1, 2$, $x \in \Omega_i \subset \Omega$, $t \in (0, T)$, where the time horizon T is a given real number in $(0, \infty)$. Let $\Omega_T = (0, T) \times \Omega$. Let χ_{Ω_i} be the characteristic function of Ω_i.

The dynamics of the pollutant in the underground is described by the state system. For the sake of simplicity, only one pollutant is assumed of interest. Its concentration in the groundwater is denoted by c. Its displacement is mainly driven by the velocity v of the mixture. A diffusion process also occurs. Due to microscopic heterogeneities in the soil, this diffusion process may depend on v. The mass and momentum conservation principles then lead to the following system ruling the transport of the pollutant in Ω_T (see Augeraud-Véron, Choquet, and Comte [2] for more details):

$$R\psi\partial_t c + v \cdot \nabla c - \operatorname{div}(\psi S(v)\nabla c) = -r(c) + p_1\chi_{\Omega_1} + p_2\chi_{\Omega_2} - gc + \gamma, \tag{2.1}$$

$$\operatorname{div}(v) = g, \quad v = -\kappa\nabla\phi. \tag{2.2}$$

According to Scheidegger [20], the dispersion tensor $S(v)$ can be written[1]

$$S(v) = S_m\mathrm{Id} + S_p(v), \quad S_p(v) = |v|\left(\frac{\alpha_L}{|v|^2}v\otimes v + \alpha_T\left(\mathrm{Id} - \frac{1}{|v|^2}v\otimes v\right)\right), \tag{2.3}$$

where S_m, α_L, and α_T are, respectively, the diffusion coefficient and the longitudinal and transverse dispersion factors. The dispersion depends on the velocity v of the fluid in the aquifer, which itself is related to the hydraulic head ϕ according to the Darcy law (second equation in (2.2)). The structure of the soil is described by the porosity function ψ and by the fluid mobility tensor κ which rates the permeability of the underground with the viscosity of the fluid. The eventual adsorption of the pollutant by the soil is assumed to be a linear and instantaneous reaction, following the arguments in Miquel and de Marsily ([18, page 251]). The corresponding retardation factor is the real number $R > 0$. The other chemical reactions are resumed in the term $r(c)$. Classical isotherms (see, e.g., Williams [24]) are described by linear functions in the form $r(c) = kc$ or by Freundlich functions, $r(c) = kc^{k'}$ or by Langmuir functions, $r(c) = kc/(1+k'c)$, $(k,k') \in \mathbb{R}^2_+$. Terms g and γ correspond to the other source terms. More precisely, the function g describes the water flow rate of the external sources and sinks and γ accounts for the pollutant natural load.

General boundary conditions have to be considered in order to encompass a various number of realistic situations. In order to disentangle boundary assumptions on the two state variables, we consider two nonoverlapping decompositions of the boundary $\partial\Omega$ of Ω: $\Gamma_1 \cup \Gamma_2 = \partial\Omega$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\Gamma_3 \cup \Gamma_4 = \partial\Omega$ with $\Gamma_3 \cap \Gamma_4 = \emptyset$. The state system is then completed by the following initial and boundary conditions:

$$S(v)\nabla c \cdot n = 0 \text{ on } \Gamma_1 \times (0,T), \; c = 0 \text{ on } \Gamma_2 \times (0,T), \; c|_{t=0} = c_0 \text{ in } \Omega, \tag{2.4}$$

$$v \cdot n = -\kappa\nabla\phi \cdot n = -\kappa v_1 \cdot n \text{ on } \Gamma_3 \times (0,T), \; \phi = \phi_b \text{ on } \Gamma_4 \times (0,T), \tag{2.5}$$

where we denote by n the unit exterior normal vector to the boundary $\partial\Omega$. Notice that we have chosen homogeneous boundary conditions for the concentrations, only for enlightening the computations below. The results naturally extend to nonhomogeneous boundary conditions.

We now describe the objectives of the two players. The set of values of the contamination flow rate, namely, $\{p_i(t,x);\,(t,x) \in \Omega_T\}$, characterizes the strategy of Player i. In the present paper we only consider open-loop Nash equilibrium solutions

[1] Here $u \otimes v$ denotes the tensor product, $(u \otimes v)_{ij} = u_i v_j$, while $u \cdot v$ denotes the scalar product, $u \cdot v = \sum_{i=1}^N u_i v_i$. Let $|u|^2 = u \cdot u$. The identity matrix is denoted by Id.

(Dockner, Feichtinger, and Jørgensen [10]). The objective of each player, denoted J_i, $i = 1, 2$, is given by a Bolza form as follows:

$$J_i(p_i; p_{-i}, c(\cdot; p_1, p_2)) = -\nu e^{-\rho T} \int_\Omega D_i(x, c(T, x; p_1, p_2))\, dx$$
$$+ \int_0^T \left(\int_\Omega (f_i(x, p_i(t, x))\chi_{\Omega_i}(x) - D_i((x, c(t, x; p_1, p_2)))\, dx \right) e^{-\rho t}\, dt, \quad (2.6)$$

where $c(\cdot; p_1, p_2)$ is defined by the state problem (2.1)–(2.5). Here and after, the notation $c(\cdot; p_1, p_2)$ is used in order to bear in mind that c defined by (2.1)–(2.5) depends on p_1 and p_2, and thus a game situation occurs. This is a nonlocal dependency, namely, $c(\cdot; p_1, p_2) : (t, x) \mapsto c(t, x; p_1, p_2)$, where

$$c(t, x; p_1, p_2) = c\left(t, x; \bigcup_{i=1,2} \{p_i(s, y) \text{ a.e. } (s, y) \in (0, T) \times \Omega_i\} \right).$$

Instantaneous welfare depends on the one hand on private benefits of the player and on the other hand on the cost of the remediation of the environmental damage due to the pollution. However, due to the spreading of pollutant in groundwater, a differential game situation appears, the pollutant concentration in groundwater depending on both players' uses.

In (2.6), function f_i $(i = 1, 2)$ is the benefit function of Player i, and D_i is the decontamination costs function he faces. Thus the term $f_i(x, p_i(t, x))\chi_{\Omega_i}(x) - D_i(x, c(t, x; p_1, p_2))$ is the profit function of player i, $i = 1, 2$. An actualization parameter $0 < \rho < 1$ is used to consider the net present value of the profit. Notice that the profit is evaluated over all the time period $(0, T)$. Indeed, the use of the PDEs model (2.1)–(2.5) as a constraints system allows to consider a fully adaptative strategy; i.e., it allows to compute explicitly the value $p_i(t, x)$ a.e. in Ω_T. Notice also that the space dependancy of the cost function D_i, $x \mapsto D_i(x, c)$ may especially include the characteristic function of a subset of the domain Ω. So we may either consider the pollution in the whole domain, or, in a less responsible way, we may only focus on the pollution in the freshwater production wells (see the numerical illustrations in section 6). The planing horizon $T \in (0, \infty)$ is finite. Thus in order to take into account the decontamination cost of the pollution remaining at time T, a scrap value is introduced: $\nu e^{-\rho T} \int_\Omega D_i(x, c(T, x; p_1, p_2))\, dx$. The scraping parameter ν, $\nu \geq 0$, enables to weight the scrap value part in the objective J_i. Basically, the more polluted is the groundwater at time T, the more the objective is lowered.

It remains to define the set of controls. Let $\overline{p} > 0$ be the maximal pollutant load that can be applied. This value exists due to saturation limits. We define the admissible sets of control as follows for $i = 1, 2$:

$$E_i = \{q \in L^2(\Omega_i \times (0, T)); 0 \leq q(t, x) \leq \overline{p} \text{ a.e. in } \Omega_i \times (0, T)\}.$$

Now, basically, the strategy of each player for the social welfare, assuming that the one of the other players is known, is resumed by[2]

$$\max_{p_i \in E_1} J_i(p_i; p_{-i}, c(\cdot; p_1, p_2)), \text{ where } p_{-i} \in E_{-i} \text{ is given.}$$

But in the present paper we consider the case of a noncooperative game, that is, when the strategy of the other player is not known.

[2]Here and after, we use the following convention for the indexes: $x_{-1} = x_2$ and $x_{-2} = x_1$.

3. Mathematical assumptions and the main result. The main result of the paper is an existence result for a Nash equilibrium. It will be further characterized by a PDE's problem in section 5 below, then allowing a uniqueness result. We begin by listing in the following subsection all the mathematical assumptions used in the paper.

3.1. Mathematical assumptions. First consider the assumptions related to the hydrogeological description thus to the state problem. For the soil porosity, the dispersion, and the mobility tensors, we assume that there exist real numbers $(\psi_-,\psi_+)\in\mathbb{R}^2$, $(S_m,\alpha_L,\alpha_T)\in\mathbb{R}^3$ with $S_m>0$, $\alpha_L\geq\alpha_T\geq 0$ and $(\kappa_-,\kappa_+)\in\mathbb{R}^2$, $0<\kappa_-\leq\kappa_+$, such that

$$\begin{aligned}&0<\psi_-\leq\psi(x)\leq\psi_+ \text{ a.e. } x\in\Omega,\\ &S(v)\xi\cdot\xi\geq(S_m+\alpha_T|v|)|\xi|^2,\ |S(v)\xi|\leq(S_m+\alpha_L|v|)|\xi| \text{ for all } \xi\in\mathbb{R}^N,\\ &\kappa\xi\cdot\xi\geq\kappa_-|\xi|^2,\ |\kappa\xi|\leq\kappa_+|\xi| \text{ for all } \xi\in\mathbb{R}^N.\end{aligned}$$

Moreover, in order to ensure that the velocity v of the flow belongs to $L^\infty(\Omega_T)$, we suppose that one of the following assumption is true:

$$\kappa\in(\mathcal{C}^1(\bar\Omega))^{N\times N} \text{ and } \phi_b\in W^{2,p}(\bar\Omega) \text{ with } p>N,$$
$$\kappa=\kappa^*\mathrm{Id} \text{ with } \kappa^*:\bar\Omega\to\mathbb{R},\ \kappa^*\in\mathcal{C}^1(\Omega), \text{ and } \phi_b\in W^{2,p}(\bar\Omega) \text{ with } p>\frac{N}{2}.$$

The nonnegative functions g and γ are assumed bounded. The retardation factor $R>0$ is a given real number. The reaction function r, possibly nonlinear, is supposed to be concave, with a bounded derivative in $\mathbb{R}_+$ and satisfying $r(0)=0$. In particular, there exists $r_+\in\mathbb{R}^+$ such that

$$|r(x)|\leq r_+|x| \text{ for all } x\in\mathbb{R}_+.$$

We now provide the assumptions for the initial and boundary values. Initial condition c_0 is given in $H^1(\Omega)$ with $c_0(x)\geq 0$ a.e. in Ω. Functions ϕ_b and v_1 in (2.5) are, respectively, in $H^1(\Omega)$ and in $(L^\infty(\Omega))^N$, with $\mathrm{div}(v_1)\in L^\infty(\Omega_T)$ and the compatibility condition $\int_\Omega g\,dx=\int_{\partial\Omega}\kappa v_1\cdot d\sigma$ for the solvability of the elliptic equation (2.2).

