N^o d'ordre : 3212

THÈSE

PRÉSENTÉE À

L'UNIVERSITÉ BORDEAUX I

ÉCOLE DOCTORALE DE MATHÉMATIQUES ET D'INFORMATIQUE

Par **Olivier Bernardi**

POUR OBTENIR LE GRADE DE

DOCTEUR

SPÉCIALITÉ : INFORMATIQUE

Combinatoire des cartes et polynôme de Tutte

Soutenue le : Jeudi 7 septembre 2006

Après avis des rapporteurs :

Philippe Flajolet	Directeur de recherche INRIA
Christian Krattenthaler	Professeur
Gilles Schaeffer	Chargé de recherche CNRS, HDR

Devant la commission d'examen composée de :

Philippe Flajolet	Directeur de recherche INRIA ...	Rapporteur
Christian Krattenthaler	Professeur	Rapporteur
Gilles Schaeffer	Chargé de recherche CNRS, HDR	Rapporteur
Robert Cori	Professeur	Examinateur
Michel Las Vergnas	Directeur de recherche CNRS	Examinateur
Eric Sopena	Professeur	Examinateur
Mireille Bousquet-Mélou	Directrice de recherche CNRS ...	Directrice

– 2006 –

à mes parents

Remerciements

Je remercie ma directrice Mireille pour son encadrement sans faille, sa gentillesse et sa bonne humeur. Je lui dois beaucoup et en premier lieu mon goût pour la combinatoire. J'ai bénéficié de toute la réalité de son sens pédagogique légendaire depuis la clarté hypnotisante des cours dispensés dans les sous-sols de Jussieu jusqu'aux plus innocentes questions nées de son scepticisme amusé. Je garderai longtemps pour modèle sa curiosité, sa sérénité scientifique et son aptitude à adoucir les pires calculs par de la *fraise sauvage*.

Philippe Flajolet, Christian Krattenthaler et Gilles Schaeffer m'ont fait un grand honneur en acceptant de sacrifier une partie de leur été à la lecture de mon manuscrit. Cette bienveillance a été précédée d'autres attentions dont je leur suis profondément reconnaissant.

Les travaux présentés dans cette thèse ont bénéficié de l'ambiance stimulante de mon laboratoire, le LaBRI, et de l'enthousiasme des membres de l'équipe combinatoire. J'ai aussi eu la chance de parcourir le monde pour partager connaissances et méconnaissances avec de nombreux collègues, parmi lesquels Nicolas Bonichon, Srecko Brlek, Philippe Duchon, Éric Fusy, Emeric Gioan, Ian Goulden, Michel Las Vergnas, Yvan Le Borgne, Pierre Leroux, Jean François Marckert, Igor Pak, Gilles Schaeffer et Nick Wormald. Chacun d'eux m'a aidé, encouragé et accueilli avec une remarquable bienveillance. Je remercie en particulier Nicolas pour l'indulgence avec laquelle il a accepté les délais que la rédaction de ma thèse a imposés à nos collaborations.

Je remercie enfin ma bonne étoile et tous ceux qui m'ont aidé à tenir le cap. La route entre mes premières fautes d'orthographes et celles contenues dans ce manuscrit a été longue. Je remercie ma mère, mon père, mes deux soeurs et mon frère pour la confiance qu'ils m'ont apportée. Sans eux je n'aurais probablement pas franchi toutes les étapes. Merci à Marta pour son soutien quotidien pendant mes années de doctorat et en particulier pour m'avoir supporté alors que je me métamorphosais en lapin d'Alice regardant convulsivement sa montre et répétant "*en retard, en retard*". Je n'oublie pas mes amis du LaBRI et d'ailleurs. Qu'ils sachent combien leur présence m'a été précieuse.

Résumé

Cette thèse est constituée d'un chapitre préliminaire suivi de trois parties. Dans le chapitre préliminaire nous introduisons les notions et outils fondamentaux, et en premier lieu les *cartes* et le *polynôme de Tutte*. Une *carte* est un plongement sans intersection d'arêtes d'un graphe dans une surface. Les cartes constituent une discrétisation naturelle des surfaces et, à ce titre, apparaissent aussi bien en informatique (pour le codage d'informations visuelles) qu'en physique (comme surfaces aléatoires de la gravitation quantique et de la physique statistique). Les premiers travaux sur les cartes datent du début des années soixante lorsque W.T. Tutte et ses disciples développèrent la méthode récursive pour l'énumération des cartes. À la même époque, Tutte découvrit le polynôme qui porte aujourd'hui son nom. Le *polynôme de Tutte* est un invariant fondamental de la théorie des graphes qui généralise à la fois le polynôme chromatique et le polynôme des flots. Les résultats présentés dans cette thèse mettent en lumière des propriétés énumératives et structurelles importantes des cartes et établissent un lien profond entre les cartes et le polynôme de Tutte.

Dans la première partie de cette thèse, nous énumérons trois familles de *triangulations* (cartes planaires dont les faces sont homéomorphes à des triangles) par une approche récursive. Plus précisément, nous démontrons l'algébricité des séries génératrices des familles de triangulations dont le degré des sommets est au moins égal à une certaine valeur d choisie parmi $\{3,\ 4,\ 5\}$. Nous déterminons aussi le développement asymptotique du nombre de triangulations dans chaque famille. L'originalité de nos résultats tient au fait que nos familles de cartes sont définies par des restrictions de degrés portant simultanément sur les faces et sur les sommets.

Dans la seconde partie, nous établissons deux bijections entre des familles de cartes et des objets dont la combinatoire est plus simple. La première bijection établit un lien entre les triangulations et les *chemins de Kreweras*, soit les chemins dans le quart de plan constitués de pas Sud, Ouest et Nord-Est. Nous obtenons, par ce biais, la premier comptage bijectif des chemins de Kreweras. La deuxième bijection établit un lien entre les cartes dont un arbre couvrant est distingué et les couples formés d'un arbre et d'une partition non-croisée. Nous établissons également un lien entre notre bijection et une construction récursive antérieure

due à Cori, Dulucq et Viennot et définie sur les mélanges de mots de parenthèses. Ces bijections révèlent des propriétés structurelles importantes des cartes et permettent leur comptage, leur codage et leur génération aléatoire.

Dans la troisième partie, nous établissons une caractérisation du polynôme de Tutte des graphes basée sur la structure de carte. Plus précisément, nous définissons les *activités de plongement* des arbres couvrants des cartes et nous montrons que le polynôme de Tutte est égal à la série génératrice des arbres couvrants comptés selon leurs activités de plongement. La caractérisation du polynôme de Tutte par les activités de plongement est mise à contribution pour définir une bijection entre les sous-graphes et les orientations. En spécialisant cette bijection nous obtenons des interprétations combinatoires pour plusieurs évaluations du polynôme de Tutte en termes d'orientations et de suites de degrés. Par exemple, nous obtenons une bijection entre les arbres couvrants (comptés par l'évaluation $T_G(1,1)$ du polynôme de Tutte) et les suites de degrés racine-accessibles. Nous établissons également une nouvelle bijection entre les arbres couvrants et les configurations récurrentes du modèle du tas de sable.

Contents

Introduction

0.1 Les cartes

0.1.1 Les cartes planaires

La notion de *carte* est à la base de tous les travaux présentés dans cette thèse. En préalable aux définitions concernant les cartes il faudrait rappeler les définitions classiques de la théorie des graphes. Nous renvoyons le lecteur aux nombreux ouvrages de référence concernant les graphes (par exemple [Boll 98] ou [Tutt 84]). Nous nous contenterons de définir les notions dont le vocabulaire polymorphe est sujet à confusion.

Un *graphe* est formé d'un ensemble fini de *sommets*, d'un ensemble fini d'*arêtes* et d'une relation d'*incidence* entre sommets et arêtes. Chaque arête est incidente à un ou à deux sommets qui sont ses *extrémités*. On peut dessiner un graphe en représentant chaque sommet par un point et chaque arête par une ligne reliant ses extrémités. Deux dessins d'un même graphe sont représentés en figure 1. Dans le dessin de droite, les arêtes ne se rencontrent qu'au niveau des sommets. Ce type de dessin est appelé *plongement*.

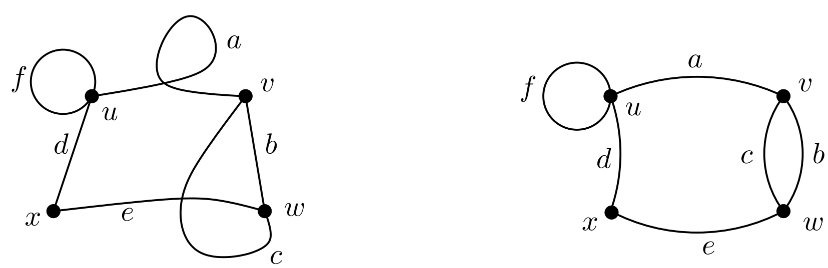


Figure 1: Deux dessins d'un même graphe dont les sommets sont u, v, w, x et les arêtes sont a, b, c, d, e, f.

Seuls certains graphes admettent un plongement dans le plan. On les appellent *planaires*. Alternativement au plan, il peut être agréable de plonger ces graphes dans la sphère. On peut aisément passer d'un plongement dans la sphère à un plongement dans le plan et inversement par projection stéréographique (voir figure 2).

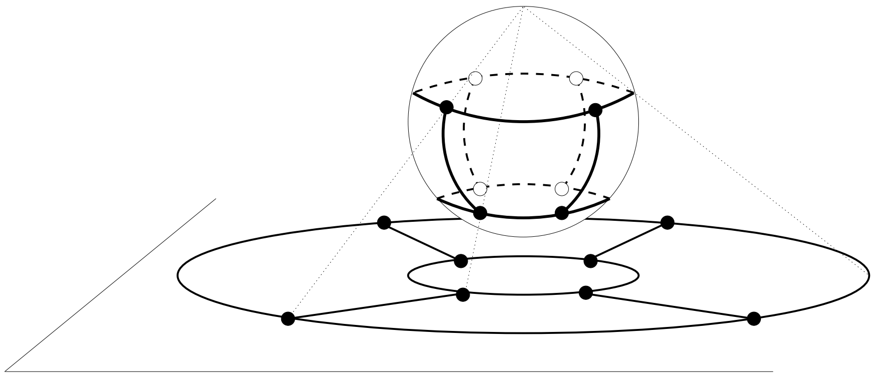


Figure 2: De la sphère au plan : la projection stéréographique.

Une *carte planaire* est un plongement d'un graphe planaire connexe sur la sphère. Un exemple de carte est donné en figure 3. Pour être précis, une carte planaire définit la *topologie* du plongement et non sa *métrique*. Ainsi nos cartes planaires sont définies à *déformation continue* près. Malgré cela, un même graphe peut donner lieu à plusieurs cartes. Ainsi sur la figure 4, la carte de gauche et celle du milieu sont identiques mais différentes de celle de droite qui correspond pourtant au même graphe.

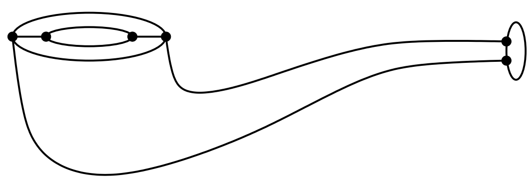


Figure 3: Ceci est une carte planaire.

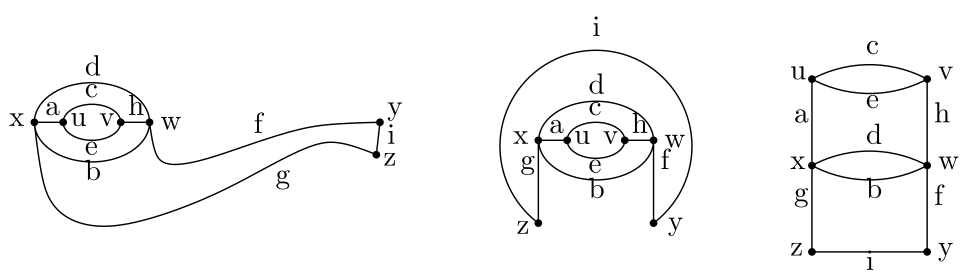


Figure 4: La carte de gauche et celle du milieu sont identiques (on peut passer de l'une à l'autre par déformation de la *sphère*) mais la carte de droite est différente.

0.1.2 Cartes en genre supérieur

Considérons à présent d'autres surfaces. Nous nous limiterons aux surfaces bidimensionnelles, compactes, orientables et sans bords que nous appellerons simplement *surfaces* (voir par exemple [Moha 01] pour les définitions concernant les surfaces). Ces surfaces sont entièrement caractérisées (à homéomorphisme près) par la donnée de leur *genre* (un entier positif). La surface de genre 0 est la sphère et la surface de genre k est le tore à k trous. Les surfaces de genre 0, 1 et 2 sont représentées en figure 5.

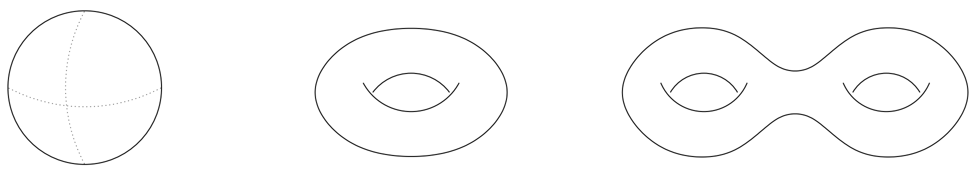


Figure 5: Surfaces de genre 0, 1 et 2.

Considérons le découpage d'une surface induit par le plongement d'un graphe. Les composantes connexes de la surface après découpage, c'est-à-dire les composantes connexes du complémentaire du graphe, sont appelées *faces*. Si les faces sont simplement connexes (i.e. homéomorphes au disque unité ouvert de $\mathbb{R}^2$), le plongement est dit *cellulaire*. Par exemple, le plongement de gauche dans la figure 6 est cellulaire mais celui de droite ne l'est pas (une des faces est un *tube*). Une *carte de genre g* est un plongement cellulaire d'un graphe connexe dans la surface de genre g. La carte est considérée à *homéomorphisme* (de la surface *orientée*) près. Observons que tout plongement d'un graphe connexe (non vide) dans la sphère est cellulaire. D'autre part, dans la sphère, une transformation est homéomorphe si et seulement si elle s'obtient par déformation continue de la sphère. La notion de *carte planaire* coïncide donc avec la notion de *carte de genre 0*.

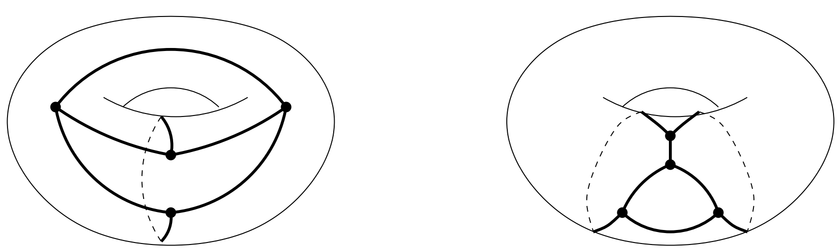


Figure 6: Un plongement cellulaire et un plongement non cellulaire du graphe complet K_4.

La notion de face amène à compléter nos relations d'*incidences*. Notons que les relations d'incidence entre arêtes et sommets peuvent se définir *topologiquement* : une arête a est incidente à un sommet s si la frontière de a contient s. De même, une face f est *incidente* à une arête a (resp. un sommet s) si la frontière de f contient a (resp. s). Une arête qui n'est incidente qu'à un seul sommet (resp. une seule face) est *doublement* incidente à ce sommet (resp. cette face). Le *degré* d'un sommet ou d'une face est le nombre d'arêtes qui lui sont incidentes, comptées avec multiplicité.

On peut également définir la notion de *demi-arête*. En supprimant un point intérieur d'une arête, on obtient deux *demi-arêtes*, c'est-à-dire deux cellules de dimension 1, chacune étant incidente à une des extrémités de l'arête. On définit un *coin* comme un couple de demi-arêtes consécutives autour d'un sommet.

Notons que le nombre de faces, de sommets et d'arêtes sont conservés par homéomorphisme de cartes, de même que les relations d'incidence. On définit la caractéristique d'Euler d'une carte C par

$$\chi(C) = s(C) + f(C) - a(C),$$

où $s(C)$, $f(C)$ et $a(S)$ sont respectivement le nombre de sommets, de faces et d'arêtes de la carte C. La caractéristique d'Euler d'une carte ne dépend en réalité que du genre de la surface dans laquelle elle est plongée. En effet, pour toute carte de genre g la *relation d'Euler* s'écrit :

$$\chi(C) = 2 - 2g. \tag{1}$$

Par exemple, la carte représentée en figure 6 (gauche) est de genre $g = 1$ et a $s = 4$ sommets, $f = 2$ faces et $a = 6$ arêtes. On vérifie donc bien la relation d'Euler $\chi = s + f - a = 0 = 2 - 2g$.

0.1.3 Représentation combinatoire

Jusqu'ici nous avons présenté les cartes de manière topologique : les plongements cellulaires définis à homéomorphisme près. Nous allons maintenant définir les cartes de manière combinatoire (discrète) et expliquer l'équivalence des deux définitions.

Considérons une carte C de genre quelconque. La carte C est considérée à homéomorphisme près. Un homéomorphisme agit localement comme une déformation continue. Par conséquent, l'ordre cyclique *positif* (ou *anti-horaire*) des arêtes autour de chaque sommet est préservé par homéomorphisme (de la surface orientée). Ainsi toute carte définit un *système de rotation* c'est-à-dire l'ordre cyclique des arêtes autour de chaque sommet. Pour la carte de gauche en figure 4, le système de rotation autour du sommet x est (g, b, a, d).

Sur la figure 4, les plongements de gauche et du milieu correspondent à une même carte, leur système de rotation sont identiques. Par contre le système de rotation de la carte de droite est différent ce qui prouve qu'il ne s'agit pas de la même carte. Nous voyons poindre une propriété fondamentale des cartes : elles sont entièrement déterminées par leur système de rotation. Cette propriété est à la base de la définition combinatoire des cartes.

Théorème 0.1 *[Moha 01, Thm. 3.2.4] Il y a correspondance bijective entre les cartes et les graphes connexes munis d'un système de rotation.*

Le système de rotation est parfois appelée *plongement combinatoire* du graphe. Plutôt que de travailler avec un graphe et un système de rotation, il est plus élégant de considérer un ensemble de *demi-arêtes*, une permutation qui correspond à l'action de tourner autour d'un sommet et une involution qui correspond à l'action de traverser une arête (de passer d'une demi-arête à la demi-arête opposée). Ceci nous amène à la définition de *carte combinatoire* telle qu'elle a été introduite par Cori et Machì [Cori 92]. Une *carte combinatoire* $C = (H, \sigma, \alpha)$ est formée d'un ensemble de *demi-arêtes* H, d'une permutation σ et d'une involution sans point fixe α sur H telles que le groupe engendré par σ et α agit transitivement sur H.

Étant donnée une carte combinatoire, on définit le *graphe sous-jacent* dont les sommets sont les cycles de σ, les arêtes sont les cycles de α et la relation d'incidence est d'avoir une demi-arête commune. La figure 7 représente le graphe sous-jacent à la carte combinatoire $C = (H, \sigma, \alpha)$ où l'ensemble des demi-arêtes est $H = \{a, a', b, b', c, c', d, d', e, e', f, f'\}$, la permutation σ est $(a, f', b, d)(d')(a', e, f, c)(e', b', c')$ en notation cyclique et l'involution α est $(a, a')(b, b')(c, c')(d, d')(e, e')(f, f')$. Le graphe sous-jacent à une carte combinatoire est toujours connexe puisque le groupe engendré par les permutations σ et α agit transitivement sur l'ensemble H des demi-arêtes. Graphiquement, on représente les cycles de σ par l'ordre anti-horaire autour des sommets (et on représente les cycles de α par les arêtes). Par conséquent, la carte combinatoire est entièrement déterminée par sa représentation graphique.

Une carte combinatoire $C = (H, \sigma, \alpha)$ définit un graphe (à réétiquetage des sommets et des arêtes prés) et un système de rotation (les cycles de σ). Réciproquement, une carte combinatoire est entièrement définie (à réétiquetage des demi-arêtes près) par la donnée d'un graphe connexe et d'un système de rotation. D'après le théorème 0.1, il y a équivalence entre la notion de carte topologique (un plongement cellulaire d'un graphe considéré à homéomorphisme près) et la notion de carte combinatoire (une permutation et une involution sans point fixe agissant transitivement sur un même ensemble). Lorsque cela est utile nous parlerons de carte *topologique* ou de carte *combinatoire* pour préciser le point de vue adopté.

Considérons une carte combinatoire $C = (H, \sigma, \alpha)$. Les cycles de la permutation $\sigma\alpha$ décrivent le tour (dans le sens négatif) des faces de la carte topologique correspondant à C. Ainsi, les faces de la carte combinatoire sont en bijection avec les cycles de la permutation $\sigma\alpha$. De plus, la relation d'incidence entre faces et arêtes (resp. sommets) est d'avoir une

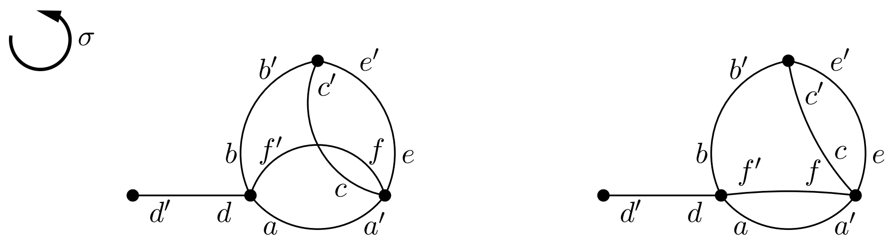


Figure 7: Deux cartes combinatoires distinctes correspondant au même graphe.

demi-arête commune. La caractéristique d'Euler d'une carte combinatoire $C = (H, \sigma, \alpha)$ est

$$\chi(C) = c(\sigma) + c(\sigma\alpha) - c(\alpha)$$

où $c(\pi)$ est le nombre de cycles de la permutation π. La relation d'Euler (1) permet de connaître le genre de la surface sur laquelle est plongée la carte topologique correspondant à C. Par exemple, une carte combinatoire C est planaire si et seulement si $\chi(C) = 2$.

0.1.4 Cartes non-étiquetées et enracinements

Jusqu'à présent, nous avons considéré des cartes *étiquetées*. En effet, nos cartes portent des *étiquettes* (sur les sommets et les arêtes pour les cartes topologiques, sur les demi-arêtes pour les cartes combinatoires). Nous allons voir comment nous affranchir de l'étiquetage. Avant cela, nous définissons l'enracinement des cartes.

On *enracine* une carte en distinguant une demi-arête comme étant la *racine*. De manière équivalente, on peut définir l'enracinement d'une carte en distinguant un coin ou encore en distinguant une *arête racine* et en l'orientant. C'est cette dernière convention qui est le plus couramment utilisée pour représenter l'enracinement. Quatre enracinements d'une même carte sont représentés en figure 8.

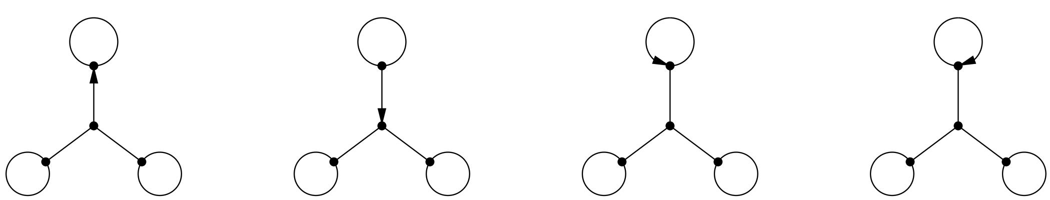


Figure 8: Quatre cartes enracinées.

Nous passons maintenant aux cartes *non-étiquetées*. Rappelons tout d'abord la notion d'isomorphisme entre graphes. Un *isomorphisme* entre deux graphes G_1 et G_2 est formé d'une bijection entre les sommets de G_1 et ceux de G_2 et d'une bijection entre les arêtes de G_1 et celles de G_2 qui *préservent les relations d'incidence* (un sommet et une arête sont incidents si et seulement si leurs images le sont). Autrement dit, un isomorphisme de graphes est un *réétiquetage* (ou *renommage*) des sommets et des arêtes. De même, un *isomorphisme* de cartes est un réétiquetage de cette carte. Autrement dit, un isomorphisme entre deux cartes combinatoires $C_1 = (H_1, \sigma_1, \alpha_1)$ et $C_2 = (H_2, \sigma_2, \alpha_2)$ est une bijection ϕ entre H_1 et H_2 telle que $\sigma_2 = \phi \circ \sigma_1 \circ \phi^{-1}$ et $\alpha_2 = \phi \circ \alpha_1 \circ \phi^{-1}$. Un isomorphisme de cartes combinatoires est représenté en figure 9. Un graphe *non-étiqueté* est un graphe considéré à isomorphisme prés. De même, une carte *non-étiquetée* est une carte considérée à isomorphisme prés. Dans la sous-section précédente nous avons vu que les notions de cartes topologiques non-étiquetées et de cartes combinatoires non-étiquetées sont équivalentes.

Figure 9: Un isomorphisme ϕ entre les cartes C_1 et C_2. L'isomorphisme ϕ associe respectivement aux demi-arêtes a, b, c, d, e, f, g, h de la carte C_1, les demi-arêtes d, c, b, a, e, g, f, h de la carte C_2.

On s'intéresse maintenant aux relations entre le nombre de cartes étiquetées, non-étiquetées, enracinées et non-enracinées. Ces relations dépendent des *symétries* des cartes, ou encore de leur groupe d'automorphismes. Un *automorphisme* d'une carte étiquetée est un isomorphisme de la carte sur elle-même, c'est-à-dire un réétiquetage qui laisse la carte inchangée. Un automorphisme est représenté en figure 10. L'ensemble des automorphismes d'une carte est un groupe (pour la composition) qui contient l'identité.

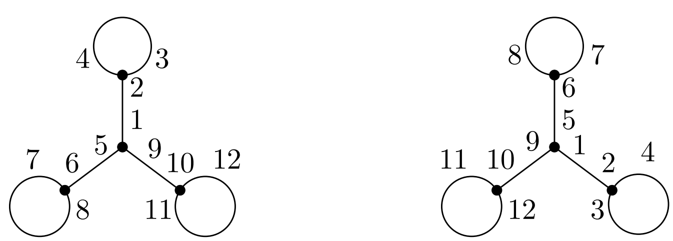


Figure 10: L'automorphisme ϕ qui aux demi-arêtes $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$ associe les demi-arêtes $5, 6, 7, 8, 9, 10, 11, 12, 1, 2, 3, 4$ respectivement.

Soit $C = (H, \sigma, \alpha)$ une carte combinatoire étiquetée à n arêtes. On s'intéresse à l'action du groupe $\mathfrak{S}_H$ des permutations de H interprétées comme des isomorphismes sur la carte C. On rappelle un résultat classique de la théorie des groupes.

Lemme 0.2 *Soit S un ensemble et I un groupe. Le cardinal de l'orbite d'un élément x de S par l'action du groupe I est égal au cardinal du groupe I divisé par le cardinal du sous-groupe $A \subseteq I$ des éléments laissant x invariant (i.e. le stabilisateur de x).*

Le groupe $\mathfrak{S}_H$ des permutations de H a cardinal $(2n)!$. On note ξ_C le cardinal du groupe d'automorphismes de la carte étiquetée C. Par exemple, le groupe d'automorphismes de la carte représenté en figure 10 a cardinal 3 puisqu'il contient l'identité, l'automorphisme ϕ et l'automorphisme ϕ^2. D'après le lemme 0.2, le nombre de cartes différentes obtenues à partir de la carte C par action du groupe $\mathfrak{S}_H$ est $\frac{(2n)!}{\xi_C}$. Puisque le paramètre ξ_C est invariant par isomorphisme de la carte C, il n'y a pas de conflit à définir ce paramètre pour une carte non étiquetée (comme étant le cardinal du groupe des automorphismes pour l'une de ses représentantes étiquetées). Ainsi, une carte non-étiquetée C à n arêtes donne lieu à $\frac{(2n)!}{\xi_C}$ cartes étiquetées sur l'ensemble $H = \{1, 2, \ldots, 2n\}$. Par exemple, la carte de la figure 10 admet $\frac{(2n)!}{\xi_C} = \frac{(12)!}{3}$ étiquetages différents sur $H = \{1, 2, \ldots, 12\}$.

Il y a $2n$ façons d'enraciner une carte étiquetée à n arêtes. Par conséquent, une carte C non-étiquetée à n arêtes donne lieu à $2n \cdot \frac{(2n)!}{\xi_C}$ cartes enracinées étiquetées sur $H = \{1, 2, \ldots, 2n\}$. Nous allons montrer (lemme 0.4) que les cartes enracinées n'ont pas d'autres automorphismes que l'identité. Par conséquent, chaque carte enracinée non-étiquetée à n arêtes donne lieu à $(2n)!$ cartes enracinées étiquetées sur $H = \{1, 2, \ldots, 2n\}$. On en déduit le résultat suivant.

Proposition 0.3 *Une carte C non-étiquetée non-enracinée à n arêtes donne lieu à $\frac{2n}{\xi_C}$ cartes non-étiquetées enracinées.*

Par exemple, la carte de la figure 10 donne lieu à $\frac{2n}{\xi_C} = \frac{12}{3} = 4$ cartes non-étiquetées enracinées différentes. Ces cartes sont représentées en figure 8. Il ne nous reste qu'à prouver que les cartes enracinées n'ont pas d'autres automorphismes que l'identité.

Lemme 0.4 *Le groupe d'automorphismes d'une carte combinatoire étiquetée enracinée est réduit à l'identité.*

Preuve : Soit $C = (H, \sigma, \alpha)$ une carte combinatoire étiquetée dont h_0 est la demi-arête racine. Soit ϕ un automorphisme de C. Par définition, les permutations σ et α commutent avec ϕ : pour toute demi-arête h, on a $\phi \circ \sigma(h) = \sigma \circ \phi(h)$ et $\phi \circ \alpha(h) = \alpha \circ \phi(h)$. Donc si une demi-arête h est telle que $\phi(h) = h$, alors $\phi(\sigma(h)) = \sigma(h)$ et $\phi(\alpha(h)) = \alpha(h)$. Puisque l'automorphisme ϕ préserve la demi-arête racine h_0 ($\phi(h_0) = h_0$) et que les permutations σ et α agissent transitivement sur l'ensemble H des demi-arêtes, on obtient $\phi(h) = h$ pour toute demi-arête h.

□

Dans cette thèse, nous énumérons plusieurs familles de cartes. Nous ne considérerons que des familles de cartes enracinées et non-étiquetées. Nous venons de voir que le nombre de cartes *étiquetées* (enracinées ou non-enracinées) à n arêtes (dont l'ensemble des demi-arêtes est $H = \{1, 2, \dots, 2n\}$) est proportionnel au nombre de cartes *non-étiquetées enracinées* à n arêtes. Comme le suggère la proposition 0.3, le passage d'un résultat d'énumération du cas enraciné au cas non-enraciné est délicat. Cependant, pour la plupart des familles de cartes, la probabilité qu'une carte de taille n ait un groupe d'automorphismes qui ne soit pas réduit à l'identité décroît exponentiellement vite avec n [Rich 95]. Ainsi, pour la plupart des familles de cartes, le rapport du nombre c_n de cartes non-enracinées de taille n au nombre c'_n de cartes enracinées de taille n vérifie $\dfrac{c_n}{c'_n} = \dfrac{1 + o(\epsilon^n)}{2n}$ où $0 < \epsilon < 1$.

0.1.5 Cartes enrichies

Dans cette thèse nous étudions plusieurs familles de cartes. Les familles de cartes sont souvent définies par des critères portant sur le degré des faces ou celui des sommets. Par exemple, les *triangulations* (resp. *cartes cubiques*) sont les cartes dont les faces (resp. sommets) ont degré 3. Les *quadrangulations* (resp. *cartes tétravalentes*) sont les cartes dont les faces (resp. sommets) ont degré 4. Enfin, les cartes *biparties* (resp. *eulériennes*) sont les cartes dont les faces (resp. sommets) ont degré pair. Des exemples sont donnés en figure 11.

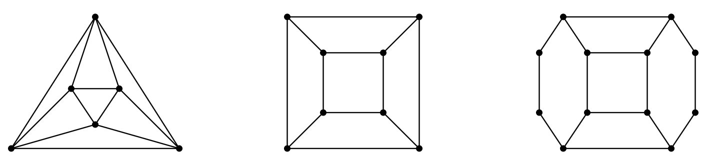


Figure 11: La carte de gauche est une triangulation (qui est aussi tétravalente), la carte du milieu est une quadrangulation (qui est aussi cubique) et la carte de droite est bipartie.

Il existe de nombreuses relations entre les différentes familles de cartes. Une relation fondamentale est la *dualité*. Étant donnée une carte C, on construit la *carte duale* C^* en plaçant un sommet de C^* dans chaque face de C et une arête de C^* à travers chaque arête de C. Sur la figure 12, la carte duale est indiquée en traits discontinus. La dualité peut aussi se définir de manière combinatoire : la carte duale de la carte $C = (H, \sigma, \alpha)$ est $C^* = (H, \alpha\sigma, \alpha)$. Les sommets d'une carte correspondent aux faces de sa carte duale et vice-versa. Ainsi, la dualité envoie la famille des triangulations (resp. quadrangulations, cartes biparties) sur la famille des cartes cubiques (resp. tétravalentes, eulériennes).

Les cartes peuvent servir de support à d'autres structures combinatoires, une coloration ou un arbre couvrant par exemple. D'une certaine manière, l'ajout d'un arbre couvrant *simplifie* souvent la combinatoire des cartes. C'est ce que démontrent les approches bijectives

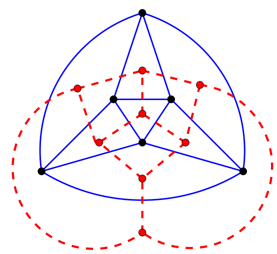


Figure 12: Une triangulation (traits continus) et la carte cubique duale (traits pointillés).

initiées par Gilles Schaeffer et basées sur les *conjugaison d'arbres* [Scha 98]. Au chapitre 2, nous verrons que l'ajout d'un arbre d'exploration en profondeur sur les cartes cubiques permet de les mettre en bijection avec une classe de chemins planaires appelés *chemins de Kreweras*. Au chapitre 3, nous verrons que l'ensemble des *cartes boisées* (cartes dont un arbre couvrant est distingué) est en bijection avec les couples formés d'un arbre et d'une partition non-croisée.

Les cartes peuvent aussi servir de support à des modèles de physique statistique. D'un point de vue physique, les cartes sont des espaces bidimensionnels discrets. Sur ces surfaces, on peut placer des particules occupant les sommets et interagissant à travers les arêtes. Un modèle simple consiste à considérer que chaque particule (sommet) peut être dans l'un des états $1, 2, \ldots, q$ mais que deux particules adjacentes ne peuvent pas être dans le même état. On obtient ainsi les *coloriages* de la carte en q couleurs. Au chapitre 9, nous amorcerons le comptage des cartes munies d'un coloriage. Le *modèle de Potts* [Baxt 82] correspond à la situation plus générale où toutes les *configurations* (attributions d'un état parmi $\{1, 2, \ldots, q\}$ à chaque particule) sont possibles et où leur probabilité d'apparition dépend du nombre d'arêtes *unicolores*, c'est-à-dire dont les deux extrémités ont même état. Plus précisément, dans le modèle de Potts la probabilité d'une configuration θ est proportionnelle à

$$W(\theta) = \exp(K \cdot u(\theta)) \tag{2}$$

où K est un paramètre et $u(\theta)$ est le nombre d'arêtes unicolores. Le modèle de Potts est un modèle important en physique statistique [Baxt 01, Baxt 82, Bonn 99, Daul 95]. Le *modèle d'Ising* (sans champ extérieur) qui correspond au modèle de Potts à $q = 2$ états est lui-même largement étudié [Boul 87, Bous 03b]. Nous verrons en section 0.3.2 que la fonction de partition du modèle de Potts sur une carte C est équivalent au *polynôme de Tutte* de cette carte [Fort 72].

Étant donné un modèle de physique statistique, le modèle de Potts par exemple, on cherche à en déterminer le comportement moyen. Le modèle fournit une mesure de probabilité W *non normalisée* (dont la somme des poids n'est pas égale à 1) sur l'ensemble des configurations. On appelle *fonction de partition* le facteur de renormalisation, c'est-à-dire la somme des poids des configurations :

$$Z = \sum_{\theta \text{ configuration}} W(\theta). \tag{3}$$

Cette fonction de partition dépend en général d'un ou plusieurs paramètres $K_1, \ldots, K_i$ contrôlant le comportement moyen du modèle. Afin d'étudier ce comportement il faut aussi enrichir la fonction de partition afin de prendre en compte de nouveaux paramètres comme, par exemple, le nombre de particules dans chaque état. On obtient une fonction de partition à plusieurs variables $Z(K_1, \ldots, K_i, x_1, \ldots, x_j)$ dont on peut (théoriquement) déduire les propriétés moyennes du système. Les principales questions concernent l'existence et la nature des *transitions de phase*. Une *transition de phase* est un changement non-analytique des propriétés moyennes du système (considérées comme fonctions des paramètres $K_1, \ldots, K_i$).

Pour l'instant nous avons considéré des modèles de physique statistique *à carte fixée*. Cependant, l'étude d'un modèle sur une carte fixée est rarement instructive (en particulier, il ne peut y avoir de transitions de phase). En réalité, les phénomènes que l'on cherche à appréhender ne se produisent que lorsque l'on approche la *limite thermodynamique*, c'est-à-dire quand la taille du système (le nombre de particules) tends vers l'infini. Pour étudier ce comportement limite, on considère généralement une carte régulière (le réseau carré par exemple) que l'on fait grossir. La fonction de partition du modèle limite est donné par

$$Z_{\lim} = \lim_{n\to\infty} Z_n^{1/n},$$

où Z_n est la fonction de partition du modèle sur la carte régulière de taille n. Alternativement, on peut considérer une famille de cartes et faire la moyenne sur les cartes de taille n. On cherche alors à évaluer la fonction de partition

$$\mathcal{Z}_n = \sum_C Z_C$$

où la somme porte sur les cartes de taille n et Z_C est la fonction de partition du modèle sur la carte C. On fait ensuite tendre la taille n des cartes vers l'infini afin d'étudier le comportement limite du modèle. Cet objectif n'est pas dépourvu de sens physique puisque les réseaux physiques sont rarement (sinon jamais) totalement réguliers. Même les structures cristallines contiennent des défauts et il convient de faire la moyenne sur ces défauts. Faire la moyenne d'un modèle sur une famille de cartes constitue l'extrême opposé à l'étude de ce modèle sur un réseau régulier. On parle alors de modèle sur une *surface aléatoire*. Il semble qu'il existe une sorte de *dualité* entre le comportement d'un modèle sur un réseau régulier et le comportement du même modèle sur une surface aléatoire (en particulier, une relation liant les exposants critiques d'un modèle par rapport à l'autre). Cette dualité est connue en physique sous le nom de *relation KPZ* d'après les initiales des physiciens Knizhnik, Polyakov et Zamolodchikov qui l'ont découverte.