Finally, consider the economic part (2.6) of the problem. We already mentioned that ν and ρ are given real numbers satisfying $\nu\geq 0$ and $0<\rho<1$. Assumptions on benefit and damage functions are the following ones: For $i=1,2$, function $f_i:\Omega\times[0,\bar p]\to\mathbb{R}$ is bounded and such that, for almost every $x\in\Omega$, $p\mapsto f_i(x,p)$ is continuous and strictly concave on $[0,\bar p]$. Function D_i: $\Omega\times\mathbb{R}_+\to\mathbb{R}_+$ is bounded such that, for almost every $x\in\Omega$, $c\in\mathbb{R}_+\mapsto D_i(x,c)$ is a hemicontinuous and convex function.

3.2. Main result. Consider first the state problem alone. A pair (c,ϕ), satisfying $c\in\mathcal{C}([0,T];L^2(\Omega))$ and $\phi\in L^\infty(0,T;H^2(\Omega))$, is a weak solution of (2.1)–(2.2), (2.4)–(2.5) if for any test function $\varphi\in H^1(0,T;H^1(\Omega))$ such that $\varphi|_{t=T}=0$ in Ω and $\varphi=0$ on $\Gamma_2\times(0,T)$,

$$\begin{aligned}(3.1)\quad &-\int_{\Omega_T}R\psi c\partial_t\varphi\,dx\,dt-\int_\Omega Rc_0\varphi|_{t=0}\,dx+\int_{\Omega_T}\psi S(v)\nabla c\cdot\nabla\varphi\,dx\,dt\\ &+\int_{\Omega_T}(v\cdot\nabla c)\varphi\,dx\,dt=\int_{\Omega_T}\Big(-r(c)+p_1\chi_{\Omega_1}+p_2\chi_{\Omega_2}-gc+\gamma\Big)\varphi\,dx\,dt,\end{aligned}$$

where $v = -\kappa\nabla\phi$ is such that for any $\theta \in L^1(0,T;H^1(\Omega))$ with $\theta = 0$ on $\Gamma_4 \times (0,T)$

$$\int_{\Omega_T} \kappa\nabla\phi \cdot \nabla\theta \, dx\, dt + \int_0^T \int_{\Gamma_3} (\kappa v_1 \cdot n)\theta \, ds dt = \int_{\Omega_T} g\theta \, dx\, dt, \tag{3.2}$$

$$\phi = \phi_b \text{ on } \Gamma_4 \times (0,T). \tag{3.3}$$

The well-posedness of the latter formulation is provided by the following lemma (we refer to Augeraud-Véron, Choquet, and Comte [2] for its proof).

LEMMA 3.1. *Let* (p_1, p_2) *be given in* $E_1 \times E_2$ *. There exists a unique solution* (c, ϕ) *of* (2.1)–(2.2)*,* (2.4)–(2.5) *associated to* (p_1, p_2) *in the sense* (3.1)–(3.3)*. Moreover,* $\alpha_T\, |v|^{\frac{1}{2}}\, \nabla c \in (L^2(\Omega_T))^N$*,* $c \in H^1(0,T;H^{-1}(\Omega))$*, and* $c \geq 0$ *a.e. in* Ω_T*.*

Remark 3.2. The velocity v is an exogenous unknown since it is defined by a PDE's problem which does not depend on p_1 and p_2, nor on the concentration c. The existence of v is thus ensured by the latter result once and for all, and we will not anymore mention this question.

Thanks to the uniqueness result in Lemma 3.1, we are allowed to simplify the notation $J_i(p_i;p_{-i},c(t,x;p_1,p_2))$ into $J_i(p_i;p_{-i})$. The players' reaction functions are then defined as follows.

DEFINITION 3.3 (players' reaction functions). *For almost every* $(t,x) \in \Omega_T$*, the reaction functions* $p_i^*(t,x;p_{-i})$*,* $i = 1, 2$*, are defined by*

$$p_1^*(t,x;p_2) = \underset{q_1 \in E_1}{\operatorname{argmax}}\, J_1(q_1;p_2) \text{ for all } p_2 \in E_2, \tag{3.4}$$

$$p_2^*(t,x;p_1) = \underset{q_2 \in E_2}{\operatorname{argmax}}\, J_2(q_2;p_1) \text{ for all } p_1 \in E_1. \tag{3.5}$$

The existence of the reaction functions is ensured by the following result. Notice their uniqueness that will turn out to be fundamental for some of our proofs. Lemma 3.4 is a direct consequence of the well-posedness of the optimal control problem in the cooperative setting proved in Augeraud-Véron, Choquet, and Comte [3] (Theorem 2.6).

LEMMA 3.4. *For a given* p_i *in* E_i*,* $i = 1, 2$*, there exists a unique pair of functions* $(p_{-i}^*, c^*(t,x;p_{-i}^*(t,x;p_i),p_i))$ *such that* $p_{-i}^*(t,x;p_i) = argmax_{q_{-i} \in E_{-i}} J_{-i}(q_{-i};p_i)$ *and* $(c^*(t,x;p_{-i}^*(t,x;p_i),p_i),\phi)$ *is the weak solution of* (2.1)–(2.2)*,* (2.4)–(2.5) *associated with the loads* (p_{-i}^*, p_i)*.*

The Nash equilibrium is defined as the intersection of the reaction functions.

DEFINITION 3.5 (Nash equilibrium). *The quadruplet* $(p_1^\flat, p_2^\flat, c^\flat, \phi)$ *is a Nash equilibrium iff:*

$$J_1(p_1^\flat;p_2^\flat) \geq J_1(p_1;p_2^\flat) \text{ for all } p_1 \in E_1, \tag{3.6}$$

$$J_2(p_2^\flat;p_1^\flat) \geq J_2(p_2;p_1^\flat) \text{ for all } p_2 \in E_2, \tag{3.7}$$

where $c^\flat = c^\flat(\cdot;p_1^\flat,p_2^\flat)$ *is the solution given by Lemma* 3.1 *of the following problem:*

$$\begin{aligned}
&R\psi\partial_t c^\flat - \operatorname{div}(S(v)\psi\nabla c^\flat) + v\cdot\nabla c^\flat = -r(c^\flat) + p_1^\flat\chi_{\Omega_1} + p_2^\flat\chi_{\Omega_2} - gc^\flat + \gamma \text{ in } \Omega_T,\\
&\operatorname{div}(v) = g,\ v = -\kappa\nabla\phi \text{ in } \Omega_T,\\
&S(v)\nabla c^\flat \cdot n = 0 \text{ on } \Gamma_1 \times (0,T),\ c^\flat = 0 \text{ on } \Gamma_2 \times (0,T),\ c^\flat|_{t=0} = c_0 \text{ in } \Omega,\\
&v\cdot n = -\kappa v_1 \cdot n \text{ on } \Gamma_3 \times (0,T),\ \phi = \phi_b \text{ on } \Gamma_4 \times (0,T).
\end{aligned}$$

The main result of the paper is the following existence result.

THEOREM 3.6. *There exists a Nash equilibrium in the sense of Definition* 3.5.

The latter will be completed by a uniqueness result under some additional assumptions in section 5 below. We begin by proving Theorem 3.6 in the next section.

4. Proof of Theorem 3.6. The proof of Theorem 3.6 is based on a fixed point strategy. Let $(p_1, p_2) \in E_1 \times E_2$. Let $p_1^* = argmax_{q_1 \in E_1} J_1(q_1; p_2)$ and $p_2^* = argmax_{q_2 \in E_2} J_2(q_2; p_1)$ be uniquely defined by Lemma 3.4. Set $c_1^*(t,x) = c_1(t,x;p_1^*,p_2)$, $c_2^*(t,x) = c_2(t,x;p_1,p_2^*)$. We aim at proving that the application $\mathcal{C}$ defined by

$$\mathcal{C} : (p_1, p_2) \in E_1 \times E_2 \mapsto (p_1^*, p_2^*) \in E_1 \times E_2$$

admits a fixed point.

First of all, we prove that $\mathcal{C}$ is continuous for the weak topology of $L^2(\Omega_T)$. Consider a sequence $(p_1^n, p_2^n)_{n \geq 0}$ of functions in $E_1 \times E_2$ such that

$$p_1^n \rightharpoonup p_1 \text{ and } p_2^n \rightharpoonup p_2 \text{ weakly in } L^2(\Omega_T). \tag{4.1}$$

They are associated with $(p_1^{*,n}, p_2^{*,n}) := \mathcal{C}(p_1^n, p_2^n)$, $c_1^{*,n}(t,x) = c(t,x;p_1^{*,n},p_2^n)$, and $c_2^{*,n}(t,x) = c(t,x;p_1^n,p_2^{*,n})$, which are, according to Lemmas 3.1 and 3.4, uniquely defined by

$$p_i^{*,n} = \underset{q_i \in E_i}{\operatorname{argmax}}\, J_i(q_i; p_{-i}^n), \tag{4.2}$$

$$\begin{aligned} &R\psi\partial_t c_i^{*,n} - \operatorname{div}(\psi S(v)\nabla c_i^{*,n}) + v \cdot \nabla c_i^{*,n} \\ &\qquad = -r(c_i^{*,n}) + p_i^{*,n}\chi_{\Omega_i} + p_{-i}^n \chi_{\Omega_{-i}} - g c_i^{*,n} + \gamma \text{ in } \Omega_T, \end{aligned} \tag{4.3}$$

$$S(v)\nabla c_i^{*,n} \cdot n = 0 \text{ on } \Gamma_1 \times (0,T),\ c_i^{*,n} = 0 \text{ on } \Gamma_2 \times (0,T),\ c_i^{*,n}|_{t=0} = c_0 \text{ in } \Omega \tag{4.4}$$

for $i = 1, 2$. We have to prove that $p_i^{*,n} \rightharpoonup p_i^*$ weakly in $L^2(\Omega_T)$, $i = 1, 2$, where

$$p_i^* = \underset{q_i \in E_i}{\operatorname{argmax}}\, J_i(q_i; p_{-i}) \tag{4.5}$$

is associated to $c_i^* = c_i(\cdot; p_i^*, p_{-i})$ solution in Ω_T of

$$R\psi\partial_t c_i^* - \operatorname{div}(\psi S(v)\nabla c_i^*) + v \cdot \nabla c_i^* = -r(c_i^*) + p_i^*\chi_{\Omega_i} + p_{-i}\chi_{\Omega_{-i}} - g c_i^* + \gamma, \tag{4.6}$$

$$S(v)\nabla c_i^* \cdot n = 0 \text{ on } \Gamma_1 \times (0,T),\ c_i^* = 0 \text{ on } \Gamma_2 \times (0,T),\ \ c_i^*|_{t=0} = c_0 \text{ on } \Omega. \tag{4.7}$$