0.2 Comptons!

La combinatoire énumérative est l'art de compter des objets. En toute généralité, on considère un ensemble d'objets (des graphes, des arbres, des mots, ...) muni d'une fonction *taille*. L'ensemble est dit *gradué* s'il existe un nombre fini d'objets de taille n. Un ensemble gradué est aussi appelé *classe combinatoire*. L'*énumération* d'une classe combinatoire consiste à déterminer le nombre d'objets de chaque taille.

Considérons, par exemple, l'ensemble des mots de Dyck. Les *mots de Dyck* (ou *mots de parenthèses*) sont les mots sur l'alphabet $\{x, \overline{x}\}$ ayant autant de lettres x que de lettres $\overline{x}$ et tels que tout préfixe a au moins autant de lettres x que de lettres $\overline{x}$. Par exemple, $xx\overline{x}x\overline{x}\overline{x}$ est un mot de Dyck. Alternativement, on peut considérer les mots de Dyck comme des chemins unidimensionnels fait de pas +1 et -1 qui partent de 0, restent positifs et retournent en 0 (on parle aussi de *chemins de Dyck*). Les premiers mots de Dyck sont représentés en figure 13. L'ensemble (infini) des mots de Dyck est muni de la fonction *taille* définie comme étant la demi longueur du mot. Il est clair que le nombre C_n de mots de Dyck de taille n est fini. La suite $(C_n)_{n\in\mathbb{N}}$ est appelée *suite de Catalan*. Une exploration rapide (voir figure 13) permet de montrer que $C_0 = C_1 = 1$, $C_2 = 2$ et $C_3 = 5$. Le travail de l'énumérateur consiste à déterminer la valeur de la suite $(C_n)_{n\in\mathbb{N}}$ ou, à défaut, son comportement asymptotique. Il existe des techniques générales pour réaliser ce travail et dont nous traçons les grandes lignes ci-dessous. Nous verrons, en particulier, comment montrer que le nombre de mots de Dyck de taille n est

$$C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{4}$$

Auparavant, nous allons tenter de répondre à la légitime question *pourquoi compter?*

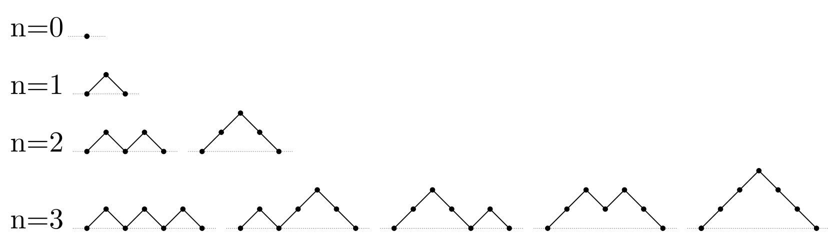


Figure 13: Les chemins de Dyck de taille 0, 1, 2 et 3.

0.2.1 Pourquoi compter ?

L'énumération est avant tout un moyen de calculer des probabilités dans des systèmes discrets. Les techniques d'énumération sont donc essentielles aussi bien en mathématique qu'en informatique (pour l'étude de la complexité moyenne d'un algorithme) ou en physique statistique.

Avant cela, le comptage est une première étape naturelle dans l'appréhension une classe combinatoire. En effet, les similitudes numériques constituent souvent des pistes précieuses pour comprendre la structure des objets étudiés. Considérons, par exemple, l'ensemble des arbres binaires à n noeuds ou encore l'ensemble des triangulations d'un polygone à $n+2$ sommets. L'analyse des premiers cas (figure 14) montre que le nombre d'objets de taille n coïncide avec la suite de Catalan: 1,1,2,5,14,.... Cette coïncidence n'en est pas une puisqu'il existe des bijections bien connues entre les mots de Dyck, les triangulations d'un polygone et les arbres binaires. Par exemple, on passe des triangulations d'un polygone aux arbres binaires par *dualité* (voir figure 15). L'énumération permet donc de découvrir des relations entre plusieurs classes combinatoires sans rapport évident. Ces découvertes sont facilitées par l'existence d'*encyclopédies de nombres* répertoriant les suites connues et leurs interprétations combinatoires [Sloa].

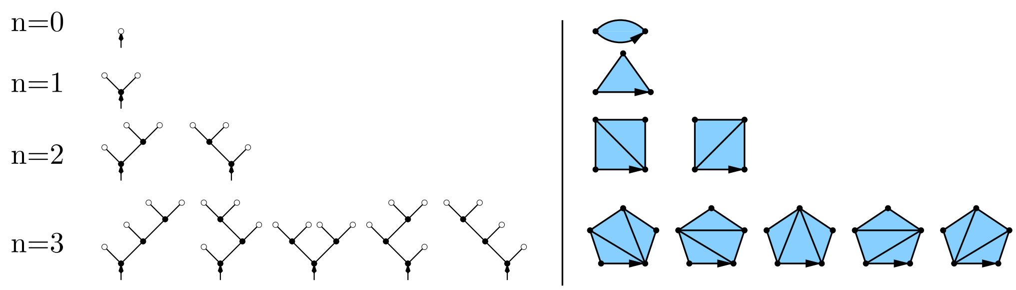


Figure 14: Les arbres binaires et les triangulations du polygone de taille 0, 1, 2 et 3.

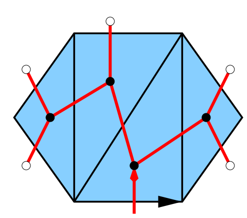


Figure 15: Bijection entre arbres binaires et les triangulations de polygone par dualité.

Comme le suggère l'exemple précédent, le comptage exact d'une classe combinatoire est un bon moyen d'acquérir des informations sur sa structure. De fait, les similitudes numériques ont constitué le point de départ de deux bijections présentées dans cette thèse.

Énumération exacte dans cette thèse. Au chapitre 2, nous définissons une bijection entre les cartes cubiques de taille n (il y en a $\frac{2^n}{(n+1)(2n+1)}\binom{3n}{n}$) munies d'un arbre d'exploration (il y en a 2^n) et les mots de Kreweras de taille n (il y en a $\frac{4^n}{(n+1)(2n+1)}\binom{3n}{n}$). Au chapitre 3, nous définissons une bijection entre les cartes boisées de taille n (il y en a C_nC_{n+1}) et les couples formés d'un arbre binaire de taille n (il y en a C_n) et d'une partition non-croisée de taille $n+1$ (il y en a C_{n+1}). Une bijection entre deux classes combinatoires facilite bien souvent l'étude de ces classes puisque certains paramètres seront plus facilement accessibles avec l'une ou l'autre des représentations. D'autre part, nos bijections donnent lieu à des algorithmes de *génération aléatoire* efficaces. Un algorithme de *génération aléatoire* pour une classe combinatoire prend en paramètre une taille et retourne un objet de taille n avec une distribution uniforme sur l'ensemble des objets de cette taille. Ces algorithmes sont utiles pour l'étude expérimentale des propriétés statistiques de la classe combinatoire.

Revenons maintenant aux motivations probabilistes du comptage. Voici quelques questions auxquelles on peut être confronté :

1. Quelle est la probabilité pour un mot de longueur n sur l'alphabet $\{A, C, G, T\}$ d'éviter le motif TAC?

2. Quelle est la distance moyenne de la racine à une feuille dans un arbre binaire de taille n?

3. Quelle est le nombre de bits nécessaire au codage d'une carte planaire de taille n?

4. Quelle est la distance moyenne entre deux sommets dans une triangulation? Quelle loi de probabilité suit le degré d'un sommet?

5. Comment varie le nombre d'arêtes unicolores dans le modèle de Potts sur réseau aléatoire en fonction du paramètre K?

Ces questions se ramènent toutes à des problèmes d'énumération d'une classe combinatoire (dont les objets sont éventuellement pondérés). Elles ont des applications évidentes en biologie, en informatique ou en physique statistique. Ainsi, savoir que le nombre d'arbres binaires de taille n est donné par la formule (4), implique que le nombre de bits nécessaire à leur codage est $\log(C_n) = 2n - \frac{3}{2}\log(n) + o(1)$. Ainsi le codage direct par un mot de Dyck est asymptotiquement optimal (2 bits par noeud).

Les questions probabilistes que nous venons d'évoquer ne nécessitent pas toujours un comptage *exact*. En pratique, le comptage *asymptotique* (le développement asymptotique du nombre d'objets de taille n lorsque n tend vers l'infini) est souvent suffisant. Nous allons maintenant présenter quelques outils et méthodes pour le comptage exact ou asymptotique d'une classe combinatoire.

0.2.2 Comment compter ?

L'approche la plus systématique pour l'énumération d'une classe combinatoire est basée sur l'utilisation des *séries génératrices*. Cette approche, dite *analytique* (ou *symbolique*) sera effective dès lors que la classe combinatoire admet une description *récursive* (en terme d'opérations élémentaires comme l'union, le produit cartésien *etc.*) [Flaj].

Considérons une classe combinatoire $\mathbf{S}$ *comptée* par la suite $(s_n)_{n\in\mathbb{N}}$ (s_n est le nombre d'objets de taille n). À la classe $\mathbf{S}$ on associe la *série génératrice* (ordinaire[1])

$$S(z) = \sum_{n\in\mathbb{N}} s_n z^n.$$

En notant $|.|$ la fonction taille de la classe $\mathbf{S}$, la série génératrice se définit de manière équivalente par $S(z) = \sum_{s\in\mathbf{S}} z^{|s|}$. Pour l'instant, la série génératrice $S(z)$ est une série formelle en la variable z et on ne se soucie pas des questions de convergence de cette série lorsqu'un nombre complexe est substitué à z. On notera $[z^n]S(z)$ le coefficient de z^n dans la série $S(z)$. L'approche analytique pour l'énumération de la classe $\mathbf{S}$ consiste à traduire une description récursive de la classe $\mathbf{S}$ en une équation vérifiée par la série génératrice $S(z)$.

Prenons l'exemple de la classe $\mathbf{D}$ des mots de Dyck comptée par la suite de Catalan $(C_n)_{n\in\mathbb{N}}$. On cherche d'abord une description récursive de la classe $\mathbf{D}$. Les mots de Dyck partent de 0, restent positifs et retournent en 0. En considérant le premier retour en 0, on peut décomposer le mot de Dyck D sous la forme $D = xD_1\overline{x}D_2$ où D_1 et D_2 sont deux mots de Dyck. Cette décomposition est illustrée par la figure 16. On obtient une bijection entre les mots de Dyck de taille $n+1$ et les couples de mots de Dyck de taille k et $n-k$ respectivement, où k est un entier compris entre 0 et n. Par conséquent la classe $\mathbf{D}$ des mots de Dyck admet la description récursive

$$\mathbf{D} = \{\epsilon\} \cup \{x\} \times \mathbf{D} \times \{\overline{x}\} \times \mathbf{D}, \tag{5}$$

où ϵ est le mot vide.

La description récursive (5) implique la relation de récurrence

$$C_0 = 1 \qquad \text{et} \qquad C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}, \tag{6}$$

[1]Il existe d'autres séries génératrices : *exponentielles*, *de Dirichlet* etc. mais nous nous limiterons aux séries génératrices ordinaires qui sont adaptées à l'énumération des cartes non-étiquetées

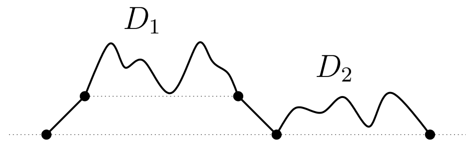


Figure 16: Décomposition récursive des mots de Dyck.

qui définit la suite de Catalan. On peut écrire (et résoudre) cette relation de récurrence à l'aide des séries génératrices. Rappelons que la série génératrice $C(x) = \sum_{n\in\mathbb{N}} C_n z^n$ est une série formelle, c'est-à-dire que la série génératrice n'est guère plus qu'une autre façon d'écrire la suite $(C_n)_{n\in\mathbb{N}}$. L'avantage de cette écriture est que l'on dispose des opérations *somme*, *produit* et *substitution* définies de façon usuelle sur les séries. Par exemple, le produit de deux séries formelles $F(z) = \sum_{n\in\mathbb{N}} f_n z^n$ et $G(z) = \sum_{n\in\mathbb{N}} g_n z^n$ est la série formelle $H(z) = FG(z) = \sum_{n\in\mathbb{N}} \left(\sum_{k=0}^{n} f_k g_{n-k}\right) z^n$. Ainsi, la relation de récurrence (6) se traduit par l'équation

$$C(z) = 1 + zC(z)^2. \tag{7}$$

Puisque l'équation (7) est équivalente à la relation de récurrence (6), elle définit la suite de Catalan de manière unique. Autrement dit, cette équation admet une unique solution dans l'espace des séries formelles. Nous verrons plus tard comment résoudre cette équation, c'est-à-dire en déduire la forme close $C_n = \frac{1}{n+1}\binom{2n}{n}$. Notons pour l'instant que l'équation (7) capture de façon élégante la description récursive (5) de la classe $\mathbf{D}$ des mots de Dyck. Cette propriété est l'un des atouts majeurs de l'approche analytique: toute description récursive d'une classe combinatoire s'appuyant sur les opérations usuelles (union, produit cartésien, etc.) se traduit de manière *automatique* en une équation satisfaite par la série génératrice correspondante [Flaj] (sous réserve que les fonctions tailles soient définies de manière à respecter la description récursive). La traduction utilise un dictionnaire dont un extrait est donné ci-dessous.

Construction combinatoire		Opération sur la série génératrice
Union disjointe	: $A = B \cup C$	$A(z) = B(z) + C(z)$
Produit cartésien	: $A = B \times C$	$A(z) = B(z) \cdot C(z)$
Suite	: $A = \sigma(B)$	$A(z) = \dfrac{1}{1 - B(z)}$
Pointage	: $A = B^\bullet$	$A(z) = z\dfrac{d}{dz}B(z)$
Substitution	: $A = B \circ C$	$A(z) = B(C(z))$

En traduisant une description récursive d'une classe combinatoire, on obtient une équation qui définit entièrement la série génératrice de cette classe. On appelle *rationnelles* les séries formelles de la forme $F(z) = P(z)/Q(z)$ où P et Q sont des polynômes. On appelle *algébriques* les séries formelles solutions d'une équation de la forme $P(F(z), z) = 0$, où $P(x, y)$ est un polynôme non nul. On appelle *différentiellement finies* les séries formelles solutions d'une équation de la forme $\sum_{i=0}^{k} p_i(z) F^{(i)}(z)$, où $p_i(z)$ est un polynôme en z et $F^{(i)}$ est la dérivée $i^{\text{ème}}$ de la série formelle F. Clairement, les séries rationnelles sont algébriques et on peut montrer que les séries algébriques sont différentiellement finies[2]. Informellement, on s'attend à ce que la *nature* d'une série génératrice (rationnelle, algébrique, différentiellement finie) reflète la *complexité* de la classe combinatoire qu'elle énumère [Bous 06]. Ainsi, les langages rationnels (resp. algébriques) donnent lieu à des séries rationnelles (resp. algébriques).

L'équation définissant une série génératrice fournit une méthode effective pour le calcul des coefficients de la série. Par exemple, l'équation de la suite de Catalan (7) fournit un algorithme quadratique pour le calcul des n premiers coefficients de la série. En fait, dès que la série est différentiellement finie (c'est le cas de la suite de Catalan qui est algébrique), on dispose d'un algorithme calculant les n premiers coefficients en un nombre *linéaire* d'opérations arithmétiques [Stan 80b].

On s'intéresse maintenant à la *résolution* des équations, c'est-à-dire au passage de l'équation définissant une série génératrice $F(z)$ à la détermination (exacte ou asymptotique) de ses coefficients. Essayons de résoudre l'équation (7) définissant la suite de Catalan. Puisque cette équation est quadratique, ses deux solutions (dans l'espace des séries de Laurent) s'expriment par radicaux. L'une des deux solutions n'est pas une série formelle car elle fait intervenir le terme z^{-1}. Cette solution est donc rejetée et on en déduit l'expression de la série génératrice $C(z)$:

$$C(z) \quad = \quad \frac{1 - \sqrt{1 - 4z}}{2z}. \tag{8}$$

Ensuite, le théorème du binôme de Newton montre que le développement de la série $C(z)$ s'écrit

$$C(z) = \sum_{n \in \mathbb{N}} \frac{1}{n+1} \binom{2n}{n} z^n.$$

Voilà donc prouvée l'expression (4) des nombres de Catalan.

La résolution explicite par radicaux n'étant pas vouée à un très grand avenir nous allons maintenant introduire d'autres méthodes.

Théorème 0.5 (Inversion de Lagrange 1770) *Soit* $\Phi(x) = \sum_{n \in \mathbb{N}} \phi_n x^n$ *une série formelle dont le coefficient constant* ϕ_0 *est non nul. Il existe une unique série formelle* $F(z)$

[2] En effet, les dérivées de F s'expriment toutes comme des fractions rationnelles en F et puisque les puissances de F ne sont pas linéairement indépendantes (sur le corps des fractions rationnelles en z) les fractions rationnelles en F engendrent un espace linéaire de dimension fini.

solution de l'équation $F(z) = z\Phi(F(z))$. Cette solution est donnée par

$$F(z) = \sum_{n\geq 1} f_n z^n \qquad \text{où} \qquad f_n = \frac{1}{n}[x^{n-1}](\Phi(x))^n.$$

De plus,

$$[z^n]f^k(z) = \frac{k}{n}[x^{n-k}](\Phi(x))^n.$$

On peut facilement mettre l'équation (7) sous la forme prescrite par le théorème d'inversion de Lagrange :

$$C(z) - 1 = \Phi(C(z) - 1) \qquad \text{où} \qquad \Phi(x) = (x+1)^2,$$

et obtenir ainsi une seconde preuve de l'expression des nombres de Catalan :

$$C_n = \frac{1}{n}[x^{n-1}](1+x)^{2n} = \frac{1}{n}\binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}.$$

Il n'est malheureusement pas toujours possible de se ramener au théorème d'inversion de Lagrange ni d'obtenir des résultats d'énumération exacte. Nous verrons bientôt (sous-section 0.2.4) comment réaliser l'énumération asymptotique à partir des équations définissant la série génératrice. Avant cela, nous allons voir comment obtenir des équations définissant la série génératrice d'une classe combinatoire dont nous n'avons qu'une compréhension partielle.

0.2.3 Les variables catalytiques

Nous avons vu comment traduire une description récursive d'une classe combinatoire en une équation définissant sa série génératrice. Malheureusement, il est souvent difficile de trouver une description récursive d'une classe combinatoire. Du moins, les descriptions récursives *naïves* nécessitent souvent de prendre en compte de nouveaux paramètres sur nos objets en plus de la taille.

Considérons, par exemple, les chemins unidimensionnels fait de pas +1 et -1 qui partent de 0 et restent positifs. Une description naïve consiste à dire qu'un chemin non-vide se décompose en un chemin plus un pas. On doit juste faire attention à ne pas ajouter un pas -1 lorsque le chemin est à hauteur 0. Cette approche nous oblige donc à prendre en compte la hauteur finale des chemins en plus de leur taille. On considère alors la série génératrice *bivariée* $F(x,z) = \sum f_{n,k} x^k z^n$, ou $f_{n,k}$ est le nombre de chemins de taille (longueur) n et de hauteur k. Les chemins à hauteur 0 sont comptés par $F(0,z)$. Par conséquent, la description

$$\{\text{chemins}\} = \{\text{chemin vide}\} \cup \{\text{chemins}\} \times \{+1\} \cup \{\text{chemins à hauteur strictement positive}\} \times \{-1\},$$

se traduit par l'équation

$$F(x,z) = 1 + xzF(x,z) + \frac{z}{x}(F(x,z) - F(0,z)). \tag{9}$$

Cette équation définit bien la série $F(x,z)$ comme série formelle en la variable z dont les coefficients sont polynomiaux en x. En particulier, la série $F(0,z)$ qui compte les mots de Dyck (selon leur longueur) est entièrement déterminée par l'équation (9).

La variable x qui compte la hauteur du chemin était *nécessaire* pour écrire l'équation (9). La preuve en est que si nous assignons une valeur à x (0 ou 1 par exemple) l'équation devient insuffisante pour définir $F(x,z)$. Par analogie à la chimie, Zeilberger [Zeil 00] qualifie la variable x de *catalytique* : elle permet d'écrire l'équation mais on aimerait s'en débarrasser a posteriori.

On cherche à *résoudre* l'équation fonctionnelle (9), c'est-à-dire à en déduire une équation pour la série $F(0,z)$ ne faisant pas intervenir la variable catalytique x. L'équation fonctionnelle (9) est linéaire en la série bivariée $F(x,z)$. On peut, dans ce cas particulier, appliquer la *méthode du noyau* [Band 02, Bous 00b, Prod 04]. On commence par mettre la série bivariée en facteur :

$$(x - z(1+x^2))F(x,y) = x - zF(0,z). \tag{10}$$

On cherche ensuite à annuler le *noyau*, c'est-à-dire le coefficient $N(x,z) = x - z(1+x^2)$ de la série bivariée $F(x,z)$. Plus précisément, on cherche une série $X(z)$ substituable à x dans l'équation (10) et telle que $N(X(z),z) = 0$. Dans notre cas, l'équation $X(z)-z(1+X(z)^2) = 0$ admet deux solutions

$$X_1(z) = \frac{1-\sqrt{1-4z^2}}{2z} \qquad \text{et} \qquad X_2(z) = \frac{1+\sqrt{1-4z^2}}{2z}.$$

La série $X_1(z) = z+o(z)$ est substituable à x dans l'équation (10). On obtient alors le système

$$\begin{cases} X_1(z) - z(1+X_1(z)^2) &= 0, \\ X_1(z) - zF(0,z) &= 0, \end{cases}$$

et finalement,

$$F(0,z) = 1 + z^2F(0,z)^2. \tag{11}$$

Cette équation définit la série $F(0,z)$ qui compte les mots de Dyck selon leur longueur. Elle est équivalente à l'équation (7) régissant la série $C(z)$ qui compte les mots de Dyck selon leur demi-longueur. Le détour par les variables catalytiques nous a donc permis d'obtenir l'équation (7) de la suite de Catalan en nous basant sur une approche très naïve de la combinatoire des mots de Dyck.

La description récursive naïve des cartes planaires enracinées consiste à décomposer une carte en une racine et une ou plusieurs cartes plus petites. Cette approche initiée par Tutte dans les années 60 permet l'énumération de nombreuses familles de cartes [Tutt 62b, Tutt 62a, Tutt 62c, Tutt 63]. Nous verrons en section 0.4 que cette approche s'adapte aussi au cas des cartes coloriées et plus généralement aux cartes pondérées par leur polynôme de Tutte.

Énumération récursive de cartes dans cette thèse. Au chapitre 1, nous utiliserons la méthode récursive pour l'énumération des familles de triangulations dont le degré des sommets est au moins égal à une valeur d choisie parmi $\{3, 4, 5\}$. Nous obtenons des équations fonctionnelles pour les séries génératrices de ces familles au prix de l'introduction d'une variable catalytique. La résolution des équations fonctionnelles utilise les méthodes présentées ci-dessous et mène à une caractérisation algébrique pour la série génératrice de chacune des familles.

Tutte a montré [Tutt 63] que la série génératrice $G(x, z)$ des cartes planaires enracinées vérifie l'équation fonctionnelle

$$G(x, z) = 1 + x^2 z G(x, z)^2 + xz \left(\frac{xG(x, z) - G(1, z)}{x - 1} \right). \tag{12}$$

Nous verrons comment obtenir l'équation (12) en section 0.4. Dans cette équation, la variable principale z se rapporte à la taille (le nombre d'arêtes) de la carte et la variable catalytique x se rapporte au degré de la face *externe* (à droite de l'arête racine). L'équation (12) n'étant pas linéaire en la variable z nous ne pouvons appliquer la méthode du noyau. Nous pourrions utiliser la *méthode quadratique* [Brow 65, Goul 83] mais nous préférons présenter une méthode plus générale qui sera utile au chapitre 1. Cette méthode due à Bousquet-Mélou et Jehanne [Bous 05b] s'applique à toute équation de la forme

$$P(G(x), G_1, \ldots, G_k, x, z) = 0, \tag{13}$$

où P est un polynôme en $k + 3$ variables, $G(x) \equiv G(x, z)$ est la série génératrice bivariée et les séries $G_i \equiv G_i(z)$ sont des séries univariées. Dans notre cas, $k = 1$, $G_1 = G(1, z)$ et

$$P(G(x), G_1, x, z) = x^2(x - 1)zG(x)^2 + (x^2 z - x + 1)G(x) - xzG_1 + x - 1. \tag{14}$$

On suppose que l'équation (13) définit $G(x) \equiv G(x, z)$ de manière unique comme série formelle en z dont les coefficients sont polynomiaux en x. Par exemple, en remplaçant x par 1 dans (14) on obtient $G_1 = G(1)$. Une fois cette information acquise, on peut calculer récursivement les coefficients de $G(x) \equiv G(x, z)$. La série $G(x)$ est donc bien définie.

Sous ces hypothèses, on dispose d'une méthode générale pour la résolution de l'équation (13). La première étape consiste à chercher les séries $X \equiv X(z)$ qui satisfont

$$P_1'(G(X(z)), G_1, \ldots, G_k, X(z), z) = 0, \tag{15}$$

où P'_1 est la dérivée de P par rapport à sa première variable. Il est démontré (sous des hypothèses assez générales) dans [Bous 05b] qu'il existe k séries (de Puiseux) $X_1, \ldots, X_k$ substituables à x dans l'équation (13) et satisfaisant l'équation (15). Dans notre cas, l'équation (15) s'écrit

$$X = 1 + zX^2 + 2zX^2(X-1)G(X)$$

et admet bien une solution $X_1 = 1 + o(1)$ substituable.

En dérivant l'équation (13) par rapport à x, on obtient

$$\frac{d}{dx}G(x) \cdot P'_1(G(x), G_1, \ldots, G_k, x, z) + P'_{k+2}(G(x), G_1, \ldots, G_k, x, z) = 0,$$

où P'_{k+2} est la dérivée de P par rapport à sa $(k+2)^{\text{ème}}$ variable. Les séries $X_i, i = 1 \ldots k$ satisfont donc aussi l'équation

$$P'_{k+2}(G(X), G_1, \ldots, G_k, X, z) = 0. \tag{16}$$

On obtient donc un système de $3k$ équations polynomiales

$$\left\{\begin{array}{lcl} P(G(X_i), G_1, \ldots, G_k, X_i, z) & = & 0, \\ P'_1(G(X_i), G_1, \ldots, G_k, X_i, z) & = & 0, \\ P'_{k+2}(G(X_i), G_1, \ldots, G_k, X_i, z) & = & 0, \end{array}\right. \qquad i = 1 \ldots k$$

pour les $3k$ séries inconnues $G_i,\ X_i(z),\ G(X_i), i = 1 \ldots k$. Il est démontré que ce système définit bien l'ensemble des séries inconnues (13). La résolution peut se faire en utilisant des techniques d'éliminations par résultants ou par bases de Gröbner. On obtient alors une équation polynomiale (ne faisant pas intervenir la variable catalytique x) pour chacune des séries inconnues $G_i, i = 1 \ldots k$. Pour l'équation des cartes, la résolution du système aboutit à l'équation :

$$27z^2G_1^2 + (1-18z)G_1 + 16z - 1. \tag{17}$$

0.2.4 Énumération asymptotique

Il n'est pas toujours possible (ni utile) de réaliser l'énumération exacte d'une classe combinatoire. Bien souvent, on s'estime heureux avec une *énumération asymptotique*, c'est-à-dire un développement asymptotique du nombre d'objets de taille n. La combinatoire analytique (basée sur les séries génératrices) fournit une collection de méthodes permettant de réaliser l'énumération asymptotique d'une classe combinatoire lorsque l'on dispose d'une équation définissant la série génératrice correspondante.

L'énumération asymptotique demande un changement de perspective sur notre façon de considérer les séries génératrices. Jusqu'à présent, nous avons considéré les séries génératrices comme des séries formelles. Cependant, si le rayon de convergence de la série génératrice est non nul, il y a beaucoup à gagner à la considérer comme une fonction holomorphe sur une partie du plan complexe. En effet, cette perspective permet d'espérer obtenir des informations sur l'asymptotique des coefficients en étudiant les *singularités* de la série génératrice (cet espoir est fondé sur le théorème des résidus de Cauchy). Cette approche extrêmement féconde est à la base de toutes les techniques d'énumération asymptotique reposant à la base sur des résultats exacts [Flaj 90, Flaj]. Nous renvoyons le lecteur à l'ouvrage de Flajolet et Sedgewick [Flaj] (en préparation) pour une exposition très complète du domaine. Nous nous contenterons pour notre part de tracer les lignes directrices des résultats et méthodes qui s'appliquent aux séries génératrices *algébriques*.

> **Énumération asymptotique dans cette thèse.** Au chapitre 1, nous établissons des équations algébriques caractérisant les séries génératrices de plusieurs familles de triangulations. Nous obtenons ensuite le développement asymptotiques du nombre de triangulations dans chaque famille à l'aide des techniques présentées ci-dessous.

Principes généraux de l'énumération asymptotique : Avant d'étudier le cas des séries génératrices algébriques nous rappelons quelques principes généraux pour l'extraction asymptotique des coefficients d'une fonction holomorphe.

1. Si une fonction holomorphe $F(z) = \sum_{n\in\mathbb{N}} f_n z^n$ a une unique singularité dominante ρ (non nulle) les coefficients f_n ont une croissance du type $f_n = \theta(n)\rho^{-n}$ où la croissance de la fonction θ est sous-exponentielle. Le facteur sous-exponentiel θ est déterminé par le développement asymptotique de la fonction $F(z)$ au voisinage de ρ.

   On peut se ramener au cas $\rho = 1$ en remarquant $f_n = [z^n]F(z) = \rho^{-n}F(\frac{z}{\rho})$.

2. Il y a une correspondance (préservant les ordres de grandeur) entre développement asymptotique d'une fonction holomorphe au voisinage de sa singularité dominante et l'asymptotique des coefficients de cette fonction. En effet, sous une condition (Δ) concernant le domaine de définition de la fonction holomorphe F, on a l'identité fondamentale (non triviale) $[z^n]O(F(z)) = O([z^n]F(z))$.

3. On dispose d'une échelle de fonctions, les fonctions du type $(1-z)^\alpha \log(1-z)^\beta$, dont on connaît l'asymptotique. Par conséquent, sous l'hypothèse (Δ), on déduit l'asymptotique des coefficients de toute fonction F dont le comportement asymptotique autour de la singularité dominante 1 est comparable à cette échelle. En répétant l'opération on obtient le *développement* asymptotique des coefficients de F.

4. Si une fonction holomorphe a un nombre fini de singularités dominantes, leurs contributions à l'asymptotique des coefficients s'additionnent.

Le cas des séries génératrice algébriques : Nous passons maintenant à des propriétés spécifiques aux séries algébriques. Les coefficients f_n d'une série génératrice $F(z) = \sum_{n\in\mathbb{N}} f_n z^n$ sont des entiers positifs et on supposera (sans grande perte) qu'il existe une infinité de coefficients non nuls. Dans ce cas, le rayon de convergence ρ de la fonction holomorphe F est fini (et inférieur à 1). De plus, par le théorème de Pringsheim, le réel positif ρ est une singularité (dominante) de la fonction F. Supposons que la série génératrice F est algébrique, c'est-à-dire solution d'une équation de la forme $P(F(z), z) = 0$, ou $P(y, z)$ est un polynôme non nul. Dans ce cas, la fonction F est une des *branches* de la *courbe algébrique complexe* $P(y, z) = 0$ (l'ensemble des couples de complexes (y, z) vérifiant cette équation). Nous allons voir que l'algébricité de F fournit des renseignements sur la *position* des singularités ainsi que sur la *nature* de celles-ci.

• Position des singularités :
On considère le discriminant $D(z)$ du polynôme $P(z, y)$ par rapport à la variable y et le coefficient dominant $d(z)$ de ce polynôme. Toute singularité de la fonction F est une racine du polynôme $d(z)D(z)$. En particulier, F a un nombre fini de singularités dominantes. De plus, les singularités dominantes de F peuvent être déterminées algorithmiquement. Le principe général consiste à déterminer pour chaque racine z_0 de $d(z)D(z)$ un développement de chacune des branches de la courbe $P(z, y) = 0$ qui soit valide dans un voisinage de z_0. Il faut ensuite être capable de faire correspondre la branche $F(z)$ à l'un de ces développements pour savoir si z_0 est une singularité de F. Un algorithme complet décrivant comment résoudre ces problèmes de branchement est présenté dans [Flaj]. Si F est une série génératrice, la positivité des coefficients simplifie grandement les problèmes de branchement auxquels on est confronté. En effet on sait qu'il existe une singularité dominante ρ qui est un réel positif. Pour trouver ρ on peut utiliser un algorithme par balayage permettant de suivre la branche correspondant à F sur l'axe des réels jusqu'à sa singularité dominante ρ. Cet algorithme ne prend en compte que le classement par ordre croissant des courbes admettant un développement réel sur l'axe des réels. En effet, les croisements entre branches, l'apparition, la disparition de branche à développement réel ou le passage de branche par l'infini ne peuvent se produire que pour des valeurs de z racine du polynôme $d(z)D(z)$.

• Nature des singularités :
Soit z_0 une singularité de $F(z)$. La fonction $F(z)$ admet un développement asymptotique dans un voisinage de z_0 coupé d'une demi-droite émanant de z_0 (en particulier, la condition

(Δ) sus-mentionnée est vérifiée). Ce développement prend la forme (de Puiseux) :

$$F(z) = \sum_{k \geq k_0} c_k (z - z_0)^{k/\kappa},$$

où k_0 est un entier relatif et κ est un entier positif. Ce développement peut être déterminé algorithmiquement par la méthode du *polygone de Newton* implémentée dans la librairie *gfun* de *Maple* [Salv 94] (il faut tout de même déterminer à quel développement correspond la branche F). Toutes les conditions sont réunies pour appliquer les principes généraux de l'extraction asymptotique des coefficients mentionnés plus haut. Si $z_0 \in \mathbb{R}^+$ est l'unique singularité dominante on obtient :

$$f_n \sim c_{k_0} \frac{n^{\alpha-1}}{\Gamma(\alpha)} z_0^n,$$

ou $\alpha = -k_0/\kappa$ et Γ est la fonction Gamma. On peut, en fait, déterminer un développement asymptotique des coefficients f_n aussi poussé que nécessaire.

0.3 Polynôme de Tutte et modèle de Potts

Le polynôme de Tutte est un invariant fondamental des graphes. Il généralise à la fois le *polynôme chromatique* (comptant les coloriages) et le *polynôme des flots* (comptant les flots partout non-nuls). Afin d'introduire le polynôme de Tutte en douceur, nous commençons par quelques rappels concernant le polynôme chromatique.

0.3.1 Polynôme chromatique

On s'intéresse au nombre de façons de *colorier* un graphe avec q couleurs. Par *coloriage* nous entendons une attribution d'une *couleur* parmi $\{1, \ldots, q\}$ à chaque sommet telle que deux sommets adjacents soient toujours de couleur différente (il n'est pas exigé que toutes les couleurs soient utilisées).

Le graphe de gauche en figure 17 admet $q(q-1)(q-2)^2$ coloriages avec q couleurs. En effet, il y a q couleurs possibles pour le sommet s, après quoi il reste $q-1$ couleurs possibles pour le sommet t, puis $q-2$ couleurs pour les sommets u et v. Remarquons, sur cet exemple, que le nombre de coloriages en q couleurs s'exprime comme un *polynôme* en la variable q. Cette propriété est en fait générale et se prouve facilement par récurrence sur le nombre d'arêtes. Pour cela on introduit deux opérations fondamentales sur les graphes : la suppression et la contraction d'une arête. Étant donné un graphe G et une arête e, on note $G \backslash e$ et G/e les graphes obtenus respectivement en supprimant l'arête e et en *contractant* l'arête e (i.e, en supprimant l'arête e et en identifiant ses deux extrémités). (La suppression et la contraction coïncident lorsque l'arête e est une boucle.) Ces deux opérations sont illustrées en figure 17.

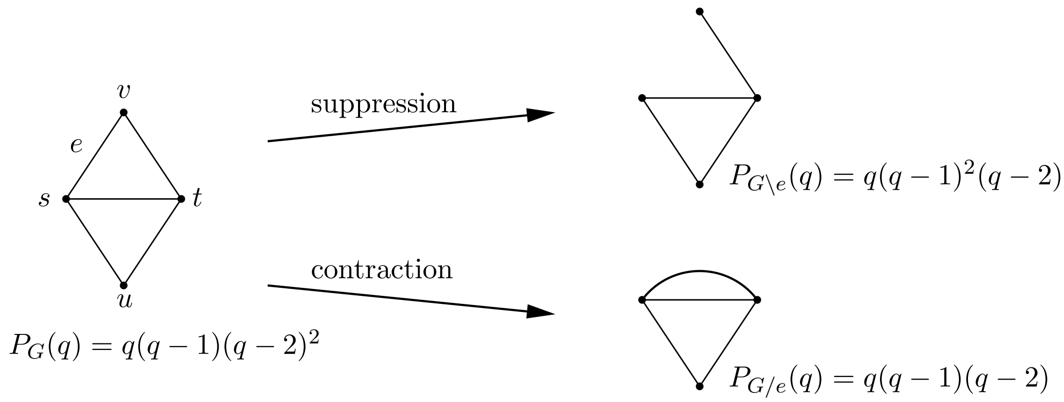


Figure 17: Relation de récurrence pour le polynôme chromatique.