The first step for proving the desired convergence results consists in stating some estimates that do not depend on n. Hereafter we denote by C a generic constant which only depends on the data of the problem, namely, on the coefficients of the PDEs (2.1)–(2.2), on the initial and boundary data in (2.4)–(2.5), on $\bar{p}$, and on Ω and T. Due to the definition of the admissible control sets E_i, $0 \leq p_i^{*,n} \leq \bar{p}$ a.e. in Ω_T, and the following estimates thus hold true:

$$\left\| p_i^{*,n} \right\|_{L^2(\Omega_T)} \leq C,\ \ i = 1, 2. \tag{4.8}$$

Multiplying (4.3) by $c_i^{*,n}$, integrating by parts over $\Omega \times (0,\tau)$, $\tau \in (0,T)$, and using (4.4) we get

$$\begin{aligned} &\frac{R}{2}\int_0^\tau \frac{d}{dt}\int_\Omega \psi \left| c_i^{*,n} \right|^2 dx\,dt + \int_0^\tau \int_\Omega \psi S(v)\nabla c_i^{*,n} \cdot \nabla c_i^{*,n}\, dx\,dt \\ &\quad + \int_0^\tau \int_\Omega (v \cdot \nabla c_i^{*,n}) c_i^{*,n}\, dx\,dt + \int_0^\tau \int_\Omega r(c_i^{*,n}) c_i^{*,n}\, dx\,dt \\ &\quad - \int_0^\tau \int_\Omega (p_i^{*,n}\chi_{\Omega_i} + p_{-i}^n \chi_{\Omega_{-i}} + \gamma) c_i^{*,n}\, dx\,dt + \int_0^\tau \int_\Omega g |c_i^{*,n}|^2\, dx\,dt = 0. \end{aligned} \tag{4.9}$$

Our assumptions ensure that

$$\int_0^\tau \int_\Omega \psi S(v)\nabla c_i^{*,n} \cdot \nabla c_i^{*,n}\, dx\, dt \geq \psi_- \int_0^\tau \int_\Omega (S_m + \alpha_T\, |v|) \left|\nabla c_i^{*,n}\right|^2\, dx\, dt,$$

$$\int_0^\tau \int_\Omega \left|r(c_i^{*,n})c_i^{*,n}\right|\, dx\, dt \leq r_+ \int_0^\tau \int_\Omega \left|c_i^{*,n}\right|^2\, dx\, dt,$$

$$\int_0^\tau \int_\Omega g|c_i^{*,n}|^2\, dx\, dt \geq 0.$$

Using the Cauchy–Schwarz and Young inequalities, we get

$$\left|\int_0^\tau \int_\Omega (p_i^{*,n}\chi_{\Omega_i} + p_{-i}^n\chi_{\Omega_{-i}} + \gamma)c_i^{*,n}\, dx\, dt\right| \leq \frac{1}{2}\int_0^\tau \int_\Omega (p_i^{*,n}\chi_{\Omega_i} + p_{-i}^n\chi_{\Omega_2} + \gamma)^2\, dx\, dt$$
$$+\frac{1}{2}\int_0^\tau \int_\Omega (c_i^{*,n})^2\, dx\, dt \leq C + C\int_0^\tau \int_\Omega (c_i^{*,n})^2\, dx\, dt$$

and, since $v \in (L^\infty(\Omega_T))^N$,

$$\left|\int_0^\tau \int_\Omega (v \cdot \nabla c_i^{*,n})c_i^{*,n}\, dx\, dt\right| \leq \frac{\psi_- S_m}{2}\int_{\Omega_T} \left|\nabla c_i^{*,n}\right|^2\, dx\, dt + C\int_{\Omega_T} \left|c_i^{*,n}\right|^2\, dx\, dt.$$

Inserting the latter estimates in (4.9), we obtain

$$\frac{\psi_- R}{2}\int_\Omega \left|c_i^{*,n}(\tau,x)\right|^2\, dx + \int_0^\tau \int_\Omega \psi_- \left(\frac{S_m}{2} + \alpha_T\, |v|\right) \left|\nabla c_i^{*,n}\right|^2\, dx\, dt$$
$$\leq C\int_0^\tau \int_\Omega \left|c_i^{*,n}\right|^2\, dx\, dt + \frac{\psi_+ R}{2}\int_\Omega |c_0(x)|^2\, dx + C$$

for any $\tau \in (0,T)$. We infer from the latter relation and from the Gronwall lemma that

$$\|c_i^{*,n}\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} \leq C, \quad i = 1,2. \tag{4.10}$$

Now, we multiply (4.3) by a test function $\varphi \in L^2(0,T;H_0^1(\Omega))$. We get

$$|\langle \psi R\partial_t c_i^{*,n}, \varphi\rangle_{L^2(0,T;H^{-1}(\Omega))\times L^2(0,T;H_0^1(\Omega))}| = \Bigg| -\int_0^T \int_\Omega \psi S(v)\nabla c_i^{*,n} \cdot \nabla\varphi\, dx\, dt$$
$$+ \int_0^T \int_\Omega (-v \cdot \nabla c_i^{*,n}) - r(c_i^{*,n}) + p_i^{*,n}\chi_{\Omega_i} + p_{-i}^n\chi_{\Omega_{-i}} + \gamma - gc_i^{*,n})\varphi\, dx\, dt\Bigg|.$$

Since $v \in (L^\infty(\Omega_T))^N$, $r(c_i^{*,n})$ is bounded in $L^2(\Omega_T)$ by $r_+\|c_i^{*,n}\|_{L^2(\Omega_T)}$, $p \in L^\infty(\Omega_T)$, and $g \in L^\infty(\Omega_T)$; the latter with (4.10) shows that the left-hand side is uniformly bounded for any $\varphi \in L^2(0,T;H_0^1(\Omega))$, that is,

$$\|\psi R\partial_t c_i^{*,n}\|_{L^2(0,T;H^{-1}(\Omega))} \leq C, \quad i = 1,2. \tag{4.11}$$

From (4.8)–(4.11), we deduce the existence of limit functions $p_i^0 \in L^2(\Omega_T)$ and $c_i^0 \in L^2(0,T;H^1(\Omega))$ and of a subsequence, not relabeled for convenience, such that

$$p_i^{*,n} \rightharpoonup p_i^0 \text{ weakly in } L^2(\Omega_T),\ i = 1,2,$$
$$c_i^{*,n} \to c_i^0 \text{ in } L^2(\Omega_T) \text{ and a.e. in } \Omega_T,\ i = 1,2,$$
$$c_i^{*,n} \rightharpoonup c_i^0 \text{ weakly in } L^2(0,T;H^1(\Omega)),\ i = 1,2,$$

the compactness of $c_i^{*,n}$ being established by an Aubin-type argument thanks to (4.11) and to $R\psi \geq R\psi_- > 0$ (see Galusinski and Saad [12]).

Bear in mind that we aim to prove that $p_i^0 = p_i^*$ and $c_i^0 = c_i^*$, $i = 1, 2$. First of all, convergence results are sufficient (according to the Lebesgue's dominated convergence theorem and the continuity of the application $c \in \mathcal{C}(0, T; L^2(\Omega)) \mapsto c_{|t=0}$) to pass to the limit in the weak formulation of (4.3)–(4.4). According to the uniqueness result in Lemma (3.1), it means that

$$c_1^0(\cdot) = c(\cdot; p_1^0, p_2) \text{ and } c_2^0(\cdot) = c(\cdot; p_1, p_2^0). \tag{4.12}$$

It remains to prove that $p_i^0 = p_i^*$, $i = 1, 2$. We will use the limit behavior of the control problem. The difficulty is that we have no compactness result for $p_i^{*,n}$, whereas it appears in the nonlinear function f_i. Nevertheless we can use convexity arguments. On the one hand, according to the definition of the optimum p_i^*,

$$J_i(p_i^0; p_{-i}) \leq J_i(p_i^*; p_{-i}). \tag{4.13}$$

On the other hand, because of the concavity of f_i, we write

$$\begin{aligned} J_i(p_i^0; p_{-i}) = &\int_0^T \left(\int_{\Omega_i} f_i(x, p_i^0)\, dx - \int_\Omega D_i(x, c_i^0)\, dx \right) e^{-\rho t}\, dt \\ &- \nu e^{-\rho T} \int_\Omega D_i(x, c_i^0(T, x))\, dx \geq \overline{\lim}_{n\to\infty} \int_0^T \int_{\Omega_i} f_i(x, p_i^{*,n}) e^{-\rho t}\, dx\, dt \\ &- \int_0^T \int_\Omega D_i(x, c_i^0) e^{-\rho t}\, dx\, dt - \nu e^{-\rho T} \int_\Omega D_i(x, c_i^0(T, x))\, dx. \end{aligned}$$

Since $c_i^{*,n} \to c_1^0$ in $\mathcal{C}([0, T]; L^2(\Omega))$ and a.e. in Ω_T, we know, thanks to the smoothness of D_i, that $\int_0^T \int_\Omega D_i(x, c_i^0) e^{-\rho t}\, dx\, dt = \lim_{n\to\infty} \int_0^T \int_\Omega D_i(x, c_i^{*,n}) e^{-\rho t}\, dx\, dt$ and that $\nu e^{-\rho T} \int_\Omega D_i(x, c_i^0(T, x))\, dx = \lim_{n\to\infty} \nu e^{-\rho T} \int_\Omega D_i(x, c_i^{*,n}(T, x))\, dx$. The latter relation thus also reads

$$\begin{aligned} J_i(p_i^0; p_{-i}) \geq &\overline{\lim}_{n\to\infty} \int_0^T \left(\int_{\Omega_i} f_i(x, p_i^{*,n})\, dx - \int_\Omega D_i(x, c_i^{*,n})\, dx \right) e^{-\rho t}\, dt \\ &- \overline{\lim}_{n\to\infty} \nu e^{-\rho T} \int_\Omega D_i(x, c_i^{*,n}(T, x))\, dx = \overline{\lim}_{n\to\infty} J_i(p_i^{*,n}; p_{-i}^n). \end{aligned} \tag{4.14}$$

By definition of the optimum $p_i^{*,n}$, we know that $J_i(p_i^{*,n}; p_{-i}^n) \geq J_i(q_i; p_{-i}^n)$ for any $q_i \in E_i$. It yields in (4.14):

$$\begin{aligned} J_i(p_i^0; p_{-i}) \geq \overline{\lim}_{n\to\infty} &\left(\int_0^T \left(\int_{\Omega_i} f_i(x, q_i)\, dx - \int_\Omega D_i(x, c(t, x; q_i, p_{-i}^n))\, dx \right) e^{-\rho t}\, dt \right. \\ &\left. - \nu e^{-\rho T} \int_\Omega D_i(x, c(T, x; q_i, p_{-i}^n))\, dx \right) \end{aligned} \tag{4.15}$$

for all $q_i \in E_i$. We will infer from (4.15) that $J_i(p_i^0; p_{-i})$ is an upper bound of $J_i(q_i; p_{-i})$ for any $q_i \in E_i$. Indeed, we can establish the same estimates for $c(\cdot; q_i, p_{-i}^n)$

than those proved for $c_i^{*,n}$ and conclude that $c(\cdot;q_i,p_{-i}^n) \to c(\cdot;q_i,p_{-i})$ in $L^2(\Omega_T)$ and a.e. in Ω_T. Therefore, we are able to compute the following limit:

$$\lim_{n\to\infty}\left(\int_0^T \left(\int_{\Omega_i} f_i(x,q_i)\,dx - \int_\Omega D_i(x,c(t,x;q_i,p_{-i}^n))\,dx\right) e^{-\rho t}\,dt \right.$$
$$\left. -\nu e^{-\rho T}\int_\Omega D_i(x,c(T,x;q_i,p_{-i}^n))\,dx\right)$$
$$= \int_0^T \left(\int_{\Omega_1} f_i(x,q_i)\,dx - \int_\Omega D_i(x,c(t,x;q_i,p_{-i}))\,dx\right) e^{-\rho t}\,dt$$
$$- \nu e^{-\rho T}\int_\Omega D_i(x,c(T,x;q_i,p_{-i}))\,dx = J_i(q_i,p_{-i}).$$

Including this result in (4.15), we get

$$J_i(p_i^0;p_{-i}) \geq J_i(q_i;p_{-i}) \text{ for any } q_i \in E_i.$$

In particular, this inequality holds true for $q_i = p_i^*$, that is,

$$J_i(p_i^0;p_{-i}) \geq J_i(p_i^*;p_{-i}). \tag{4.16}$$

We conclude from (4.13) and (4.16) that $J_i(p_i^0;p_{-i}) = J_i(p_i^*,p_{-i})$. Function p_i^0 is then solution of the same optimal control problem than p_i^*. The solution of this problem being unique (Lemma 3.4) we conclude that

$$p_i^0 = p_i^*,$$

and the sequence $p_i^{*,n}$ converges weakly to p_i^* in $L^2(\Omega_T)$ for $i=1,2$. The continuity of $\mathcal{C}$ for the weak topology of $L^2(\Omega_T)\times L^2(\Omega_T)$ is proved.