Remarquons que les coloriages de $G\backslash e$ qui ne sont pas des coloriages de G sont ceux pour lesquels les deux extrémités de l'arête e sont de même couleur, c'est-à-dire les coloriages de G/e. Cette propriété est la clef pour démontrer le résultat classique suivant.

Proposition 0.6 *Pour tout graphe G il existe un unique polynôme $P_G(q)$, appelé polynôme* chromatique*, tel que pour tout entier q, l'évaluation $P_G(q)$ soit le nombre de coloriages de G avec q couleurs. De plus, pour toute arête e du graphe G qui n'est pas une boucle, le polynôme $P_G(q)$ satisfait la relation*

$$P_G(q) = P_{G\backslash e}(q) - P_{G/e}(q). \tag{18}$$

La proposition 0.6 est illustrée par la figure 17. On peut aussi exprimer le polynôme chromatique d'un graphe G par sommation sur les *sous-graphes couvrants*. Un graphe H est un *sous-graphe couvrant* de G si les sommets de H sont les sommets de G et les arêtes de H sont un sous-ensemble des arêtes de G. Nous ne considérerons que des sous-graphes couvrants et nous les appellerons simplement *sous-graphes*.

Proposition 0.7 *Le polynôme chromatique du graphe G est égal à*

$$P_G(q) = \sum_{H\subseteq G} (-1)^{|H|} q^{c(H)}, \tag{19}$$

où la somme porte sur les sous-graphes de G, et les exposants $|H|$ et $c(H)$ sont respectivement le nombre d'arêtes et de composantes connexes du sous-graphe H.

La proposition 0.7 peut être prouvée par une méthode de *crible* (*sieving methods*) [Stan 86, Chap. 2]). On considère les *pseudo-coloriages* du graphe G, soit l'attribution d'une couleur à chaque sommet (sans contrainte sur les sommets adjacents). Pour un sous-graphe H, on note $f(H)$ le nombre pseudo-coloriages en q couleurs tels que les sommets adjacents dans H soient de couleur identique mais que les sommets adjacents dans G et non dans H soient de couleurs différentes. Pour le sous-graphe sans-arête $H_\emptyset$ le paramètre $f(H_\emptyset)$ n'est autre que $P_G(q)$. Pour tout sous-graphe H, il est clair que $g(H) = \sum_{H\subseteq K\subseteq G} f(K)$ compte les pseudo-coloriages en q couleurs tels que les sommets adjacents dans H soient de couleurs identiques. On obtient donc $g(H) = q^{c(H)}$. Le principe du crible (qui consiste à inverser la matrice donnant les valeurs de g en fonction de celles de h) montre que $f(H) = \sum_{H\subseteq K\subseteq G}(-1)^{|K|-|H|}g(K)$. En particulier, pour le sous-graphe sans arête $H_\emptyset$ on retrouve l'expression (19).

L'expression (19) du polynôme chromatique invite à étudier le polynôme bivarié $\sum\limits_{H\subseteq G} v^{|H|}q^{c(H)}$. Nous verrons en sous-section 0.3.4 qu'il y a équivalence (à changement de variables près) entre cette généralisation du polynôme chromatique et la fonction de partition du modèle de Potts. Ce polynôme est aussi équivalent (à changement de variables près) à un invariant fondamental de la théorie de graphe que Tutte baptisa *polynôme dichromatique* et qui est couramment appelé *polynôme de Tutte*.

0.3.2 Polynôme de Tutte : définition et spécialisations

Le polynôme de Tutte est un polynôme bivarié qui généralise à la fois le polynôme chromatique et le polynôme des flots et admet de nombreuses autres spécialisations intéressantes. Depuis sa découverte par William T. Tutte dans les années 1950, plusieurs caractérisations du polynôme de Tutte ont été proposées. Dans la définition originale de Tutte, son polynôme est défini comme la série génératrice des arbres couvrants comptés selon leurs *activités* [Tutt 54]. Au chapitre 5 de cette thèse nous établissons une nouvelle caractérisation, toujours comme série génératrice des arbres couvrants. Notre caractérisation s'appuie sur une nouvelle notion d'*activité* basée sur la structure de carte combinatoire. Cependant, la caractérisation la plus rassurante du polynôme de Tutte est comme série génératrice des sous-graphes comptés selon le nombre d'arêtes et de composantes connexes [Bryl 91].

Definition 0.8 *Soit G un graphe ayant c composantes connexes et s sommets. Le polynôme de Tutte du graphe G est*

$$T_G(x,y) = \sum_{H\subseteq G}(x-1)^{c(H)-c}(y-1)^{|H|+c(H)-s}, \tag{20}$$

où la somme porte sur les sous-graphes de G, et les exposants $|H|$ et $c(H)$ sont respectivement le nombre d'arêtes et de composantes connexes du sous-graphe H.

Par exemple, le graphe complet K_3 (le triangle) a 8 sous-graphes. Le sous-graphe sans arête a contribution $(x-1)^2$, chaque sous-graphe à une arête a contribution $(x-1)$, chaque sous-graphe à deux arêtes a contribution 1, et le sous-graphe à trois arêtes a contribution $(y-1)$. En additionnant ces contributions, on obtient $T_{K_3}(x,y) = (x-1)^2+3(x-1)+3+(y-1) = x^2+x+y$.

Il est facile de voir que le polynôme de Tutte est multiplicatif sur les composantes connexes : lorsque G est l'union disjointe de deux graphes G_1 et G_2, alors $T_G(x,y) = T_{G_1}(x,y) \times T_{G_2}(x,y)$. Cette remarque nous autorise à restreindre notre attention aux graphes connexes. À partir de maintenant *tous nos graphes sont connexes.*

Le *développement par sous-graphes* (20) du polynôme de Tutte prend en compte deux paramètres : le nombre (renormalisé) de composantes connexes $c(H)-1$ et le nombre *cyclomatique* $|H|+c(H)-s$. Le nombre cyclomatique est le nombre maximal d'arêtes pouvant être supprimées de H sans augmenter le nombre de composantes connexes. En particulier, le nombre cyclomatique est nul si et seulement si H est une *forêt* (i.e. sans cycle). Plusieurs spécialisations du polynôme de Tutte sont immédiates à partir de l'expression (20). Par exemple, $T_G(2,2) = 2^{|G|}$ compte tous les sous-graphes de G (i.e. les sous ensembles d'arêtes), $T_G(1,2)$ compte les sous-graphes connexes, $T_G(2,1)$ compte les forêts et $T_G(1,1)$ compte les arbres couvrants. Le polynôme chromatique est lui aussi une spécialisation du polynôme de Tutte puisque l'équation (19) donne

$$P_G(q) = \sum_{H\subseteq G} (-1)^{|H|} q^{c(H)} = q^c(-1)^s \; T_G(1-q,0).$$

Le polynôme de Tutte admet encore bien d'autres spécialisations intéressantes. La voie la plus rapide (mais aussi la moins satisfaisante) pour démontrer une telle spécialisation est souvent d'utiliser les relations de récurrence du polynôme de Tutte.

Proposition 0.9 *Soit G un graphe et e une arête de G. Le polynôme de Tutte satisfait les relations*

$$T_G(x,y) \;=\; \left| \begin{array}{ll} x\cdot T_{G/e}(x,y) & \textit{si } e \textit{ est un } \text{isthme} \textit{ (i.e. sa suppression déconnecte } G\textit{),} \\ y\cdot T_{G\backslash e}(x,y) & \textit{si } e \textit{ est une boucle,} \\ T_{G/e}(x,y) \;+\; T_{G\backslash e}(x,y) & \textit{si } e \textit{ n'est ni un isthme ni une boucle.} \end{array} \right. \qquad (21)$$

Ces relations permettent, par exemple, de montrer par récurrence que le polynôme chromatique est une spécialisation du polynôme de Tutte en utilisant (18). Par une induction similaire on montre aussi que le polynôme des flots est une spécialisation du polynôme de Tutte.

0.3.3 Polynôme de Tutte et activités des arbres couvrants

Comme nous l'avons mentionné le polynôme de Tutte n'est pas né sous la forme (20) d'une série génératrice des sous-graphes, mais comme une série génératrice des arbres couvrants. Historiquement, Tutte définit le polynôme qui porte son nom après s'être amusé à réduire des graphes à néant par une suite de suppressions et de contractions d'arêtes[3]. Le jeu est le suivant : on ordonne linéairement l'ensemble des arêtes d'un graphe puis on considère les arêtes dans l'ordre décroissant. Si une arête est un isthme on la contracte, si c'est une boucle on la supprime et dans les autres cas on choisit soit de la contracter soit de la supprimer. L'ensemble des exécutions possibles est représenté sur un exemple en figure 18.

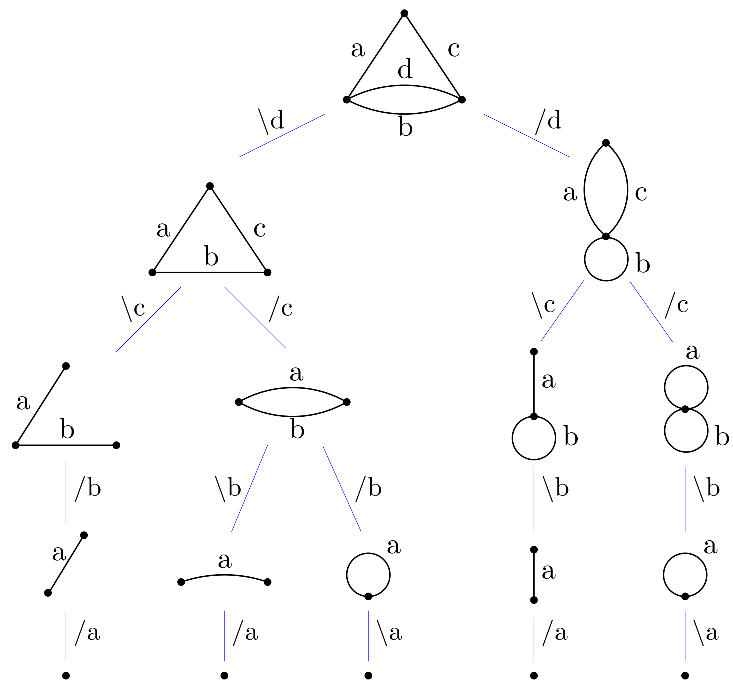


Figure 18: Jeu des suppressions et contractions pour l'ordre $a < b < c < d$ des arêtes.

Observons que pour toute exécution, l'ensemble des arêtes contractées est un arbre couvrant du graphe G. Par exemple l'exécution la plus à gauche en figure 18 correspond à l'arbre couvrant $\{a, b\}$. Pour chaque exécution E, on considère le nombre $i(E)$ de contractions forcées (l'arête était un isthme) et le nombre $e(E)$ de suppressions forcées (l'arête était une boucle). Par exemple, pour l'exécution la plus à gauche en figure 18 on a $i(E) = 2$ et $e(E) = 0$. Si on associe à chaque exécution E le monôme $x^{i(E)}y^{e(E)}$ et que l'on en fait la somme, on obtient un polynôme qui n'est autre que le polynôme de Tutte du graphe. Sur notre exemple, $T_G(x, y) = x^2 + x + y + xy + y^2$. Il mérite d'être souligné que le polynôme obtenu *ne dépend pas de l'ordre choisi sur les arêtes.*

Plutôt que de caractériser le polynôme de Tutte en terme d'exécutions du jeu de suppression/contraction, il est sans doute plus agréable de considérer les arbres couvrants associés. C'est ce que fit Tutte dans l'article fondateur [Tutt 54]. Étant donné un graphe et un arbre couvrant, on appelle *internes* les arêtes qui sont dans l'arbre et *externes* les arêtes qui n'y

[3]Cette petite histoire de la pensée est relatée par Ruth Bari en appendice de [Bari 79].

sont pas. Le *cycle fondamental* d'une arête externe est le cycle qu'elle forme avec les arêtes de l'arbre. Le *cocycle fondamental* d'une arête interne est le cocycle qu'elle forme avec les arêtes qui ne sont pas dans l'arbre. Autrement dit, le cocycle fondamental d'une arête interne e est l'ensemble des arêtes qui relient les deux sous-arbres obtenus en supprimant l'arête e de l'arbre couvrant. Un exemple est représenté en figure 19. On suppose maintenant que les arêtes du graphe sont linéairement ordonnées. Une arête externe (resp. interne) est *active* si elle est minimale dans son cycle (resp. cocycle) fondamental. Pour le graphe de la figure 19, les arêtes actives sont 1, 4, 6 et 9. Avec ces notations, le polynôme de Tutte s'obtient comme série génératrice des arbres couvrants comptés selon leurs activités.

Théorème 0.10 *[Tutt 54] Soit G un graphe dont les arêtes sont linéairement ordonnées. Le polynôme de Tutte du graphe G est*

$$T_G(x,y) = \sum_{A \text{ arbre couvrant}} x^{i(A)} y^{e(A)}, \tag{22}$$

où la somme porte sur les arbres couvrants A de G et $i(A)$ (resp. $e(A)$) est le nombre d'arêtes internes (resp. externes) actives.

Le développement par arbres (22) du polynôme de Tutte est d'autant plus étonnant qu'il implique l'invariance de la somme (22) par rapport à l'ordre choisi sur les arêtes. Si on identifie l'arbre couvrant à une exécution du jeu de suppression/contraction, alors les arêtes internes (resp. externes) actives correspondent aux arêtes dont la contraction (resp. la suppression) est forcée durant l'exécution du jeu.

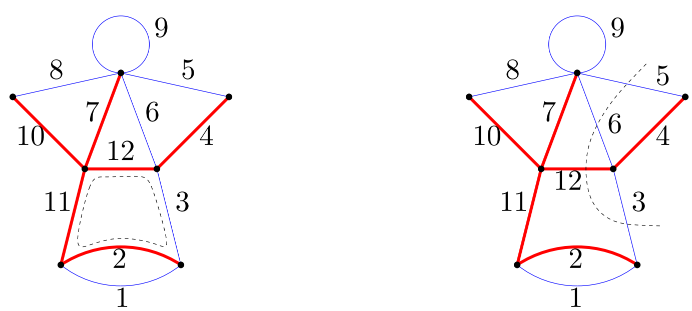


Figure 19: Le cycle fondamental de l'arête externe 3 est $\{2, 3, 11, 12\}$. Le cocycle fondamental de l'arête interne 12 est $\{3, 5, 6, 12\}$.

Il existe une autre caractérisation du polynôme de Tutte comme série génératrice des orientations comptées selon leurs *activités cycliques* [Las 84b]. Cette caractérisation due à Las Vergnas demande, tout comme celle due à Tutte, d'ordonner linéairement les arêtes du graphe. Un lien entre le *développement par orientations* de Las Vergnas et le *développement par arbres* de Tutte est établi dans [Gioa 05]. Mentionnons enfin, qu'il existe une autre définition de l'*activité externe* des arbres couvrants due à Gessel et Wang [Gess 79]. Une comparaison entre les différentes caractérisations du polynôme de Tutte est effectuée au chapitre 5, section 5.3.

> **Le polynôme de Tutte dans cette thèse.** Au chapitre 5, nous établirons un autre développement par arbres du polynôme de Tutte qui utilise non pas un ordre linéaire sur les arêtes mais une carte combinatoire enracinée. Dans la perspective du jeu de suppression/contraction, cette nouvelle caractérisation revient à considérer l'arête racine à chaque étape de l'exécution (et à choisir entre la suppression et la contraction). On doit ré-enraciner la carte à chaque étape de l'exécution. Si la carte $C = (H, \sigma, \alpha)$ est enracinée sur la demi-arête h et que l'on choisi de contracter (resp. supprimer) l'arête racine, la racine de la nouvelle carte sera $\sigma\alpha(h)$ (resp. $\sigma(h)$). Nous appellerons *activité de plongement* la notion d'activité qui résulte de cette nouvelle règle du jeu. Les activités de plongement nous servent ensuite à définir une bijection entre les sous-graphes et les orientations (chapitre 6). Cette bijection se spécialise agréablement à différentes classes de sous-graphes (sous-graphes connexes, forêts, arbres couvrants *etc.*). L'étude de ces spécialisations est effectué au chapitre 7 et permet d'obtenir bijectivement l'interprétation de plusieurs évaluations du polynôme de Tutte en termes d'orientations.

0.3.4 Polynôme de Tutte et modèle de Potts

Nous présentons maintenant l'équivalence (découverte par Fortuin et Kasteleyn [Fort 72]) entre le polynôme de Tutte et la fonction de partition du modèle de Potts.

Soit G un graphe dont S est l'ensemble des sommets. On considère le modèle de Potts (défini en sous-section 0.1.5) sur le graphe G. On rappelle que la fonction de partition du modèle de Potts à q états s'écrit

$$Z_G(q, K) = \sum_{\theta\ :S\mapsto\{1,\ldots,q\}} \exp(K \cdot u(\theta)),$$

où la somme porte sur l'ensemble des *configurations* θ (l'attribution d'un état parmi $\{1, \ldots, q\}$ à chaque sommet) et $u(\theta)$ est le nombre d'arêtes *unicolores* (dont les deux extrémités ont même état). Le paramètre $u(\theta)$ peut se définir par une sommation sur l'ensemble A des arêtes :

$$u(\theta) = \sum_{a\in A} \delta_\theta(a),$$

où $\delta_\theta(a)$ vaut 1 si l'arête est unicolore et 0 sinon. En reportant cette expression dans la fonction de partition et en développant la fonction exponentielle on obtient

$$Z_G(q, K) \;=\; \sum_{\theta\ :S\mapsto\{1,\ldots,q\}} \prod_{a\in A} \exp(K\delta_\theta(a)) \;=\; \sum_{\theta\ :S\mapsto\{1,\ldots,q\}} \prod_{a\in A} \left(1 + v\delta_\theta(a)\right),$$

où $v = \exp(K) - 1$. On considère le développement du produit. Chaque terme de ce développement correspond à un sous-graphe H de G dont les arêtes sont celles pour lesquelles le terme $v\delta_\theta(a)$ est pris. On fait ensuite la somme sur toutes les configurations. Le terme correspondant au sous-graphe H est non nul si et seulement si la configuration c est constante sur chaque composante connexe de H. Il y a $q^{c(H)}$ telles configurations ($c(H)$ est le nombre de composantes connexes de H). On obtient donc

$$Z_G(q, K) = \sum_{H \subseteq G} v^{|S|} q^{c(H)}. \tag{23}$$

On reconnaît maintenant le développement par sous-graphes du polynôme de Tutte (20), et on trouve

$$\frac{Z_G(q, K)}{q v^s} = T_G(x, y),$$

pour $q = (x-1)(y-1)$ et $v \equiv \exp(K) - 1 = y - 1$.

La relation (23) entre la fonction de partition du modèle de Potts et le polynôme de Tutte explique l'intérêt suscité par ce polynôme chez certains physiciens [Baxt 01, Soka 05]. Résoudre le modèle de Potts sur réseau aléatoire revient, dans une perspective combinatoire, à compter les cartes pondérées par leur polynôme de Tutte.

0.4 Comptage des cartes

Nous présentons maintenant les principales méthodes utilisées pour le comptage des cartes planaires. Nous avons déjà évoqué le comptage par décomposition récursive *à la Tutte* [Tutt 62b, Tutt 62a, Tutt 62c, Tutt 63]. Nous présentons également trois autres méthodes *par substitution* [Mull 68, Tutt 62d, Tutt 63], par *intégrales de matrices* [Bréz 78, Di F 04, Bout 02, Zvon 97] et par *conjugaisons d'arbres* [Scha 98, Scha 97, Poul 03a, Bous 03b].

0.4.1 Approche récursive

L'approche récursive pour l'énumération des cartes a été initiée par Tutte au début des années 60 dans sa célèbre série d'articles *census* [Tutt 62b, Tutt 62a, Tutt 62c, Tutt 63]. Tutte espérait que l'étude des cartes, qui au contraire des graphes planaires contiennent une description explicite de leur planarité, le mènerait à une preuve du théorème des quatre couleurs. Si cet espoir a été déçu, la méthode mise au point par Tutte et ses disciples permit l'énumération de nombreuses familles de cartes. C'est cette méthode que nous utiliserons au chapitre 1 pour énumérer trois familles de triangulations dont les sommets sont de degré supérieur à 3, 4 et 5 respectivement. La méthode récursive pour l'énumération d'une famille de cartes enracinées consiste, tout simplement, à exprimer ce que l'on obtient en supprimant l'arête racine d'une carte.

Considérons, par exemple, la classe **C** des cartes planaires enracinées. La fonction taille, notée $|.|$, est le nombre d'arêtes. La description récursive de la classe **C** est représentée en figure 20. On distingue deux cas suivant que l'arête racine est un isthme ou non. Si l'arête racine est un isthme, la carte se décompose en un couple de cartes enracinées (figure 21). Sinon, la carte obtenue en supprimant l'arête est une carte dont un coin de la face *externe* (à droite de la racine) est distingué (figure 22).

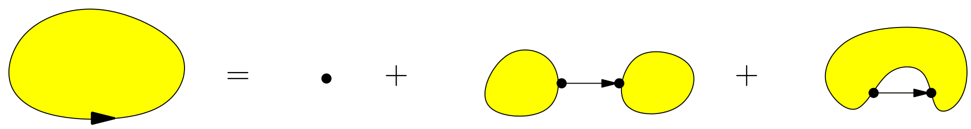


Figure 20: Description récursive des cartes planaires enracinées par suppression de la racine.

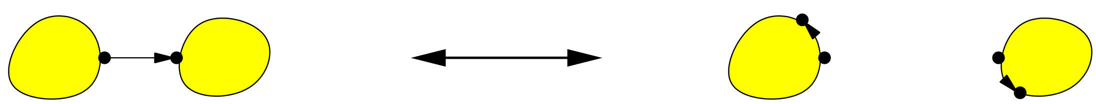


Figure 21: Cas 2: l'arête racine est un isthme.

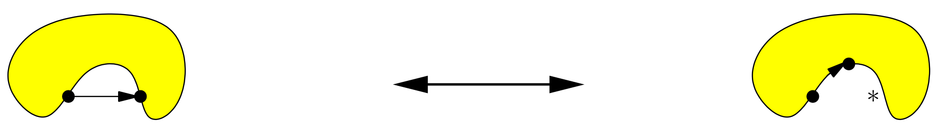


Figure 22: Cas 2: l'arête racine n'est pas un isthme.

Pour traduire la description des cartes *par suppression de la racine* il est nécessaire de prendre en compte le *degré de la face externe* (i.e. le nombre de coins). On considère donc la série génératrice bivariée

$$G(x,z) = \sum_{C \in \mathbf{C}} x^{f(C)} z^{|C|},$$

où $f(C)$ est le degré de la face externe de la carte C. La description récursive des cartes planaires *par suppression de la racine* se traduit par l'équation fonctionnelle

$$G(x,z) = 1 + x^2 z G(x,z)^2 + xz \left(\frac{xG(x,z) - G(1,z)}{x-1} \right).$$

Nous avons vu comment résoudre cette équation en sous-section 0.2.3 afin d'obtenir l'équation algébrique (17) pour la série $G_1 \equiv G(1,z)$.

Au lieu de supprimer la racine, on peut essayer de la contracter. On obtient alors une autre description récursive des cartes qui est représentée en figure 23. On distingue deux cas suivant que l'arête racine est une boucle ou non. Si l'arête racine est une boucle, la carte se décompose en un couple de cartes enracinées. Sinon, la carte obtenue en contractant la racine est une carte dont un coin du *sommet racine* (l'origine de l'arête racine) est distingué. Pour traduire la description *par contraction de la racine* il est nécessaire de prendre en compte le degré du sommet racine. En considérant la série génératrice bivariée

$$H(y,z) = \sum_{C \in \mathbf{C}} y^{s(C)} z^{|C|},$$

où $s(C)$ est le degré du sommet racine de la carte C, on obtient

$$H(y,z) = 1 + y^2 z H(y,z)^2 + yz \left(\frac{yH(y,z) - H(1,z)}{y-1} \right).$$

Cette équation est identique à la précédente (au renommage près des variables et des séries). Ceci s'explique par le fait que la famille $\mathbf{C}$ des cartes est stable par dualité et que la suppression de l'arête racine d'une carte revient à la contraction de l'arête racine de sa duale.

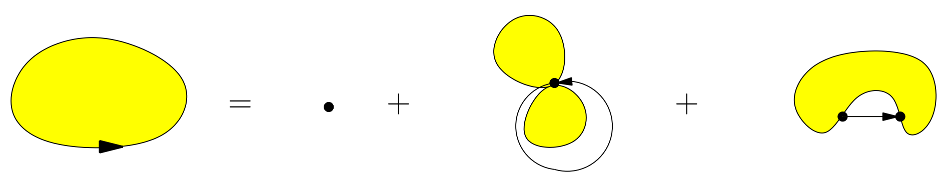


Figure 23: Description récursive des cartes planaires enracinées par contraction de la racine.

Rappelons que le polynôme chromatique et le polynôme de Tutte admettent une définition récursive par suppression et contraction d'arêtes (voir (18) et (21)). Par conséquent, savoir décrire une famille de cartes à la fois par suppression et par contraction de la racine permet d'envisager l'obtention d'équations pour les cartes pondérées par leur polynôme chromatique ou par leur polynôme de Tutte. Pour la classe $\mathbf{C}$ des cartes planaires enracinées, l'approche récursive permet de montrer [Tutt 71] que les séries génératrices

$$Q(x,y) \equiv Q(x,y,z,\lambda) = \sum_{C \in \mathbf{C}} x^{f(C)} y^{s(C)} z^{|C|} \frac{P_C(\lambda)}{\lambda},$$

et

$$F(x,y) \equiv F(x,y,z,\mu,\nu) = \sum_{C \in \mathbf{C}} x^{f(C)} y^{s(C)} z^{|C|} T_C(\mu,\nu),$$

vérifient respectivement les équations fonctionnelles

$$\begin{aligned} Q(x,y) \quad = \quad & 1 + yz(x^2(\lambda-1)+x)Q(x,y)Q(x,1) + xyz\lambda \left(\frac{xQ(x,y) - Q(1,y)}{x-1} \right) \\ & -xyzQ(x,y)Q(1,y) - xyz \left(\frac{yQ(x,y) - Q(x,1)}{y-1} \right), \end{aligned} \qquad (24)$$

et

$$F(x,y) = 1 + xyz(x\mu - 1)F(x,y)F(x,1) + xyz\left(\frac{xF(x,y) - F(1,y)}{x-1}\right)$$
$$+xyz(y\nu - 1)F(x,y)F(1,y) + xyz\left(\frac{yF(x,y) - F(x,1)}{y-1}\right). \quad (25)$$

La description récursive des cartes pondérées par leur polynôme chromatique ou par leur polynôme de Tutte nous a contraints à utiliser non pas *une* mais *deux* variables catalytiques. Au contraire des équations à une variable catalytique que l'on sait résoudre de manière systématique [Bous 05b], les équations à deux variables catalytiques s'avèrent très coriaces. À ce jour, il existe quelques équations *linéaires* (en la série trivariée) qui ont été résolues par Bousquet-Mélou [Bous 02, Bous 03a] et une unique équation non-linéaire qui a été résolue par Tutte. La résolution due à Tutte concerne le comptage des triangulations pondérées par leur polynôme chromatique. Ce tour de force lui a tout de même demandé près de dix articles étalés sur autant d'années [Tutt 73a, Tutt 73b, Tutt 73c, Tutt 73d, Tutt 74, Tutt 78, Tutt 82a, Tutt 82b, Tutt 95]. Au chapitre des perspectives de cette thèse, nous donnerons un aperçu de la méthode de résolution de Tutte. Il est tentant d'essayer d'appliquer cette méthode pour compter les cartes planaires pondérées par leur polynôme chromatique, voire leur polynôme de Tutte. Du point de vue de la physique statistique, cette tâche revient à la résolution du modèle de Potts sur réseau aléatoire.

0.4.2 Approche par substitution

Le comptage de cartes peut aussi être réalisé par des techniques de *substitution* [Mull 68, Tutt 62d, Tutt 63]. Plus précisément, l'approche par substitution permet, dans certains cas, de transférer des résultats énumératifs d'une famille de cartes à une autre. Supposons, par exemple, que l'on cherche à énumérer les cartes planaires enracinées dont les sommets sont de degré supérieur ou égal à 2. On peut alors utiliser une approche par substitution pour se ramener au cas des cartes générales.

Par commodité, nous allons relaxer quelque peu notre contrainte sur le degré des sommets et étudier la classe $\mathbf{B}$ des cartes dont les sommets *non-incidents à l'arête racine* sont de degré supérieur ou égal à 2. En prenant une carte quelconque et en supprimant récursivement tous les sommets de degré 1 (qui ne sont pas incidents à l'arête racine) on obtient une carte dans la classe $\mathbf{B}$. Cette opération est représentée en figure 24. En toute généralité, une carte planaire C se décompose en une carte B de la classe $\mathbf{B}$ appelé *noyau* et en une suite d'arbres enracinés (éventuellement vides) qui viennent se greffer sur les coins de la carte B. La classe $\mathbf{C}$ des cartes est donc en bijection avec les couples formés d'une carte $B \in \mathbf{B}$ (leur noyau) et d'une suite de $2|B|$ arbres enracinés (la carte B a $2|B|$ coins). Cette bijection se traduit par l'équation

$$G_1(z) = 1 + B(zA(z)^2). \tag{26}$$

liant la série génératrice $A(z)$ de la classe **A** des arbres enracinés et les séries génératrices $B(z)$ et $G_1(z)$ des classes **B** et **C** :

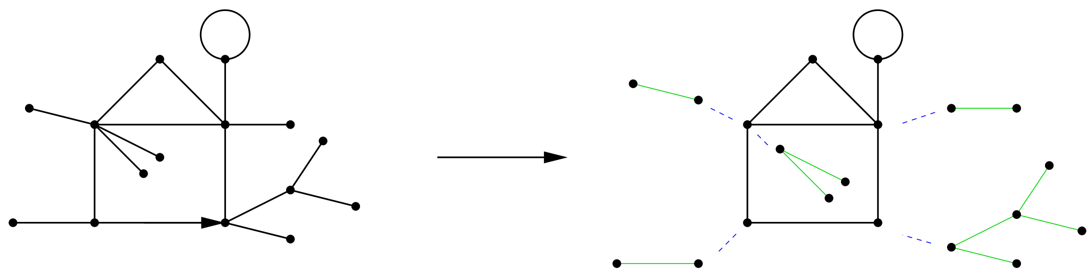


Figure 24: Élagage des sommets de degré 1.

Nous connaissons déjà une équation pour la série génératrice $G_1(z)$ des cartes générales (17) . En y substituant l'équation (26) on obtient

$$27z^2B(t)^2 + \left(1 - 18z - 54z^2\right) B(t) - 2 + 34z + 27z^2 = 0 \tag{27}$$

où $t \equiv t(z) = zA(z)^2$. On sait aussi que la classe **A** des arbres enracinés est comptée par la suite de Catalan. Elle vérifie donc l'équation $A(z) = 1 + zA(z)^2$. Par élimination (résultant), on montre que la variable z et la série $t \equiv t(z) = zA(z)^2$ sont liés par l'équation

$$z - t + 2zt + zt^2 = 0. \tag{28}$$

Par élimination (résultant) on montre aussi que $B(t)$ et t sont liés par l'équation

$$27B(t)^2t^2 + \left(t^4 - 14t^3 - 84t^2 - 14t + 1\right) B(t) - 2t^4 + 26t^3 + 83t^2 + 26t - 2 = 0. \tag{29}$$

Il ne reste qu'à réaliser que la série $t \equiv t(z)$ peut être considérée comme une variable muette dans l'équation (29). En effet, l'équation (28) montre qu'il existe une série $z = z(u)$ telle que $t(z(u)) = u$. L'équation (29) est donc une équation algébrique dont on peut vérifier qu'elle définit bien la série $B(t)$ de manière unique comme série formelle en t. On pourra aussi effectuer l'énumération asymptotique de la classe **B** par les méthodes décrites en sous-section 0.2.4.

0.4.3 Approche par intégrales de matrices

Dans les années 70, un groupe de physiciens développèrent une méthode radicalement nouvelle pour l'énumération des cartes planaires [Bréz 78]. Cette méthode, extrêmement efficace (mais pas toujours rigoureuse), est basée sur des calculs d'*intégrales de matrices* qui s'inspirent de la gravité quantique. L'article [Zvon 97] constitue une très agréable introduction au lien entre les cartes et les intégrales de matrices.

Par *intégrale de matrices* nous entendons une intégrale sur l'espace des matrices hermitiennes. Une matrice hermitienne $M = (m_{k,l})_{1 \leq k,l \leq n}$ de taille n est spécifiée par les n^2 coefficients réels $x_{i,j} = \mathrm{Re}(m_{i,j}), 1 \leq i \leq j \leq n$ et $y_{i,j} = \mathrm{Im}(m_{i,j}), 1 \leq i < j \leq n$. L'espace des matrices hermitiennes est identifié à l'espace vectoriel $\mathbb{R}^{n^2}$ et est muni de la mesure gaussienne

$$d\nu(M) = (2\pi)^{-n^2/2} \exp(\mathrm{tr}(M^2)/2) dM \qquad \text{où} \qquad dM = \prod_{1 \leq i \leq j \leq n} dx_{i,j} \prod_{1 \leq i < j \leq n} dy_{i,j}.$$

On sait que l'intégrale des polynômes dans un espace à mesure gaussienne peut se décomposer en une somme sur les *couplages de Wick*. Avec la mesure $\nu(M)$, les couplages de Wick ont une contribution égale à 0 ou 1. De plus, l'interprétation des couplages de Wick par les *diagrammes de Feynman* montre que les couplages ayant une contribution 1 peuvent s'identifier au *pseudo-cartes* (des cartes combinatoires qui ne sont pas forcément connexes). À un polynôme correspond donc un ensemble de pseudo-cartes. Le comptage d'une famille de cartes se ramène au calcul de l'intégrale d'un polynôme, ou plutôt d'une série, judicieusement choisie. Par exemple, pour énumérer les cartes planaires tétravalentes l'intégrale à calculer est $\int \exp(\mathrm{tr}(zM^4)) d\nu(M)$. Plus exactement, cette intégrale compte les pseudo-cartes tétravalentes étiquetées de genre quelconque. La série génératrice $T(z)$ des cartes planaires tétravalentes est en fait donnée par

$$T(z) = zI'(z) \qquad \text{où} \qquad I(z) = \lim_{n \to \infty} \frac{1}{n^2} \log \int \exp(\mathrm{tr}(zM^4/n)) d\nu(M).$$

(Le logarithme permet de passer des pseudo-cartes aux cartes et la limite permet de se débarrasser des cartes de genre supérieur). Il ne reste qu'à calculer l'intégrale. Ici commencent les pires désagréments pour le mathématicien consciencieux car les intégrales en question divergent... Mais les physiciens d'expérience savent comment s'en départir et arrivent à énumérer de nombreuses familles de cartes. Nous renvoyons le lecteur à [Di F 04, Eyna 01] pour les calculs d'intégrales de matrices.

0.4.4 Approche bijective par conjugaison d'arbres

La dernière (mais non la moindre) des approches que nous présentons est une approche bijective récente basée sur les *conjugaisons d'arbres*. Les premières approches bijectives pour le comptage des cartes sont dues à Cori et Vauquelin [Cori 81] et à Arques [Arqu 86]. Mais ce n'est que récemment, notamment avec les travaux de Schaeffer, que l'approche bijective s'est généralisée au point de pouvoir prétendre au titre de *méthode* [Scha 98, Scha 97, Poul 03a, Bous 03b]. La première étape pour l'énumération d'une famille

F de cartes par *conjugaison d'arbres* consiste à définir un arbre couvrant canonique pour chaque carte de cette famille. On considère ensuite la classe des arbres *bourgeonnants* non-enracinés obtenus en prenant une carte munie de son arbre couvrant canonique et en coupant chaque arête externe en deux demi-arêtes ou *bourgeons*. Si la magie s'opère, la famille **F** est en *bijection* avec la classe des arbres bourgeonnants non-enracinés ainsi obtenus (ou, plus généralement, dans une relation k à p).

À titre d'exemple, nous allons utiliser l'approche par conjugaison d'arbres afin de montrer que le nombre de cartes tétravalentes enracinées à n sommets est

$$t_n = \frac{2}{n+2}\frac{3^n}{n+1}\binom{2n}{n}. \tag{30}$$

Cet exemple (et les dessins qui l'accompagnent) est tiré de [Scha 98]. On considère la classe des arbres binaires. Les feuilles sont considérées comme des demi-arêtes et on enracine l'arbre sur une des feuilles. Nous avons mentionné en sous-section 0.2.1 que les arbres binaires enracinés à n noeuds sont comptés par le $n^{\text{ème}}$ nombre de Catalan $C_n = \frac{1}{n+1}\binom{2n}{n}$. Un arbre binaire enraciné à 8 noeuds est représenté en figure 25 (gauche). On enrichit les arbres binaires en greffant une demi-arête *bourgeon* à chaque noeud. Il y a trois coins possibles pour chaque bourgeon et donc $a_n = 3^n C_n = \frac{3^n}{n+1}\binom{2n}{n}$ arbres binaires bourgeonnants enracinés à n noeuds. Un tel arbre est représenté en figure 25 (milieu).

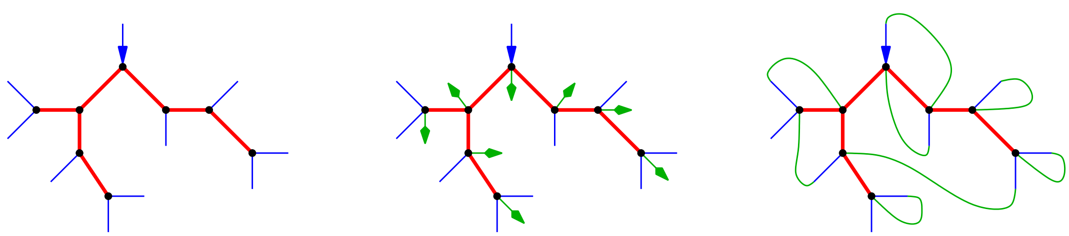


Figure 25: Gauche: un arbre binaire enraciné. Milieu: un arbre binaire bourgeonnant enraciné. Droite: l'appariement des bourgeons et des feuilles.