But we also can prove the sequential compactness of the image of $\mathcal{C}$ by going on with the latter sequences and proving that strong convergences actually hold true:

$$p_i^{*,n} \to p_i^* \text{ in } L^2(\Omega_T),\ i=1,2.$$

The key argument will be the strict concavity of f_i. Knowing that $p_i^* = p_i^0$, (4.14) now reads

$$J_i(p_i^*;p_{-i}) \geq \overline{\lim}_{n\to\infty} J_i(p_i^{*,n};p_{-i}^n). \tag{4.17}$$

By definition of $p_i^{*,n} = \operatorname{argmax}_{q_i\in E_i} J_i(q_i;p_{-i}^n)$, we have in particular, since $p_i^* \in E_i$, $J_i(p_i^{*,n};p_{-i}^n) \geq J_i(p_i^*;p_{-i}^n)$. Thus

$$\underline{\lim}_{n\to\infty} J_i(p_i^{*,n};p_{-i}^n) \geq \underline{\lim}_{n\to\infty} J_i(p_i^*;p_{-i}^n) = \int_0^T\int_{\Omega_i} f_i(x,p_i^*)e^{-\rho t}\,dx\,dt$$
$$-\underline{\lim}_{n\to\infty}\left(\int_0^T\int_\Omega D_i(x;c(t,x;p_i^*,p_{-i}^n))e^{-\rho t}\,dx\,dt\right.$$
$$\left.-\nu e^{-\rho T}\int_\Omega D_i(x;c(T,x;p_i^*,p_{-i}^n))\,dx\right), \tag{4.18}$$

where $c(t,x;p_i^*,p_{-i}^n)$ satisfies

$$R\psi\partial_t c(t,x;p_i^*,p_{-i}^n) - \operatorname{div}(\psi S(v)\nabla c(t,x;p_i^*,p_{-i}^n)) + v\cdot\nabla c(t,x;p_i^*,p_{-i}^n) =$$
$$- r(c(t,x;p_i^*,p_{-i}^n)) + p_i^*\chi_{\Omega_i} + p_{-i}^n\chi_{\Omega_{-i}} - gc(t,x;p_i^*,p_{-i}^n) + \gamma \text{ in } \Omega_T,$$
$$S(v)\nabla c(t,x;p_i^*,p_{-i}^n)\cdot n = 0 \text{ on } \Gamma_1\times(0,T),\ c(t,x;p_i^*,p_{-i}^n) = 0 \text{ on } \Gamma_2\times(0,T),$$

with $c(0,x;p_i^*,p_{-i}^n) = c_0$ in Ω. Following the lines of the estimates done for $c_i^{*,n}$, we show that there exists a subsequence of $c(\cdot;p_i^*,p_{-i}^n)$ which strongly converges in $L^2(\Omega_T)$, a.e. in Ω_T, and weakly in $L^2(0,T;H^1(\Omega))$ to the unique solution of

$$R\psi\partial_t c - \mathrm{div}(\psi S(v)\nabla c) + v\cdot\nabla c = -r(c) + p_i^*\chi_{\Omega_i} + p_{-i}\chi_{\Omega_{-i}} - gc + \gamma \text{ in } \Omega_T,$$
$$S(v)\nabla c\cdot n = 0 \text{ on } \Gamma_1\times(0,T),\ c = 0 \text{ on } \Gamma_2\times(0,T), \text{ with } c|_{t=0} = c_0 \text{ in } \Omega,$$

that is,

$$c(\cdot;p_i^*,p_{-i}^n) \to c(\cdot;p_i^*,p_{-i}) \text{ in } L^2(\Omega_T) \text{ and a.e. in } \Omega_T.$$

Thanks to the hemicontinuity of D_i, $i=1,2$, it follows that

$$\underline{\lim}_{n\to\infty} J_i(p_i^{*,n};p_{-i}^n) \ge J_i(p_i^*;p_{-i}). \tag{4.19}$$

From (4.17) and (4.19), we get

$$J_i(p_i^*;p_{-i}) = \lim_{n\to\infty} J_i(p_i^{*,n};p_{-i}^n). \tag{4.20}$$

We can also prove that $c(\cdot;p_i^{*,n},p_{-i}^n) \to c(\cdot;p_i^*,p_{-i})$ in $L^2(\Omega_T)$ and a.e. in Ω_T, and according to the hemicontinuity of D_i, as $n\to\infty$,

$$\int_{\Omega_T} D_i(x,c(t,x;p_i^{*,n},p_{-i}^n))e^{-\rho t}\,dx\,dt \to \int_{\Omega_T} D_i(x,c(t,x;p_i^*,p_{-i}))e^{-\rho t}\,dx\,dt,$$
$$\int_{\Omega} D_i(x,c(T,x;p_i^{*,n},p_{-i}^n))e^{-\rho T}\,dx \to \int_{\Omega} D_i(x,c(T,x;p_i^*,p_{-i}))e^{-\rho T}\,dx.$$

Thus from (4.20) we get

$$\lim_{n\to\infty}\int_0^T\int_{\Omega_i} f_i(x,p_i^{*,n})e^{-\rho t}\,dx\,dt = \int_0^T\int_{\Omega_i} f_i(x,p_i^*)e^{-\rho t}\,dx\,dt.$$

Function f_i being continuous and strictly concave, we infer from the latter relation and Theorem 3 in Visintin [22] that

$$p_i^{*,n} \to p_i^* \text{ in } L^2(\Omega_i\times(0,T)), \quad i=1,2.$$

We have proved that $\mathcal{C}$ is a compact application in $L^2(\Omega_T)\times L^2(\Omega_T)$.

Finally, the Schauder fixed point theorem applies: there exists $(p_1^\flat,p_2^\flat)\in E_1\times E_2$ such that

$$\mathcal{C}(p_1^\flat,p_2^\flat) = (p_1^\flat,p_2^\flat).$$

By definition of $\mathcal{C}$, it satisfies

$$\begin{cases} p_1^\flat = \mathrm{argmax}_{q_1\in E_1} J_1(q_1;p_2^\flat) \text{ then } J_1(p_1^\flat;p_2^\flat) \ge J_1(q_1;p_2^\flat) \text{ for all } q_1\in E_1,\\ p_2^\flat = \mathrm{argmax}_{q_2\in E_2} J_2(q_2;p_1^\flat) \text{ then } J_2(p_2^\flat;p_1^\flat) \ge J_2(q_2;p_1^\flat) \text{ for all } q_2\in E_2. \end{cases}$$

Thus $(p_1^\flat,p_2^\flat)$ is a Nash equilibrium. This ends the proof of Theorem 3.6.

5. Characterization of a Nash equilibrium by the adjoint problem: A uniqueness result. In view of the nonlinearities in the problem, both in the objective functions and in the state equation, proving a uniqueness result for the Nash equilibrium is a complex issue. We turn to another formulation of (3.6)–(3.7) by deriving the adjoint problems associated with the reaction functions given in Definition 3.3. Such a Pontryagin approach allows us to characterize a Nash equilibrium by a PDE's problem but at the cost of some additional assumptions on the objective functions. Notice that all the assumptions listed in subsection 3.1 remain.

5.1. Optimality conditions. First we derive the adjoint problem satisfied by a Nash equilibrium.

LEMMA 5.1. *Assume that* f_i: $p \in [0,\bar{p}] \mapsto f_i(x,p)$ *and* $c \in \mathbb{R}_+ \mapsto D_i(x,c)$, $i = 1,2$, *are* $\mathcal{C}^1$ *functions for almost every* $x \in \Omega$. *Let* $(p_1^\flat, p_2^\flat)$ *be a Nash equilibrium defined by Definition* 3.5. *Let* $c^\flat(t,x) = c(t,x;p_1^\flat,p_2^\flat)$. *There exists* $(\mu_1^\flat,\mu_2^\flat) \in (L^2(0,T;H^1(\Omega)))^2$ *such that for* $i = 1,2$

$$(5.1)\ \frac{\partial f_i}{\partial p}(x,p_i^\flat(t,x)) = \mu_i^\flat(t,x)\chi_{\Omega_i}(x) \text{ in } \Omega_T,$$

$$(5.2)\ R\psi\partial_t\mu_i^\flat + v\cdot\nabla\mu_i^\flat + \mathrm{div}(\psi S(v)\nabla\mu_i^\flat) = r'(c^\flat)\mu_i^\flat + R\mu_i^\flat\psi\rho - \frac{\partial D_i}{\partial c}(x,c^\flat) \text{ in } \Omega_T,$$

$$(5.3)\ \big(\psi S(v)\nabla\mu_i^\flat + \mu_i^\flat(\chi_{\Gamma_4}v - \chi_{\Gamma_3}\kappa v_1)\big)\cdot n = 0 \text{ on } \Gamma_1\times(0,T), \mu_i^\flat = 0 \text{ on } \Gamma_2\times(0,T),$$

$$(5.4)\ R\psi\mu_i^\flat(T,x) = \nu\frac{\partial D_i}{\partial c}(x,c^\flat(T,x)) \text{ in } \Omega.$$

Proof. The proof relies on the fact that a Nash equilibrium corresponds to an intersection of the optimality conditions associated to the players' reaction functions of Definition 3.3. Let λ_1 (resp., λ_2) be the adjoint variable associated with the state variable c for the first (resp., second) reaction function. The corresponding Lagrangian functions are, for $i = 1,2$,

$$\mathcal{L}_i(c,p_i,\lambda_i) = J_i(p_i;p_{-i}^\flat,c) + \int_0^T\int_\Omega\big(R\psi\partial_t c + v\cdot\nabla c - \mathrm{div}(\psi S(v)\nabla c) + r(c) + gc - p_i\chi_{\Omega_i} - p_{-i}^\flat\chi_{\Omega_{-i}} - \gamma\big)\lambda_i\,dx\,dt.$$

Using the relation $(v\cdot\nabla c)\lambda_i = (\mathrm{div}(vc) - c\,\mathrm{div}(v))\lambda_i$, bearing in mind that $\mathrm{div}\,v = g$, integrating by parts, and using Fubini's theorem, we get the following form of $\mathcal{L}_i$:

$$\begin{aligned}\mathcal{L}_i(c,p_i,\lambda_i) = &\int_0^T\int_{\Omega_i} f_i(x,p_i(t,x))e^{-\rho t}\,dx\,dt - \int_0^T\int_\Omega D_i(x,c(t,x;p_i,p_{-i}^\flat))e^{-\rho t}\,dx\,dt\\ &-\nu e^{-\rho T}\int_\Omega D_i(x,c(T,x;p_i,p_{-i}^\flat)))\,dx + \int_0^T\int_\Omega(-R\psi\partial_t\lambda_i c - (v\cdot\nabla\lambda_i)c\\ &+\psi S(v)\nabla c\cdot\nabla\lambda_i + r(c)\lambda_i)\,dx\,dt - \int_0^T\int_\Omega(p_i\chi_{\Omega_i} + p_{-i}^\flat\chi_{\Omega_{-i}} + \gamma)\lambda_i\,dx\,dt\\ &+R\int_\Omega\psi c(T,x)\lambda_i(T,x)\,dx - R\psi\int_\Omega c_0(x)\lambda_i(0,x)\,dx\\ &-\int_0^T\int_{\Gamma_2}(\psi S(v)\nabla c\cdot n)\lambda_i d\sigma\,dt + \int_0^T\int_{\Gamma_1}\lambda_i c(\chi_{\Gamma_4}v - \chi_{\Gamma_3}\kappa v_1)\cdot n\,d\sigma\,dt.\end{aligned}$$

We compute its variations using Taylor's first order formula for the nonlinearities. Since $S(v)$ is symmetric, we obtain

$$\begin{aligned}\delta\mathcal{L}_i(c,p_i,\lambda_i) = &\int_0^T\int_{\Omega_i}\left(\frac{\partial f_i}{\partial p}(x,p_i(t,x))e^{-\rho t} - \lambda_i\right)\delta p_i\,dx\,dt\\ &+\int_\Omega\left(R\psi\lambda_i(T,x) - \nu\frac{\partial D_i}{\partial c}(x,c(T,x;p_i,p_{-i}^\flat))e^{-\rho T}\right)\delta c(T,x)\,dx\end{aligned}$$

$$+\int_0^T\int_\Omega\left(-R\psi\partial_t\lambda_i - v\cdot\nabla\lambda_i + r'(c)\lambda_i - \frac{\partial D_i}{\partial c}(x,c(t,x;p_i,p^\flat_{-i}))e^{-\rho t}\right)\delta c\,dx\,dt$$
$$-\int_0^T\int_\Omega \operatorname{div}(\psi S(v)\nabla\lambda_i)\delta c\,dx\,dt - \int_0^T\int_{\Gamma_2}(\psi S(v)\delta(\nabla c)\cdot n)\lambda_i d\sigma\,dt$$
$$+\int_0^T\int_{\Gamma_1}(\psi S(v)\nabla\lambda_i)\cdot n\delta c d\sigma\,dt + \int_0^T\int_{\Gamma_1}\lambda_i\delta c(\chi_{\Gamma_4}v - \chi_{\Gamma_3}v_1)\cdot n\,d\sigma\,dt.$$

Set $\mu_i^\flat(t,x)=\lambda_i(t,x)e^{\rho t}$ for $(t,x)\in\Omega_T$. Canceling the Lagrangian variations with respect to $c(T,x)$, to the control p_i, and to the state variable c, respectively, provides the terminal condition (5.4), the optimality condition (5.1), the adjoint equation (5.2), and the boundary conditions (5.3). ∎

Adjoint equation (5.2) is antidiffusive. However, it is well-posed according to its terminal condition (5.4). We define the time reversal operator $\mathcal{C}_T : L^1(0,T)\to L^1(0,T)$ and the functions $F_i : f_i'([0,\bar p])\to[0,\bar p]$, $i=1,2$, by

$$\mathcal{C}_T u(t) = u(T-t) \text{ for } t\in[0,T], \quad \frac{\partial f_i}{\partial p}(x,F_i(y)) = y.$$

The existence of F_i is ensured by the strict concavity assumption on $p\mapsto f_i(\cdot,p)$. Using the latter notations, the state system (2.1)–(2.4) and Lemma 5.1, a Nash equilibrium is now characterized by the following PDE's problem.

Definition 5.2 (adjoint problem $\mathcal{P}_{\text{adj}}$). *Problem $\mathcal{P}_{\text{adj}}$ consists in finding $(c^\flat,\mu_1^\flat,\mu_2^\flat)$ satisfying*

$$R\psi\partial_t c^\flat + v\cdot\nabla c^\flat - \operatorname{div}(\psi S(v)\nabla c^\flat) = -r(c^\flat) - gc^\flat + \chi_{\Omega_1}F_1(\chi_{\Omega_1}\mathcal{C}_T\mu_1^\flat) + \chi_{\Omega_2}F_2(\chi_{\Omega_2}\mathcal{C}_T\mu_2^\flat) + \gamma \text{ in } \Omega_T, \tag{5.5}$$

$$S(v)\nabla c^\flat\cdot n = 0 \text{ on } \Gamma_1\times(0,T),\ c^\flat = 0 \text{ on } \Gamma_2\times(0,T),\ c^\flat_{|t=0} = c_0 \text{ in } \Omega, \tag{5.6}$$

$$R\psi\partial_t\mu_i^\flat - \mathcal{C}_T v\cdot\nabla\mu_i^\flat - \operatorname{div}(\psi S(\mathcal{C}_T v)\nabla\mu_i^\flat) + r'(\mathcal{C}_T c^\flat)\mu_i^\flat + R\psi\rho\mu_i^\flat - \frac{\partial D_i}{\partial c}(\cdot,\mathcal{C}_T c^\flat) = 0 \text{ in } \Omega_T, \quad i=1,2, \tag{5.7}$$

$$\big(\psi S(\mathcal{C}_T v)\nabla\mu_i^\flat + \mu_i^\flat(\chi_{\Gamma_4}\mathcal{C}_T v - \chi_{\Gamma_3}\kappa\mathcal{C}_T v_1)\big)\cdot n = 0 \text{ on } \Gamma_1\times(0,T),$$
$$\mu_i^\flat = 0 \text{ on } \Gamma_2\times(0,T), \quad i=1,2, \tag{5.8}$$

$$R\psi\mu^\flat_{i\,|t=0} = \nu\frac{\partial D_i}{\partial c}(\cdot, c^\flat{}_{|t=T}), \quad i=1,2, \tag{5.9}$$

where v is the solution of (2.2) *and* (2.5).

We state and prove the following existence result of a weak solution for problem $\mathcal{P}_{\text{adj}}$.

Proposition 5.3. *There exists a weak solution $(c^\flat,\mu_1^\flat,\mu_2^\flat)$ of problem $\mathcal{P}_{\text{adj}}$ belonging to $(\mathcal{C}([0,T];L^2(\Omega))\cap L^2(0,T;H^1(\Omega))\cap H^1(0,T;H^{-1}(\Omega)))^3$. Moreover, if $c\in\mathbb{R}_+\mapsto D_i(x,c)$ is an increasing function a.e. in Ω_T, then $\mu_i^\flat\ge 0$ a.e. in Ω_T, $i=1,2$.*

Proof. The existence result for such a system of parabolic equations coupled by continuous and bounded nonlinearities is classical. Its proof, for instance, using a fixed point approach after the linearization of (5.5)–(5.9), is thus not detailed. We check the nonnegativity of $\mu_i^\flat$. Let $\mu^- = \sup(-\mu_i^\flat, 0)$. We multiply (5.7) by μ^-, and we integrate by parts over $\Omega_\tau = \Omega \times (0,\tau)$ with $\tau \in (0,T)$. We obtain

$$\begin{aligned}
&\frac{R}{2}\int_\Omega \psi|\mu^-|^2\,dx + \int_{\Omega_\tau} (\mathcal{C}_T v\cdot\nabla\mu_i^\flat)\mu^-\,dxdt + \int_{\Omega_\tau} \psi_- S_m |\nabla\mu^-|^2\,dx\,dt \\
&\quad + \int_{\Omega_\tau} \big(r'(c^\flat) + R\psi\rho\big)|\mu^-|^2\,dxdt + \int_{\Omega_\tau} \frac{\partial D_i}{\partial c}(\mathcal{C}_T c^\flat)\mu^-\,dx\,dt \le \frac{R}{2}\int_\Omega \psi|\mu^-_{|t=0}|^2\,dx \\
&\quad - \int_0^\tau\int_{\Gamma_1} \Big((\mathcal{C}_T v\cdot n)\chi_{\Gamma_4} - (\kappa\mathcal{C}_T v_1\cdot n)\chi_{\Gamma_3}\Big)|\mu^-|^2\,ds\,dt. \qquad (5.10)
\end{aligned}$$

Notice that $\mu^-_{|t=0} = 0$ because we assume here that $\partial_c D_i$ is a nonnegative function. Moreover, integrating by parts and using $\mathcal{C}_T v\cdot\nabla\mu_i^\flat = \mathrm{div}(\mathcal{C}_T v\mu_i^\flat) - \mu_i^\flat \mathrm{div}(\mathcal{C}_T v)$, we compute

$$\begin{aligned}
\int_{\Omega_\tau} (\mathcal{C}_T v\cdot\nabla\mu_i^\flat)\mu^-\,dxdt &= \int_{\Omega_\tau} \mathrm{div}(\mathcal{C}_T v\mu_i^\flat)\mu^-\,dx\,dt + \int_{\Omega_\tau} g|\mu^-|^2\,dx\,dt \\
&= \int_{\Omega_\tau} (\mathcal{C}_T v\cdot\nabla\mu^-)\mu^-\,dx\,dt + \int_{\Omega_\tau} g|\mu^-|^2\,dx\,dt \\
&\quad - \int_0^\tau\int_{\Gamma_1} \Big((\mathcal{C}_T v\cdot n)\chi_{\Gamma_4} - (\kappa\mathcal{C}_T v_1\cdot n)\chi_{\Gamma_3}\Big)|\mu^-|^2\,ds\,dt,
\end{aligned}$$

where $\int_{\Omega_\tau} g|\mu^-|^2\,dx\,dt \ge 0$ and

$$\left|\int_{\Omega_\tau} (\mathcal{C}_T v\cdot\nabla\mu^-)\mu^-\,dx\,dt\right| \le \int_{\Omega_\tau} \frac{\psi_- S_m}{2}|\nabla\mu^-|^2\,dx\,dt + C\int_{\Omega_\tau} |\mu^-|^2\,dx\,dt$$

since v belongs to $(L^\infty(\Omega_T))^N$. The other terms in (5.10) satisfy $\int_{\Omega_\tau} R\psi\rho|\mu^-|^2dxdt \ge 0$, $\int_{\Omega_\tau} \frac{\partial D_i}{\partial c}(\mathcal{C}_T c^\flat)\mu^- dxdt \ge 0$, and $|\int_{\Omega_\tau} r'(c^\flat)|\mu^-|^2dxdt| \le r_+\int_{\Omega_\tau}|\mu^-|^2dxdt$. Inserting all the latter estimates in (5.10), we obtain

$$\frac{R}{2}\int_\Omega \psi_-|\mu^-|^2\,dx + \int_{\Omega_\tau} \frac{\psi_- S_m}{2}|\nabla\mu^-|^2\,dx\,dt \le C\int_{\Omega_\tau} |\mu^-|^2\,dx\,dt$$

for any $\tau \in (0,T)$. We conclude with the Gronwall lemma that $\int_{\Omega_T}|\mu^-|^2dxdt = 0$; thus $\sup(-\mu_i^\flat, 0) = 0$ a.e. in Ω_T. This ends the proof of the result. ◻

5.2. A uniqueness result for the Nash equilibrium. According to Lemma 5.1, the uniqueness of the Nash equilibrium will be ensured by a uniqueness result for the solution of problem $\mathcal{P}_{\mathrm{adj}}$ given in Proposition 5.3. We state and prove the following result.