On *clôt* un arbre bourgeonnant en appariant chaque bourgeon à une feuille. Plus précisément, on considère l'ordre cyclique des bourgeons et des feuilles obtenu en faisant le tour de l'arbre dans le sens anti-horaire (la racine ne joue pas de rôle particulier). Il y a n bourgeons pour $n+2$ feuilles. Si un bourgeon est suivi par une feuille on apparie ce bourgeon et cette feuille puis on recommence jusqu'à épuisement des feuilles. Après clôture il reste 2 feuilles. L'arbre bourgeonnant est dit *équilibré* si la racine est l'une de ces deux feuilles. L'arbre représenté en figure 25 n'est pas équilibré puisque la racine a été appariée à une feuille par l'opération de clôture. Le principe de *conjugaison* (consistant à considérer les arbres non-enracinés) montre qu'il existe $\frac{2}{n+2}a_n = \frac{2}{n+2}\frac{3^n}{n+1}\binom{2n}{n}$ arbres équilibrés à n sommets. Lorsque l'arbre est équilibré, on joint la feuille racine et l'autre feuille non-appariée pour créer l'arête racine. On peut montrer [Scha 98] que l'opération de clôture établit une bijection entre les arbres équilibrés et les cartes tétravalentes ce qui prouve la formule (30).

Le comptage bijectif de cartes dans cette thèse. L'approche bijective par *conjugaison d'arbres* permet d'énumérer la famille des triangulations. Nous établirons au chapitre 2 une bijection alternative qui permet l'énumération des triangulations. Notre bijection s'écarte du schéma classique par conjugaison d'arbres. Une des différences notables tient au fait que nous associons non pas *un* arbre couvrant canonique à chaque triangulation mais toute une famille d'arbres. Plus précisément, nous associons 2^n arbres couvrants à chaque triangulation de taille n. D'autre part, le schéma de clôture des arbres que nous utilisons est assez différent de celui présenté ci-dessus.

Au chapitre 3, nous présentons une bijection permettant le comptage des *cartes boisées*, c'est-à-dire des cartes dont un arbre couvrant est distingué. Ce résultat a des liens assez étroits avec les bijections par conjugaison d'arbres. Notre bijection fait correspondre à chaque carte boisée un couple formé d'un arbre et d'une partition non-croisée. Intuitivement, la partition non croisée constitue un *mode d'emploi* pour replier l'arbre en une carte. L'une des étapes de notre bijection consiste à associer une orientation de la carte à chaque arbre couvrant. Cette idée, qui apparaissait déjà dans [Fusy 03], semble prometteuse. De fait, les orientations apparaissent de plus en plus régulièrement comme outils fondamentaux pour la caractérisation et le comptage bijectif de familles de cartes [Fray 01, Fusy 05a, Fusy 05b, Scha 97].

Part I

Recursive counting of maps

Chapter 1

Triangulations with high vertex degree

Abstract: We solve three enumerative problems concerning families of planar maps. More precisely, we establish algebraic equations for the generating function of loopless triangulations in which all vertices have degree at least d, for a certain value d chosen in $\{3, 4, 5\}$.
The originality of the problem lies in the fact that degree restrictions are placed both on vertices and faces. Our proofs first follow Tutte's classical approach: we decompose maps by deleting the root-edge and translate the decomposition into an equation satisfied by the generating function of the maps under consideration. Then we proceed to solve the equation obtained using a recent technique that extends the so-called *quadratic method*.

Résumé: Nous énumérons trois familles de cartes planaires. Plus précisément, nous démontrons des résultats d'algébricité pour les familles de triangulations sans boucle dont le degré des sommets est au moins égal à une certaine valeur d choisie parmi $\{3, 4, 5\}$.
L'originalité de nos résultats tient au fait que les restrictions de degrés portent simultanément sur les faces et les sommets. Nous adoptons, dans un premier temps, la démarche classique de Tutte : nous décomposons nos cartes par suppression de la racine et traduisons cette décomposition en une équation portant sur la série génératrice correspondante. Nous résolvons ensuite l'équation obtenue en utilisant des techniques récentes qui généralisent la *méthode quadratique*.

1.1 Introduction

The enumeration of planar maps (or *maps* for short) has received a lot of attention in the combinatorists community for nearly fifty years. This field of research, launched by Tutte, was originally motivated by the four-color conjecture. Tutte and his students considered a large number of map families corresponding to various constraints on face- or vertex-degrees. These seminal works, based on elementary decomposition techniques allied to a generating function approach, gave rise to many explicit results [Tutt 62b, Tutt 62a, Tutt 62c, Tutt 63, Mull 65]. Fifteen years later, some physicists became interested in the subject and developed their own tools [Bess 80, Bréz 78, Hoof 74] based on matrix integrals (see [Zvon 97] for an introduction). Their techniques proved very powerful for map enumeration [Bout 02, Di F 04]. More recently, a bijective approach based on *conjugacy classes of trees* has emerged providing new insights on the subject [Scha 98, Bous 00a, Bous 03b, Poul 03a].

However, when one considers a map family defined by both face- and vertex-constraints, each of the above mentioned methods seems relatively ineffective and very few enumerative results are known. There are two major exceptions. First, and most importantly for this chapter, the enumeration of loopless *triangulations* (faces have degree 3) in which all vertices have degree at least 3 and of 3-connected triangulations in which all vertices have degree at least 4 were performed by Gao and Wormald using a compositional approach [Gao 02]. More recently, the enumeration of all *bipartite* maps (faces have an even degree) according to the degree distribution of the vertices was accomplished using conjugacy classes of trees [Scha 97, Bous 03b]. This result includes as a special case the enumeration of bipartite *cubic* maps (vertices have degree 3) performed by Tutte via a generating function approach [Tutt 62c, Tutt 73c].

In this chapter, we consider loopless triangulations in which all vertices have degree at least d, for a certain value d chosen in $\{3, 4, 5\}$. We establish algebraic equations for the generating function of each of these families. We also give the asymptotic behavior of the number of maps in each family. It is well-known that there is no triangulation in which all vertices have degree at least 6 (we shall prove this fact in Section 1.2). Hence, we have settled the problem of counting triangulations with 'high' vertex degree entirely.

As mentioned above, the loopless triangulations in which all vertices have degree at least 3 have already been enumerated by Gao and Wormald [Gao 02]. Our proof differs from theirs. Let us also mention that several families of triangulations defined by connectivity constraints have been enumerated, for instance: the general triangulations [Gao 91b], the loopless triangulations (i.e. non-separable triangulations) [Mull 65, Poul 03a], the 3-connected triangulations (i.e. triangulations without multiple edges) [Tutt 62b, Poul 03b], the 4-connected [Brow 64] and the 5-connected triangulations [Gao 01]. Observe that the vertices of k-connected triangulations have degree at least k (except for the degenerated case of the triangle K_3). However, there is no equivalence between connectivity constraints and vertex-degree constraints. In the present chapter, we shall focus on loopless triangulations but our approach can also be adapted to some other families of triangulations, in particular to general triangulations as well as 3-connected ones.

Our proofs first follow Tutte's classical approach, which consists in translating the decomposition obtained by deletion of the root-edge into a functional equation satisfied by the generating function. It is not clear at first sight why this approach should work here. As a matter of fact, finding a functional equation for triangulations with vertex degree at least 5 turns out to be rather complicated. But it eventually works if some of the constraints are relaxed at this stage of the solution. Our decomposition scheme requires to consider the set of near-triangulations and to take into account, beside the size of the map, the degree of its root-face. Consequently, in order to write a functional equation, we need to consider a *bivariate* generating function. We end up with an equation for the (bivariate) generating function in which the variable counting the degree of the root-face cannot be trivially eliminated. We then use a recent generalization of the *quadratic method* to get rid of the extra variable and compute an algebraic equation characterizing the *univariate* generating function (see [Brow 65] and [Goul 83, Section 2.9] for the quadratic method and [Bous 05b] for its generalization).

This chapter is organized as follows. In Section 1.2, we recall some definitions on planar maps and introduce the main notations. In Section 1.3, we recall the classic decomposition scheme due to W.T. Tutte (by deletion of the root-edge). We illustrate this scheme on the set of unconstrained non-separable near-triangulations. In Section 1.4, we apply the same decomposition scheme to the sets of near-triangulations in which any internal vertex has degree at least 3, 4, 5. We obtain functional equation in which the variable x counting the degree of the root-face cannot be trivially eliminated. In Section 1.5, we use techniques generalizing the quadratic method in order to get rid of the variable x. We obtain algebraic equations for triangulations in which *any vertex not incident to the root-edge* has degree at least 3, 4, 5. In Section 1.6, we give algebraic equations for triangulations in which *any vertex* has degree at least 3, 4. Lastly, in Section 1.7 we study the asymptotic behavior of the number of maps in each family.

1.2 Preliminaries and notations on maps

We begin with some vocabulary on maps. A map is a proper embedding of a connected graph into the two-dimensional sphere, considered up to continuous deformations. A map is *rooted* if one of its edges is distinguished as the *root-edge* and attributed an orientation. Unless otherwise specified, all maps under consideration in this chapter are rooted. The face at the right of the root-edge is called the *root-face* and the other faces are said to be *internal.* Similarly, the vertices incident to the root-face are said to be *external* and the others are said to be *internal.* Graphically, the root-face is usually represented as the infinite face when the map is projected on the plane (see Figure 26). The endpoints of the root-edge are distinguished as its *origin* and *end* according to the orientation of the root-edge. A map is a *triangulation* (resp. *near-triangulation*) if all its faces (resp. all its internal faces) have degree 3. For instance, the map of Figure 26 is a near-triangulation with root-face of degree 4. Lastly, a map is *non-separable* if it is loopless and 2-connected (the deletion of a vertex does not disconnect the map). For instance, the map in Figure 26 is non-separable. Observe that for a triangulation it is equivalent to be loopless or non-separable but this is not true for near-triangulations.

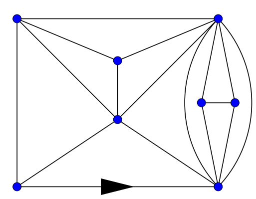


Figure 26: A non-separable near-triangulation.

In what follows, we enumerate 3 families of rooted non-separable triangulations. We recall some basic facts about these maps.

• By definition, a non-separable triangulation has no loop. Therefore, the faces of non-separable triangulations are always homeomorphic to a triangle: they have three distinct vertices and three distinct edges.

• Consider a triangulation with f faces, e edges and v vertices. Given the incidence relation between edges and faces, we have $2e = 3f$. Hence, the number of edges of a triangulation is a multiple of 3. Moreover, given the Euler relation ($v - e + f = 2$), we see that a triangulation with $3n$ edges has $2n$ faces and $n + 2$ vertices.

• Observe that a non-separable map (not reduced to an edge) cannot have a vertex of degree one. Let us now prove, as promised, that *any triangulation has a vertex of degree less than 6.* Moreover, we prove that *this vertex can be chosen not to be incident to the root-edge.* Indeed, if all vertices not incident to the root-edge have degree at least 6 the incidence relation between vertices and edges gives $2e \geq 6(v-2)+4$. This contradicts the fact that triangulations with $e = 3n$ edges have $v = n+2$ vertices. This property shows that, if one considers the sets of non-separable triangulations with vertex degree at least d, *the only interesting values of* d *are* $d = 2$ (which corresponds to unconstrained non-separable triangulations) *and* $d = 3,\ 4,\ 5$.

Let $\mathbf{S}$ be the set of non-separable rooted *near-triangulations*. By convention, we exclude the map reduced to a vertex from $\mathbf{S}$. Thus, the smallest map in $\mathbf{S}$ is the map reduced to a straight edge (see Figure 27). This map is called the *link-map* and is denoted L. The vertices of other maps in $\mathbf{S}$ have degree at least 2. We consider three sub-families $\mathbf{T},\ \mathbf{U},\ \mathbf{V}$ of $\mathbf{S}$. The set $\mathbf{T}$ (resp. $\mathbf{U}$, $\mathbf{V}$) is the subset of non-separable near-triangulations in which any *internal* vertex has degree at least 3 (resp. 4, 5). For each of the families $\mathbf{W} = \mathbf{S},\ \mathbf{T},\ \mathbf{U},\ \mathbf{V}$, we consider the bivariate generating function $\mathbb{W}(x,z)$, where z counts the size (the number of edges) and x the degree of the root-face minus 2. That is to say, $\mathbb{W}(x) \equiv \mathbb{W}(x,z) = \sum_{n,d} a_{n,d} x^d z^n$ where $a_{n,d}$ is the number of maps in $\mathbf{W}$ with size n and root-face of degree $d+2$. For instance, the link-map L, which is the smallest map in all our families, has contribution z to the generating function. Therefore, $\mathbb{W}(x) = z + o(z)$. Since the degree of the root-face is bounded by two times the number of edges, the generating function $\mathbb{W}(x,z)$ is a power series in the main variable z with polynomial coefficients in the secondary variable x. For each family $\mathbf{W} = \mathbf{S},\ \mathbf{T},\ \mathbf{U},\ \mathbf{V}$, we will characterize the generating function $\mathbb{W}(x)$ as the unique power series solution of a functional equation (see Equation (32) and Propositions 1.1, 1.2, 1.3).

Figure 27: The link-map L.

We also consider the set $\mathbf{F}$ of non-separable rooted *triangulations* and three of its subsets $\mathbf{G}$,$\mathbf{H}$,$\mathbf{K}$. The set $\mathbf{G}$ (resp. $\mathbf{H}$, $\mathbf{K}$) is the subset of non-separable triangulations in which any vertex *not incident to the root-edge* has degree at least 3 (resp. 4, 5). As observed above, the number of edges of a triangulation is always a multiple of 3. To each of the families $\mathbf{L} = \mathbf{F},\ \mathbf{G},\ \mathbf{H},\ \mathbf{K}$, we associate the univariate generating function $\mathbb{L}(t) = \sum_n a_n t^n$ where a_n is the number of maps in $\mathbf{L}$ with $3n$ edges ($2n$ faces and $n+2$ vertices). For each family we will give an algebraic equation satisfied by $\mathbb{L}(t)$ (see Equation (34) and Theorems 1.4, 1.5, 1.6).

There is a simple connection between the generating functions $\mathbb{F}(t)$ (resp. $\mathbb{G}(t)$, $\mathbb{H}(t)$, $\mathbb{K}(t)$) and $\mathbb{S}(x)$ (resp. $\mathbb{T}(x)$, $\mathbb{U}(x)$, $\mathbb{V}(x)$). Consider a non-separable near-triangulation distinct from L rooted on a digon (i.e. the root-face has degree 2). Deleting the external edge that is not the root-edge produces a non-separable triangulation (see Figure 28). This classical mapping (see e.g. [Gao 91a, Tutt 95]) establishes a one-to-one correspondence between the set of triangulations $\mathbf{F}$ (resp. $\mathbf{G}$, $\mathbf{H}$, $\mathbf{K}$) and the set of near-triangulations in $\mathbf{S} - \{L\}$ (resp. $\mathbf{T} - \{L\}$, $\mathbf{U} - \{L\}$, $\mathbf{V} - \{L\}$) rooted on a digon.

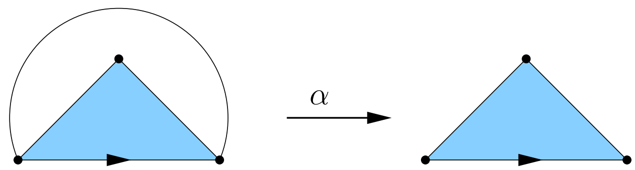


Figure 28: Near-triangulations rooted on a digon and triangulations.

For $\mathbf{W} \in \{\mathbf{S},\ \mathbf{T},\ \mathbf{U},\ \mathbf{V}\}$, the power series $\mathbb{W}(0) \equiv \mathbb{W}(0, z)$ is the generating function of near-triangulations in $\mathbf{W}$ rooted on a digon. Given that the link-map has contribution z, we have

$$\mathbb{S}(0) = z + z\mathbb{F}(z^3),\ \ \mathbb{T}(0) = z + z\mathbb{G}(z^3),\ \ \mathbb{U}(0) = z + z\mathbb{H}(z^3),\ \ \mathbb{V}(0) = z + z\mathbb{K}(z^3). \qquad (31)$$

1.3 The decomposition scheme

In the following, we adopt Tutte's classical approach for enumerating maps. That is, we decompose maps by deleting their root-edge and translate this combinatorial decomposition into an equation satisfied by the corresponding generating function. In this section we illustrate this approach on unconstrained non-separable triangulations (this was first done in [Mull 65]). We give all the details on this simple case in order to prepare the reader to the more complicated cases of constrained non-separable triangulations treated in the next section.

We recall that $\mathbf{S}$ denotes the set of non-separable near-triangulations and $\mathbb{S}(x) = \mathbb{S}(x, z)$ the corresponding generation function. As observed before, the link-map L has contribution z to the generating function $\mathbb{S}(x)$. We decompose the other maps by deleting the root-edge. Let M be a non-separable triangulation distinct from L. Since M is non-separable, the root-edge of M is not an isthmus. Therefore, the face at the left of the root-edge is internal, hence has degree 3. Since M has no loop, the three vertices incident to this face are distinct. We denote by v the vertex not incident to the root-edge. When analyzing what can happen to M when deleting its root-edge, one is led to distinguish two cases (see Figure 29).

Either the vertex v is incident to the root-face, in which case the map obtained by deletion of the root-edge is separable (see Figure 30). Or v is not incident to the root-face and the map obtained by deletion of the root-edge is a non-separable near-triangulation (see Figure 31).

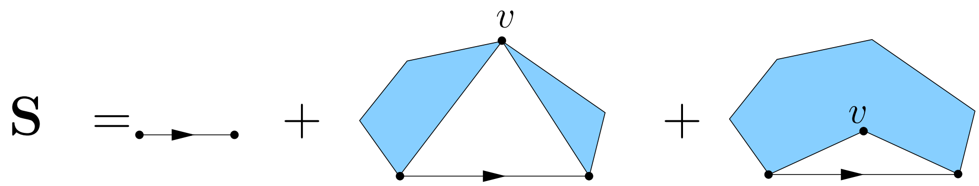


Figure 29: Decomposition of non-separable near-triangulations.

In the first case, the map obtained is in correspondence with an ordered pair of non-separable near-triangulations. This correspondence is bijective, that is, any ordered pair is the image of exactly one near-triangulation. In the second case the degree of the root-face is increased by one. Hence the root-face of the near-triangulation obtained has degree at least 3. Here again, any near-triangulation in which the root-face has degree at least 3 is the image of exactly one near-triangulation.

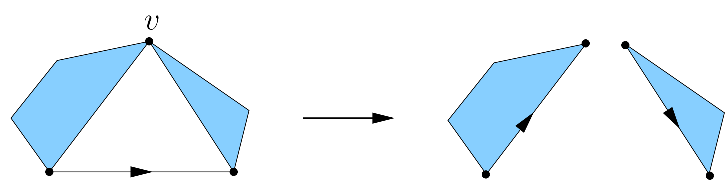


Figure 30: Case 1. The vertex v is incident to the root-face.

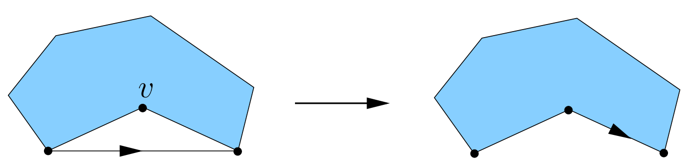


Figure 31: Case 2. The vertex v is not incident to the root-face.