THEOREM 5.4. *Assume that $\mu_i^\flat$ given in Proposition 5.3 belongs to $L^\infty(\Omega_T)$ for $i = 1, 2$. Assume that the functions r', F_i, and $\partial_c D_i$, $i = 1, 2$, are Lipschitz continuous. Then the solution of problem $\mathcal{P}_{\mathrm{adj}}$ is unique.*

Theorem 5.4 is based on the assumption that $\mu_i^\flat$ given in Proposition 5.3 belongs to $L^\infty(\Omega_T)$ for $i = 1, 2$. We claim that it makes sense because this assumption may hold in a variety of settings. An example is the following.

Assuming $c \in \mathbb{R}_+ \mapsto D_i(x, c)$ is an increasing function a.e. in Ω_T, we have already seen in Proposition 5.3 that $\mu_i^\flat \ge 0$ a.e. in Ω_T, $i = 1, 2$. Assume moreover that $\kappa v_1\cdot n \le 0$ and $\Gamma_4 = \emptyset$ (which means that pollution is trapped in the domain), or assume that

$\Gamma_1 = \emptyset$. Assume that there exists $\mu_{\max} \in \mathbb{R}_+$ such that $\nu\partial_c D_i(x,c) \leq \mu_{\max}$ and $(g(t,x) - r'(c) + \rho)\mu_{\max} - \partial_c D_i(x,c) \geq 0$ for any $c \in \mathbb{R}_+$ and for a.e. $(t,x) \in \Omega_T$. Then $0 \leq \mu_i^\flat(t,x) \leq \mu_{\max}$ for a.e. $(t,x) \in \Omega_T$.

For proving this boundedness result, notice that $\mu_i^\flat - \mu_{\max}$ solves the following problem:

$$
\begin{aligned}
&R\psi\partial_t(\mu_i^\flat - \mu_{\max}) - \mathcal{C}_T v \cdot \nabla(\mu_i^\flat - \mu_{\max}) - \operatorname{div}(\psi S(\mathcal{C}_T v)\nabla(\mu_i^\flat - \mu_{\max})) + r'(\mathcal{C}_T c)\mu_{\max} \\
&\quad + r'(\mathcal{C}_T c)(\mu_i^\flat - \mu_{\max}) + R\psi\rho(\mu_i^\flat - \mu_{\max}) + R\psi\rho\mu_{\max} - \partial_c D_i(x, \mathcal{C}_T c) = 0 \text{ in } \Omega_T, \\
&\max(0, \mu_i^\flat - \mu_{\max})_{|t=0} = 0 \text{ in } \Omega, \\
&\psi S(\mathcal{C}_T v)\nabla(\mu_i^\flat - \mu_{\max}) \cdot n + \big((\mathcal{C}_T v \cdot n)\chi_{\Gamma_4} - (\kappa\mathcal{C}_T v \cdot n)\chi_{\Gamma_3}\big)\mu_i^\flat = 0 \text{ on } \Gamma_1 \times (0,T), \\
&\max(0, \mu_i^\flat - \mu_{\max}) = \max(0, -\mu_{\max}) = 0 \text{ on } \Gamma_2 \times (0,T),
\end{aligned}
$$

the initial condition being given by the additional assumption on $\nu\partial_c D_i$. Then multiply the first equation by $\mu^+ := \max(0, \mu_i^\flat - \mu_{\max})$, and integrate by parts over $\Omega \times (0,\tau)$, $\tau \in (0,T)$. Similar computations to those in the proof of the nonnegativity of $\mu_i^\flat$ given for Proposition 5.3 then give the result.

Proof of Theorem 5.4. The difficulty lies in the strong nonlinear coupling between (5.5) and (5.7). It appears in particular that the nonlinearities do not allow in general the use of the Gronwall lemma for proving the uniqueness result. Indeed, the Gronwall lemma gives an estimate that only depends on the initial value of a functional iff this functional only appears at the power one and uncoupled from the other unknowns in the estimates. Here we thus begin by proving the uniqueness for small times. Then we will check that the argument may be reiterated until the complete time interval of study is covered.

For the Lipschitz-type assumptions in Theorem 5.4, we introduce the following notation: If function ℓ is Lipschitz continuous, we denote by ℓ_+ the real number such that $|\ell(x) - \ell(y)| \leq \ell_+|x-y|$ for any (x,y). Assume now that there exist two solutions, (c, μ_1, μ_2) and $(\underline{c}, \underline{\mu}_1, \underline{\mu}_2)$ of problem $\mathcal{P}_{\text{adj}}$. Their difference solves the following problem:

$$
\begin{aligned}
&R\psi\partial_t(c - \underline{c}) + v \cdot \nabla(c - \underline{c}) - \operatorname{div}(\psi S(v)\nabla(c - \underline{c})) = -(r(c) - r(\underline{c})) \\
&\qquad - g(c - \underline{c}) + \sum_{i=1,2} \chi_{\Omega_i}\big(F_i(\mathcal{C}_T\mu_i) - F_i(\mathcal{C}_T\underline{\mu}_i)\big) \text{ in } \Omega_T,
\end{aligned} \tag{5.11}
$$

$$\psi S(v)\nabla(c - \underline{c}) \cdot n = 0 \text{ on } \Gamma_1 \times (0,T),\ c - \underline{c} = 0 \text{ on } \Gamma_2 \times (0,T), \tag{5.12}$$

$$(c - \underline{c})_{|t=0} = 0 \text{ in } \Omega, \tag{5.13}$$

$$
\begin{aligned}
&R\psi\partial_t(\mu_i - \underline{\mu}_i) - \mathcal{C}_T v \cdot \nabla(\mu_i - \underline{\mu}_i) - \operatorname{div}(\psi S(\mathcal{C}_T v)\nabla(\mu_i - \underline{\mu}_i)) \\
&\qquad + r'(\mathcal{C}_T c)(\mu_i - \underline{\mu}_i) + \big(r'(\mathcal{C}_T c) - r'(\mathcal{C}_T \underline{c})\big)\underline{\mu}_i + R\psi\rho(\mu_i - \underline{\mu}_i) \\
&\qquad - \big(\partial_c D_i(x, \mathcal{C}_T c) - \partial_c D_i(x, \mathcal{C}_T \underline{c})\big) = 0 \text{ in } \Omega_T, \\
&\psi S(\mathcal{C}_T v)\nabla(\mu_i - \underline{\mu}_i) \cdot n + (\mathcal{C}_T v \cdot n)\chi_{\Gamma_4}(\mu_i - \underline{\mu}_i) \\
&\qquad - (\kappa\mathcal{C}_T v_1 \cdot n)\chi_{\Gamma_3}(\mu_i - \underline{\mu}_i) = 0 \text{ on } \Gamma_1 \times (0,T),
\end{aligned} \tag{5.14}
$$

$$\mu_i = \underline{\mu}_i \text{ on } \Gamma_2 \times (0,T), \tag{5.15}$$

$$(\mu_i - \underline{\mu}_i)_{|t=0} = \nu\partial_c D_i(x, c_{|t=T}) - \nu\partial_c D_i(x, \underline{c}_{|t=T}) \text{ in } \Omega. \tag{5.16}$$

Let $T_0 \in (0,T)$ and $\tau \in (0,T_0)$. First, multiply (5.11) by $(c - \underline{c})$, and integrate by parts over $\Omega_\tau = \Omega \times (0,\tau)$. Using the Cauchy–Schwarz and Young inequalities, bearing in mind the assumptions, we get the following estimate:

$$\frac{R\psi_-}{2}\int_\Omega (c-\underline{c})^2_{|t=\tau}\,dx + \frac{\psi_- S_m}{2}\int_{\Omega_\tau} |\nabla(c-\underline{c})|^2\,dx\,dt \\ \le \left(\frac{\|v\|^2_\infty}{2\psi_- S_m} + r_+ + 1\right) T_0 \sup_{[0,T_0]}\int_\Omega (c-\underline{c})^2_{|t=\tau}\,dx \\ + \frac{F^2_{i,+}T_0}{2}\sum_{i=1,2}\sup_{[0,T_0]}\int_\Omega |\mu_i - \underline{\mu}_i|^2\,dx. \tag{5.17}$$

Next, multiply (5.14) by $(\mu_i - \underline{\mu}_i)$, and integrate by parts over Ω_τ. Once again, using the Cauchy–Schwarz and Young inequalities, we get the following estimate:

$$\frac{R\psi_-}{2}\int_\Omega (\mu_i-\underline{\mu}_i)^2_{|t=\tau}\,dx + \psi_- S_m\int_{\Omega_\tau} |\nabla(\mu_i-\underline{\mu}_i)|^2\,dx\,dt + \int_{\Omega_\tau} R\psi\rho|\mu_i-\underline{\mu}_i|^2\,dx\,dt \\ + \int_0^\tau\int_{\Gamma_1} \left((\mathcal{C}_T v - \kappa\mathcal{C}_T v_1)\cdot n\right)|\mu_i-\underline{\mu}_i|^2\,ds\,dt \\ \le \int_{\Omega_\tau}(\mathcal{C}_T v\cdot\nabla(\mu_i-\underline{\mu}_i))(\mu_i-\underline{\mu}_i)\,dx\,dt + \frac{R\psi_+\nu}{2}\int_\Omega |(\partial_c D_i(c_{|t=T}) - \partial_c D_i(\underline{c}_{|t=T})|^2\,dx \\ + \left(r_+ + \frac{\|\mu_i\|_\infty r'_+}{2} + \frac{(\partial_c D_i)_+}{2}\right) T_0\sup_{[0,T_0]}\int_\Omega |\mu_i-\underline{\mu}_i|^2\,dx \\ + \left(\frac{\|\mu_i\|_\infty r'_+}{2} + \frac{(\partial_c D_i)_+}{2}\right) T_0\sup_{[0,T_0]}\int_\Omega |c-\underline{c}|^2\,dx, \tag{5.18}$$

where, since $\mathcal{C}_T v\cdot\nabla(\mu_i-\underline{\mu}_i) = \mathrm{div}\big(\mathcal{C}_T v(\mu_i-\underline{\mu}_i)\big) - (\mu_i-\underline{\mu}_i)\mathrm{div}(\mathcal{C}_T v)$,

$$\int_{\Omega_\tau}(\mathcal{C}_T v\cdot\nabla(\mu_i-\underline{\mu}_i))(\mu_i-\underline{\mu}_i)\,dx\,dt = \int_{\Omega_\tau}\mathrm{div}\big(\mathcal{C}_T v(\mu_i-\underline{\mu}_i)\big)(\mu_i-\underline{\mu}_i)\,dx\,dt \\ - \int_{\Omega_\tau} g|\mu_i-\underline{\mu}_i|^2\,dx\,dt = -\int_{\Omega_\tau}(\mu_i-\underline{\mu}_i)\big(\mathcal{C}_T v\cdot\nabla(\mu_i-\underline{\mu}_i)\big)\,dx\,dt \\ + \int_0^\tau\int_{\Gamma_1}\big((\mathcal{C}_T v - \kappa\mathcal{C}_T v_1)\cdot n\big)|\mu_i-\underline{\mu}_i|^2\,ds\,dt - \int_{\Omega_\tau} g|\mu_i-\underline{\mu}_i|^2\,dx\,dt$$

and

$$\left|\int_{\Omega_\tau}(\mu_i-\underline{\mu}_i)\big(\mathcal{C}_T v\cdot\nabla(\mu_i-\underline{\mu}_i)\big)\,dx\,dt\right| \le \frac{\psi_- S_m}{2}\int_{\Omega_\tau}|\nabla(\mu_i-\underline{\mu}_i)|^2\,dx\,dt \\ + \frac{\|v\|^2_\infty}{2\psi_- S_m} T_0\sup_{[0,T_0]}\int_\Omega |\mu_i-\underline{\mu}_i|^2\,dx\,dt.$$

Using the two latter relations in (5.18), we obtain

$$\frac{R\psi_-}{2}\int_\Omega (\mu_i-\underline{\mu}_i)^2_{|t=\tau}\,dx + \frac{\psi_- S_m}{2}\int_{\Omega_\tau}|\nabla(\mu_i-\underline{\mu}_i)|^2\,dx\,dt \\ \le \left(r_+ + \frac{\|\mu_i\|_\infty r'_+}{2} + \frac{(\partial_c D_i)_+}{2} + \frac{\|v\|^2_\infty}{2\psi_- S_m} + \|g\|_\infty\right) T_0\sup_{[0,T_0]}\int_\Omega |\mu_i-\underline{\mu}_i|^2\,dx \\ + \left(\frac{\|\mu_i\|_\infty r'_+}{2} + \frac{1}{2}(1+R\psi_+\nu)(\partial_c D_i)_+\right) T_0\sup_{[0,T_0]}\int_\Omega |c-\underline{c}|^2\,dx. \tag{5.19}$$

We pass to the sup in the sum of relations (5.17) and (5.19). We obtain

$$\frac{R\psi_-}{2} - T_0\left(\frac{\|v\|_\infty^2}{2\psi_- S_m} + r_+ + 1 + \|\mu_i\|_\infty r'_+ + (1 + R\psi_+\nu)(\partial_c D_i)_+\right) \sup_{[0,T_0]} \int_\Omega |c - \underline{c}|^2\,dx$$
$$+ \sum_{i=1,2}\left(\frac{R\psi_-}{2} - T_0\left(\frac{F_{i,+}^2}{2} + r_+ + \frac{\|\mu_i\|_\infty r'_+}{2} + \frac{(\partial_c D_i)_+}{2} + \frac{\|v\|_\infty^2}{2\psi_- S_m} + \|g\|_\infty\right)\right)$$
$$\times \sup_{[0,T_0]} \int_\Omega |\mu_i - \underline{\mu}_i|^2\,dx + \frac{\psi_- S_m}{2}\int_{\Omega_{T_0}}\left(|c - \underline{c}|^2\,dx\,dt + \sum_{i=1,2}|\mu_i - \underline{\mu}_i|^2\right)dx\,dt$$
$$\leq 0.$$

We conclude that $c = \underline{c}$ and $\mu_i = \underline{\mu}_i$, $i = 1,2$, a.e. in $\Omega \times (0, T_0)$ if T_0 is such that

$$T_0 < T_+ := \frac{R\psi_-}{2}\min\left\{\left(\frac{\|v\|_\infty^2}{2\psi_- S_m} + r_+ + 1 + \|\mu_i\|_\infty r'_+ + (1 + R\psi_+\nu)(\partial_c D_i)_+\right)^{-1};\right.$$
$$\left.\left(\frac{F_{i,+}^2}{2} + r_+ + \frac{\|\mu_i\|_\infty r'_+}{2} + \frac{(\partial_c D_i)_+}{2} + \frac{\|v\|_\infty^2}{2\psi_- S_m} + \|g\|_\infty\right)^{-1}\right\}$$

Finally, we can extend this uniqueness result in the small, for instance, from $[0, T_+/2]$ to $[0, T_+]$ by choosing the values $(c - \underline{c})_{t=T_+/2} = 0$ and $(\mu_i - \underline{\mu}_i)_{t=T_+/2} = 0$, $i = 1,2$, as new initial conditions and by using the same arguments in $[T_+/2, T_+]$. We reiterate until covering the whole $[0, T]$. The global uniqueness result of Theorem 5.4 is proved. ∎

6. Numerical illustrations. This last section is devoted to the presentation of some numerical illustrations in the context of agricultural pollution due to fertilization. Although we had to limit ourselves to a few examples of situations, they help to get insight into the applicability of the previous theoretical results. We adapt the tools developed in [8] for the numerical simulation of optimal control problems linked with groundwater pollution. More precisely, we compute the solution of (5.5)–(5.9) using a finite volume scheme based on a two-point flux approximation with upwind mobilities embedded in an iterative fixed point approximation. The time and space steps characterizing the discretization of the PDEs model have been fixed at 10^{-2}.

The aquifer is figured by the parallelepiped $(x,y) \in]0,900[^2$, $z \in]-9,0[$; the values are given here in meters. Indeed most of the groundwater reservoirs are thin geological formations. The players are two farmers who equally own the field above the aquifer: $]0,450[\times]0,900[$ for Player 1, $[450,900[\times]0,900[$ for Player 2. One or two water production wells are located on the parcel on the line $y = 450$. We will study the influence of their positions on the results. The duration of the experiment is 100 days which corresponds to the fertilizer application period for most cereal crops. The physical parameters have been chosen in agreement with the classical literature (see, for instance, Bear [5]). The soil is characterized by the parameters $\kappa = 39.04$ m·day^{-1} and $\psi = 0.3$. In such a shallow aquifer the fluid displacement is essentially horizontal. For ensuring the interpretability of the results, the fluid velocity equation (2.2) is completed by boundary conditions that are homogeneous, except at the left and right boundaries, $x = 0$ and $x = 900$, leading to a quasi-horizontal flow from left to right (see Figure 1) with a Darcy velocity of the

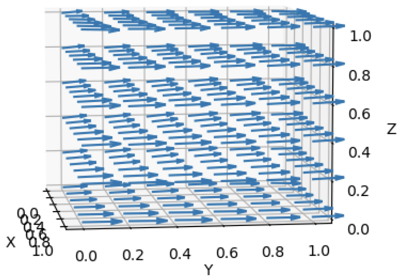

FIG. 1. *Representation of the Darcy velocity in the aquifer rescaled into the hypercube* $[0,1]^3$ *for the readability.*

order of 600 m·year^{-1}. Such a value is in the low range of the usual water displacement speed in porous media. It has been chosen to illustrate that the dynamics of the underground water, even if it is very slow, has a great influence on the results.

The water diffusion is given by $S_m = 9.4 \cdot 10^{-8}$ m^2·s^{-1}. The Péclet number is thus of about 600, and the horizontal dispersion is thus dominant (see [9]). We set $\alpha_L = 5 \cdot 10^{-2}$ m. We choose the example of nitrates fertilization. Nitrates are known to easily leach with water: they are almost not adsorbed by the soil, and we thus set $R = 1$. For the reaction term r, we use the function $r(c) = 10^{-3}c^2$. The natural input is $\gamma = 0.05$ mg·L^{-1}s^{-1} and $g = 0.005$ s^{-1}. The initial condition is $c_0 = 5$ mg·L^{-1}. The benefit function is the same for both farmers, namely,

$$f(p) = \begin{cases} K_1 - K_2K_3 10^{-4} e^{-K_3\bar{p}} e^{-10^4 p} - K_2 + K_2K_3 & \text{if } p < 0, \\ K_1 - K_2 e^{-K_3 p} & \text{if } p \in [0, 2\bar{p}], \\ K_1 - K_2K_3 10^{-4} e^{-2K_3\bar{p}} (e^{10^4(2\bar{p}-p)} - 1) - K_2 e^{-2K_3\bar{p}} & \text{if } p > 2\bar{p}, \end{cases}$$

with $K_1 = 11.7888$, $K_2 = 8.6 \cdot 10^{-3}$, $K_3 = 50.465$, and $\bar{p} = 1.5$. It comes from Godart et al. [14], where the crop yield is depending on the yield value without nitrogen input and the asymptotic value when the input becomes important. Values of the parameters depend on the crop species. Here we choose the example of wheat crops. Notice that the Godard function has been modified for ensuring that the fertilizer load p remains in the interval $[0, \bar{p}]$ without using a truncated function so that the functions F_1 and F_2 in the adjoint problem $\mathcal{P}_{\text{adj}}$ are well defined. The cost functions, on the other hand, are differentiated among farmers by a parameter that allows the pollution cost to be distributed unequally:

$$D_i(c) = \omega_i D(c), i = 1, 2, \text{ with } D(c) = 100\, c^2 \chi_{\text{wells}},\ \omega_2 = 1 - \omega_1,\ \omega_1 \in [0, 1].$$

The other parameters in the functionals J_i, $i = 1, 2$, are $\nu = 1$ and $\rho = 0.05$.

The first illustrations are given in Figures 2 and 3. We chose a situation where only geography, i.e., the position of the well(s) on the land, differentiates the farmers. The cost and benefit functions are the same for both players. Consider first the images in line 3. The first observation is that having the well in the heart of one's property is obviously penalizing from the point of view of spreading (observe the figure in column 2). But above all, by comparing column 1 and column 2, we can see that the displacement of water in the subsoil, even if it is very slow, has a real influence: farmer 1 is indeed very penalized because all that he

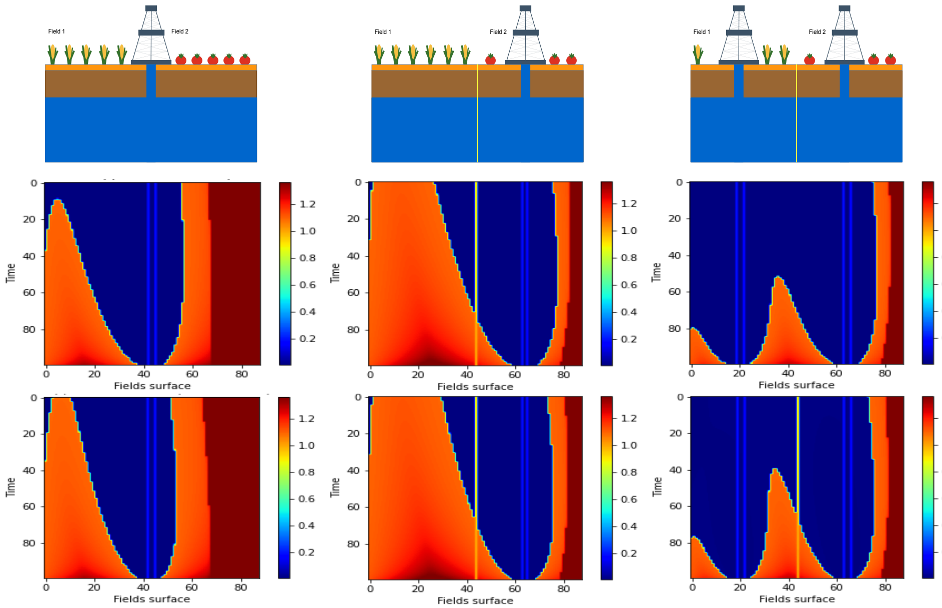

FIG. 2. *Influence of the position of the water production wells on the Nash equilibrium: fertilizer load* $(p_1^\flat, p_2^\flat)$ *on the line* $y = 450$ *as a function of time. Line* 1*: illustration of the position of the well(s). Line* 2*: cooperative case. Line* 3*: noncooperative case, with equal distribution of the cost between the two farmers* $(\omega_1 = 0.5)$*. The blue lines figure the border of the well. The yellow line at* $x = 450$ *m has been added to enhance the nonsymmetry of the results. The color bar gives the value of the pollution flux on the surface.*

spreads is directed toward the well (observe the figure in column 1). These constraints tend to decrease with time (observe the decrease of the blue area in the figures) because we consider the problem in finite time and the exponential with the discounting parameter ρ in Definition (2.6), even if it is small, has a real influence.

Figures 2 and 3 also compare the cooperative and noncooperative cases. The cooperative case actually requires considering the problem in the case where one farmer owns the entire area $]0, 900[\times]0, 900[$. The problem then comes back to the classical optimal control problem, the aim being to maximize the functional J defined by

$$J(p_1, p_2, c(\cdot; p_1, p_2)) = -\nu e^{-\rho T} \int_\Omega D(x, c(T, x; p_1, p_2))\, dx$$

$$+ \int_0^T \left(\int_\Omega \left(\sum_{i=1}^{2} f_i(x, p_i(t, x)) \chi_{\Omega_i}(x) - D((x, c(t, x; p_1, p_2)) \right) dx \right) e^{-\rho t}\, dt. \tag{6.1}$$

Using the adjoint approach, one may compute that the solution of the latter problem is given by

$$p = \sum_i \chi_{S_i} (f_i')^{-1} (\chi_{S_i} \mu),$$

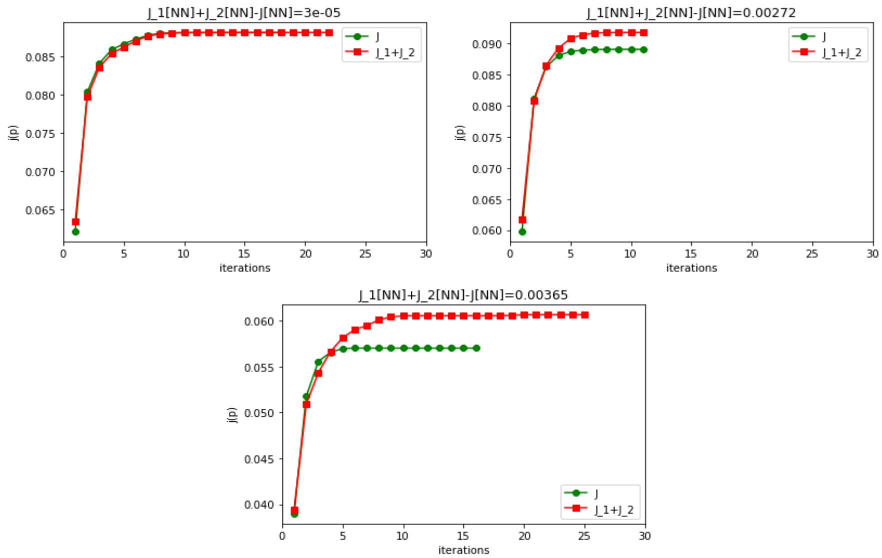

FIG. 3. *Optimal control (cooperative case) versus game theory: comparison of the values of the objectives for one well in the middle of the field, one well in the right part of the field, and two wells (see the illustration of the settings in line* 1 *of Figure* 2*). The difference of the objectives values at the last iteration is explicitly given at the top of each figure.*

where μ satisfies

$$\begin{aligned}
&R\psi\partial_t c + v\cdot\nabla c - \operatorname{div}(\psi S(v)\nabla c) = -r(c) - gc + p + \gamma \text{ in } \Omega_T,\\
&S(v)\nabla c\cdot n = 0 \text{ on } \Gamma_1\times(0,T),\ \ c = 0 \text{ on } \Gamma_2\times(0,T),\ \ c_{|t=0} = c_0 \text{ in } \Omega,\\
&R\psi\partial_t\mu - \mathcal{C}_T v\cdot\nabla\mu - \operatorname{div}(\psi S(\mathcal{C}_T v)\nabla\mu) + r'(\mathcal{C}_T c)\mu + R\psi\rho\mu\\
&\qquad\qquad - \frac{\partial D}{\partial c}(\cdot,\mathcal{C}_T c) = 0 \text{ in } \Omega_T,\\
&\big(\psi S(\mathcal{C}_T v)\nabla\mu + \mu(\chi_{\Gamma_4}\mathcal{C}_T v - \chi_{\Gamma_3}\kappa\mathcal{C}_T v_1)\big)\cdot n = 0 \text{ on } \Gamma_1\times(0,T),\\
&\mu = 0 \text{ on } \Gamma_2\times(0,T),\qquad R\psi\mu_{|t=0} = \nu\frac{\partial D}{\partial c}(\cdot, c_{|t=T}).
\end{aligned}$$

So, even in the case of homogeneous boundary conditions (where $\mu_i^\flat = \chi_{\Omega_i}\omega_i\mu$), the value of the objective J defined in (6.1) differs from the sum $J_1 + J_2$ of the objectives defined in (2.6). Nevertheless, when comparing lines 2 and 3 in Figure 2, the results in the cooperative and noncooperative cases seem very similar for the parameters chosen here. One has to compute the values of the objectives to see that noncooperation is more profitable; see the green and red values at the last iteration represented in Figure 3. The reader will note that we have taken advantage of Figure 3 to illustrate the convergence of the fixed point algorithm used for the calculation, hence the representation of the values of the objectives at the different iterations while only the value at the last iteration obviously reflects the optimal solution. We observe a stabilization of the objectives over the iterations which illustrates the convergence.

The numerical tool allows us to test the sensitivity of the model to the different parameters involved. In this article, we chose to illustrate the influence of the

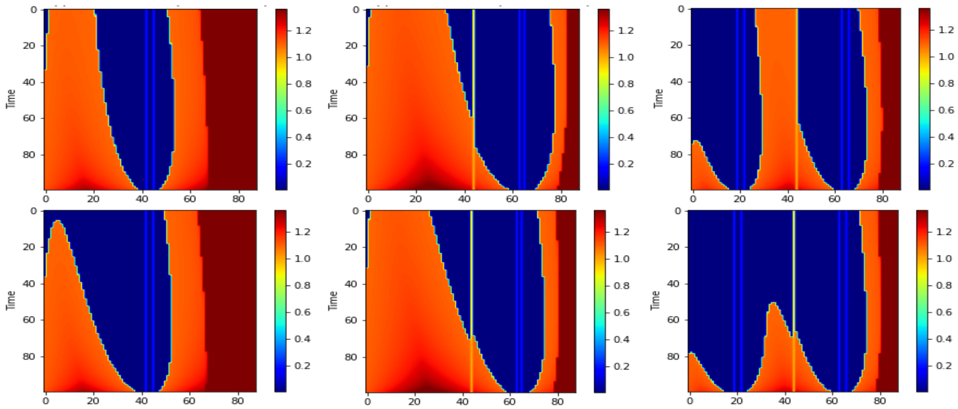

FIG. 4. *Non-symmetric repartition of the cost: Nash equilibrium. Line* 1: $\omega_1 = 0.3$. *Line* 2: $\omega_1 = 0.7$. *The symmetric case* $\omega_1 = 0.5$ *was represented in Line* 3 *of Figure* 2.

environmental cost distribution. Some results are given in Figure 4. Focus, for instance, on the third column. The important point is that Farmer 2 is penalized almost as much when he pays only less than one-third of the cost as when he pays more than two-thirds. Once again, this observation allows us to point out the importance of groundwater dynamics (which here transports all the pollution to the side of Farmer 2).

7. Conclusion. In this article the mathematical analysis of a spatial differential game of groundwater pollution is performed. The existence result for a Nash equilibrium is stated using fixed point theory, and the uniqueness result is proved using Pontryagin's maximum principle optimality conditions. Numerical simulations are provided. They illustrate the difference between the cooperative and noncooperative cases in various situations, in particular depending on the position of the wells. The extension of our work to infinite horizon, that is, setting $T = \infty$, is a challenging question. The interested reader may check that Theorem 3.6 is actually a global result. The existence result of a Nash equilibrium in infinite horizon follows. Nevertheless, our uniqueness proof then falls because the well-posedness of the adjoint problem derived through the Pontryagin's approach is not a straightforward result. More precisely, if $T = \infty$, the time reversal operator introduced for dealing with the adjoint problem is useless, and we have to tackle the antidiffusive problem (5.1)–(5.3) completed by an appropriate transversality condition as $T \to \infty$. We finally note that the latter one may be guessed. Indeed, assuming that the limit of the time derivative of the value function as time goes to infinity is equal to zero (this hypothesis has to be substituted to the long-time vanishing of the value function used in Bouccekine, Camacho, and Fabbri [7] because here, due to the differential game setting, the individual player optimization problems in Definition 3.3 are not autonomous and the corresponding Hamilton–Jacobi–Bellman equations are not stationary), Proposition 4 in Ballestra [4] holds. We thus guess that the necessary transversality condition should have a classical form, namely, stipulating the long-time vanishing of the integral in space of the Hamiltonian functions on the optimal paths:

$$\lim_{T\to\infty} \int_\Omega H_i(T, c^\flat, p_i^\flat, \lambda_i)\, dx = 0,$$

where Hamiltonians H_i are defined by

$$H_i(t, c^\flat, p_i^\flat, \lambda_i^\flat)(x) = \big(f_i(x, p_i^\flat(t,x))\chi_{\Omega_i} - D_i(x, c(t,x; p_1^\flat, p_2^\flat))\big)e^{-\rho t}$$
$$+\lambda_i^\flat(t,x)\big(\mathrm{div}(\psi S(v)\nabla c^\flat) - v \cdot \nabla c^\flat - r(c^\flat) - gc^\flat + p_1^\flat \chi_{\Omega_i} + p_2^\flat \chi_{\Omega_{-i}}\big)(t,x)$$

and $\lambda_i^\flat(t,x) = \mu_i^\flat(t,x)e^{-\rho t}$.

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