We want to translate this analysis into a functional equation. Observe that the degree of the root-face appears in this analysis. This is why we are *forced* to introduce the variable x counting this parameter in our generating function $\mathbb{S}(x,z)$. For this reason, following Zeilberger's terminology [Zeil 00], the secondary variable x is said to be *catalytic*: we need it to write the functional equation, but we shall try to get rid of it later.

In our case, the decomposition easily translates into the following equation (details will be given in Section 1.4):

$$\mathbb{S}(x,z) = z + xz\mathbb{S}(x,z)^2 + \frac{z}{x}\left(\mathbb{S}(x,z) - \mathbb{S}(0,z)\right). \tag{32}$$

The first summand of the right-hand side accounts for the link map, the second summand corresponds to the case in which the vertex v is incident to the root-face, and the third summand corresponds to the case in which v is not incident to the root-face.

It is an easy exercise to check that this equation defines the series $\mathbb{S}(x,z)$ uniquely as a power series in z with polynomial coefficients in x. By techniques presented in Section 1.5, we can derive from Equation (32) a polynomial equation satisfied by the series $\mathbb{S}(0,z)$ where the extra variable x does not appear anymore. This equation reads

$$\mathbb{S}(0,z) \;=\; z - 27z^4 + 36z^3\mathbb{S}(0,z) - 8z^2\mathbb{S}(0,z)^2 - 16z^4\mathbb{S}(0,z)^3. \tag{33}$$

Given that $\mathbb{S}(0,z) = z + z\mathbb{F}(z^3)$, we deduce the algebraic equation

$$\mathbb{F}(t) = t(1-16t) - t(48t-20)\mathbb{F}(t) - 8t(6t+1)\mathbb{F}(t)^2 - 16t^2\mathbb{F}(t)^3, \tag{34}$$

characterizing $\mathbb{F}(t)$ (the generating function of non-separable triangulations) uniquely as a power series in t. From this equation one can derive the asymptotic behavior of the coefficients of $\mathbb{F}(t)$, that is, the number of non-separable triangulations of a given size (see Section 1.7).

1.4 Functional equations

In this section, we apply the decomposition scheme presented in Section 1.3 to the families $\mathbf{T}$, $\mathbf{U}$, $\mathbf{V}$ of non-separable near-triangulations in which all internal vertices have degree at least 3, 4, 5. We obtain functional equations satisfied by the corresponding generating functions $\mathbb{T}(x)$, $\mathbb{U}(x)$, $\mathbb{V}(x)$.

Note that, when one deletes the root-edge of a map, the degree of its endpoints is lowered by one. Given the decomposition scheme, this remark explains why we are led to consider the near-triangulations where only *internal* vertices have a degree constraint. However, we need to control the degree of the origin of the root-edge since it may come from an internal vertex (see Figure 31). This leads to the following notations. Let $\mathbf{W}$ be one of the sets $\mathbf{S}$, $\mathbf{T}$, $\mathbf{U}$, $\mathbf{V}$. We define $\mathbf{W}_k$ as the set of maps in $\mathbf{W}$ such that the root-face has degree at least 3 and the origin of the root-edge has degree k. We also define $\mathbf{W}_\infty$ as the set of (separable) maps obtained by gluing the root-edge's end of a map in $\mathbf{W}$ with the root-edge's origin of a map in $\mathbf{W}$. The root-edge of the map obtained is chosen to be the root-edge of the second map. Generic elements of the sets $\mathbf{W}_k$ and $\mathbf{W}_\infty$ are shown in Figure 32. We also write $\mathbf{W}_{\geq k} \triangleq \mathbf{W}_\infty \cup \bigcup_{j\geq k} \mathbf{W}_j$. The notation $\mathbf{W}_{\geq k}$, which at first sight might seem awkward, allows to unify the two possible cases of our decomposition scheme (Figure 30 and 31). It shall simplify our arguments and equations (see for instance Equations (35-38)) which will prove a valuable property.

Figure 32: Generic elements of the sets $\mathbf{W}_k$ and $\mathbf{W}_\infty$.

The symbols $\mathbb{W}_k(x,z)$, $\mathbb{W}_\infty(x,z)$ and $\mathbb{W}_{\geq k}(x,z)$ denote the bivariate generating functions of the sets $\mathbf{W}_k$, $\mathbf{W}_\infty$ and $\mathbf{W}_{\geq k}$ respectively. In these series, as in $\mathbb{W}(x,z)$, the contribution of a map with n edges and root-face degree $d+2$ is $x^d z^n$.

We are now ready to apply the decomposition scheme to the triangulations in $\mathbf{T}$, $\mathbf{U}$, $\mathbf{V}$. Consider a near-triangulation M distinct from L in $\mathbf{W} = \mathbf{S}$, $\mathbf{T}$, $\mathbf{U}$, $\mathbf{V}$. As observed before, the face at the left of the root-edge is an internal face incident to three distinct vertices. We denote by v the vertex not incident to the root-edge. If v is external, the deletion of the root-edge produces a map in $\mathbf{W}_\infty$ (see Figure 33). If v is internal and M is in $\mathbf{S}$ (resp. $\mathbf{T}$, $\mathbf{U}$, $\mathbf{V}$) then v has degree at least 2 (resp. 3, 4, 5) and the map obtained by deleting the root-edge is in $\bigcup_{k\geq 2}\mathbf{S}_k$ (resp. $\bigcup_{k\geq 3}\mathbf{T}_k$, $\bigcup_{k\geq 4}\mathbf{U}_k$, $\bigcup_{k\geq 5}\mathbf{V}_k$). Therefore, the deletion of the root-edge induces a mapping from $\mathbf{S}-\{L\}$ (resp. $\mathbf{T}-\{L\}$, $\mathbf{U}-\{L\}$, $\mathbf{V}-\{L\}$) to $\mathbf{S}_{\geq 2}$ (resp. $\mathbf{T}_{\geq 3}$, $\mathbf{U}_{\geq 4}$, $\mathbf{V}_{\geq 5}$).

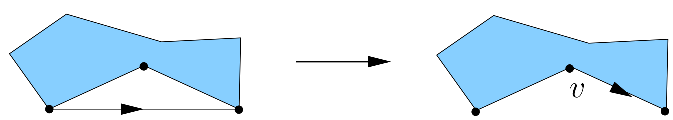


Figure 33: Mapping induced by deletion of the root-edge: the vertex v can be a separating point in which case the map is in $\mathbf{W}_\infty$.

This mapping is clearly bijective. Moreover, the map obtained after deleting the root-edge has size lowered by one and root-face degree increased by one. This analysis translates into the following equations:

$$\mathbb{S}(x) = z + \frac{z}{x}\mathbb{S}_{\geq 2}(x) , \tag{35}$$

$$\mathbb{T}(x) = z + \frac{z}{x}\mathbb{T}_{\geq 3}(x) , \tag{36}$$

$$\mathbb{U}(x) = z + \frac{z}{x}\mathbb{U}_{\geq 4}(x) , \tag{37}$$

$$\mathbb{V}(x) = z + \frac{z}{x}\mathbb{V}_{\geq 5}(x) . \tag{38}$$

In view of Equation (35), we will obtain a non-trivial equation for $\mathbb{S}(x)$ if we can express $\mathbb{S}_{\geq 2}(x)$ in terms of $\mathbb{S}(x)$. Similarly, we will obtain a non-trivial equation for $\mathbb{T}(x)$ if we can express $\mathbb{T}_{\geq 2}(x)$ and $\mathbb{T}_2(x)$ in terms of $\mathbb{T}(x)$. Similar statements hold for $\mathbb{U}(x)$ and $\mathbb{V}(x)$.

Thus, our first task will be to evaluate $\mathbb{W}_{\geq 2}(x)$ for $\mathbb{W}$ in $\{\mathbb{S},\ \mathbb{T},\ \mathbb{U},\ \mathbb{V}\}$.

By definition, $\mathbf{W}_\infty$ is in bijection with $\mathbf{W}^2$, which translates into the functional equation

$$\mathbb{W}_\infty(x) \ = \ x^2\mathbb{W}(x)^2 \ .$$

Observe that $\bigcup_{k\geq 2}\mathbf{W}_k$ is the set of maps in $\mathbf{W}$ for which the root-face has degree at least 3, that is, all maps except those rooted on a digon. Since $\mathbb{W}(0)$ is the generating function of maps in $\mathbf{W}$ rooted on a digon, we have

$$\sum_{k\geq 2}\mathbb{W}_k(x) \ = \ \mathbb{W}(x) - \mathbb{W}(0).$$

Given that $\mathbf{W}_{\geq 2} = \mathbf{W}_\infty \cup \bigcup_{k\geq 2}\mathbf{W}_k$, we obtain, for $\mathbb{W}$ in $\{\mathbb{S},\ \mathbb{T},\ \mathbb{U},\ \mathbb{V}\}$,

$$\mathbb{W}_{\geq 2}(x) \ = \ x^2\mathbb{W}(x)^2 \ + \ (\mathbb{W}(x) - \mathbb{W}(0)) \qquad \text{for } \mathbb{W} \text{ in } \{\mathbb{S},\ \mathbb{T},\ \mathbb{U},\ \mathbb{V}\} \ . \tag{39}$$

Equations (35) and (39) already prove Equation (32) announced in Section 1.3:

$$\mathbb{S}(x) = z + xz\mathbb{S}(x)^2 + z\left(\frac{\mathbb{S}(x) - \mathbb{S}(0)}{x}\right) \ .$$

In order to go further, we need to express $\mathbb{T}_2(x)$, $\mathbb{U}_2(x)$, $\mathbb{U}_3(x)$, $\mathbb{V}_2(x)$, $\mathbb{V}_3(x)$ and $\mathbb{V}_4(x)$ (see Equations (36-38)). We begin with an expression of $\mathbb{W}_2(x)$ for $\mathbb{W}$ in $\{\mathbb{S},\ \mathbb{T},\ \mathbb{U},\ \mathbb{V}\}$. Observe that for $\mathbf{W} = \{\mathbf{S},\ \mathbf{T},\ \mathbf{U},\ \mathbf{V}\}$, the set $\mathbf{W}_2$ is in bijection with $\mathbf{W}$ by the mapping illustrated in Figure 34. Consequently we can write

$$\mathbb{W}_2(x) \ = \ xz^2\mathbb{W}(x) \qquad \text{for } \mathbb{W} \text{ in } \{\mathbb{S},\ \mathbb{T},\ \mathbb{U},\ \mathbb{V}\} \ . \tag{40}$$

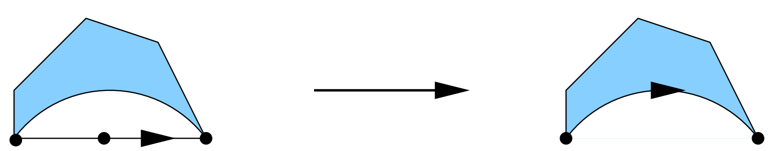


Figure 34: A bijection between $\mathbf{W}_2$ and $\mathbf{W}$.

This suffices to obtain an equation for the set $\mathbf{T}$:

$$\begin{aligned}
\mathbb{T}(x) &= z + \frac{z}{x}\mathbb{T}_{\geq 3}(x) && \text{by (36)}\\
&= z + \frac{z}{x}\left(\mathbb{T}_{\geq 2}(x) - \mathbb{T}_2(x)\right) \\
&= z + \frac{z}{x}\left(x^2\mathbb{T}(x)^2 + (\mathbb{T}(x) - \mathbb{T}(0)) - xz^2\mathbb{T}(x)\right) && \text{by (39) and (40).}
\end{aligned}$$

Proposition 1.1 *The generating function $\mathbb{T}(x)$ of non-separable near-triangulations in which all internal vertices have degree at least 3 satisfies:*

$$\mathbb{T}(x) \ = \ z \ + \ xz\mathbb{T}(x)^2 \ + \ z\left(\frac{\mathbb{T}(x) - \mathbb{T}(0)}{x}\right) \ - \ z^3\mathbb{T}(x) \ . \tag{41}$$

In order to find an equation concerning the sets **U** and **V**, we now need to express $\mathbb{U}_3(x)$ and $\mathbb{V}_3(x)$ in terms of $\mathbb{U}(x)$ and $\mathbb{V}(x)$ respectively. Let **W** be **U** or **V** and M be a map in $\mathbf{W}_3$. By definition, the root-face of M has degree at least 3 and its root-edge's origin u has degree 3. We denote by a and b the vertices preceding and following u on the root-face (see Figure 35). Since the map M is non-separable, the vertices a, b and u are distinct. Let v be the third vertex adjacent to u. Since M cannot have loops, the vertex v is distinct from a, b and u.

Suppose that M is in $\mathbf{U}_3$ (resp. $\mathbf{V}_3$) and consider the operation of deleting u and the three adjacent edges. If the vertex v is internal it has degree $d \geq 4$ (resp. $d \geq 5$) and the map obtained is in $\mathbf{U}_{d-1}$ (resp. $\mathbf{V}_{d-1}$). If it is external, the map obtained is in $\mathbf{U}_\infty$ (resp. $\mathbf{V}_\infty$). Thus, the map obtained is in $\mathbf{U}_{\geq 3}$ (resp. $\mathbf{V}_{\geq 4}$). This correspondence is clearly bijective. It gives

$$\mathbb{U}_3(x) \quad = \quad z^3\mathbb{U}_{\geq 3}(x) \quad = \quad z^3(\mathbb{U}_{\geq 2}(x) - \mathbb{U}_2(x)) \; , \tag{42}$$

$$\mathbb{V}_3(x) \quad = \quad z^3\mathbb{V}_{\geq 4}(x) \quad = \quad z^3(\mathbb{V}_{\geq 2}(x) - \mathbb{V}_2(x) - \mathbb{V}_3(x)) \; . \tag{43}$$

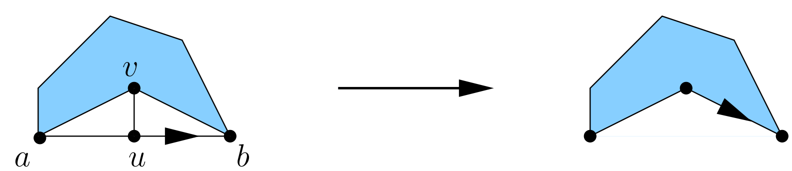


Figure 35: A bijection between $\mathbf{U}_3$ and $\mathbf{U}_{\geq 3}$ (resp. $\mathbf{V}_3$ and $\mathbf{V}_{\geq 4}$).

We are now ready to establish the functional equation concerning **U**:

$$\begin{aligned}
\mathbb{U}(x) &= z + \frac{z}{x}\mathbb{U}_{\geq 4}(x) && \text{by (37)} \\
&= z + \frac{z}{x}\left(\mathbb{U}_{\geq 2}(x) - \mathbb{U}_2(x) - \mathbb{U}_3(x)\right) \\
&= z + \frac{z(1-z^3)}{x}\left(\mathbb{U}_{\geq 2}(x) - \mathbb{U}_2(x)\right) && \text{by (42)} \\
&= z + \frac{z(1-z^3)}{x}\left(x^2\mathbb{U}(x)^2 + (\mathbb{U}(x) - \mathbb{U}(0)) - xz^2\mathbb{U}(x)\right) && \text{by (39) and (40).}
\end{aligned}$$

Proposition 1.2 *The generating function $\mathbb{U}(x)$ of non-separable near-triangulations in which all internal vertices have degree at least 4 satisfies:*

$$\mathbb{U}(x) = z + xz(1-z^3)\mathbb{U}(x)^2 + z(1-z^3)\left(\frac{\mathbb{U}(x) - \mathbb{U}(0)}{x}\right) - z^3(1-z^3)\mathbb{U}(x) \; . \tag{44}$$

We proceed to find an equation concerning the set **V**. This will require significantly more work than the previous cases. We write

$$\mathbb{V}(x) \quad = \quad z \; + \; \frac{z}{x}\mathbb{V}_{\geq 4}(x) \quad = \quad z \; + \; \frac{z}{x}\left(\mathbb{V}_{\geq 2}(x) - \mathbb{V}_2(x) - \mathbb{V}_3(x) - \mathbb{V}_4(x)\right) \tag{45}$$

and we want to express $\mathbb{V}_{\geq 2}(x)$, $\mathbb{V}_2(x)$, $\mathbb{V}_3(x)$ and $\mathbb{V}_4(x)$ in terms of $\mathbb{V}(x)$. We already have such expressions for $\mathbb{V}_{\geq 2}(x)$ and $\mathbb{V}_2(x)$ (by Equations (39) and (40)). Moreover, Equation (43) can be rewritten as

$$\mathbb{V}_3(x) \quad = \quad \frac{z^3}{1+z^3}\left(\mathbb{V}_{\geq 2}(x) - \mathbb{V}_2(x)\right) \ . \tag{46}$$

It remains to express $\mathbb{V}_4(x)$ in terms of $\mathbb{V}(x)$. Unfortunately, this requires some efforts and some extra notations. We define $\mathbf{V}_{k,l}$ as the set of maps in $\mathbf{V}$ such that the root-face has degree at least 4, the root-edge's origin has degree k and the root-edge's end has degree l (see Figure 36). The set $\mathbf{V}_{k,\infty}$ is the set of maps obtained by gluing the root-edge's end of a map in $\mathbf{V}_k$ with the root-edge's origin of a map in $\mathbf{V}$. The root-edge of the new map obtained is the root-edge of the map in $\mathbf{V}_k$. The set $\mathbf{V}_{\infty,k}$ is the set of maps obtained by gluing the root-edge's end of a map in $\mathbf{V}$ with the root-edge's origin of map in $\mathbf{V}$ for which the root-face has degree at least 3 and the root-edge's end has degree k. The root-edge of the new map obtained is the root-edge of the second map. The set $\mathbf{V}_{\infty,\infty}$ is obtained by gluing 3 maps of $\mathbf{V}$ as indicated in Figure 36.

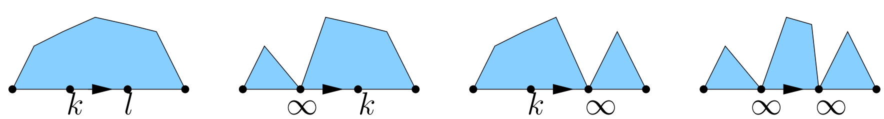


Figure 36: The sets $\mathbf{V}_{k,l}$, $\mathbf{V}_{\infty,k}$, $\mathbf{V}_{k,\infty}$ and $\mathbf{V}_{\infty,\infty}$.

We also write $\mathbf{V}_{k,\geq l} \triangleq \bigcup_{i\geq l} \mathbf{V}_{k,i} \cup \mathbf{V}_{k,\infty}$ and

$$\mathbf{V}_{\geq k,\geq l} \triangleq \bigcup_{i\geq k, j\geq l} \mathbf{V}_{i,j} \cup \bigcup_{i\geq k} \mathbf{V}_{i,\infty} \cup \bigcup_{j\geq l} \mathbf{V}_{\infty,j} \cup \mathbf{V}_{\infty,\infty}.$$

As before, if $\mathbf{W}$ is any of these sets, the symbol $\mathbb{W}$ denotes the corresponding generating function, where the contribution of a map of size n and root-face degree $d+2$ is $x^d z^n$.

Moreover, we consider the subset $\mathbf{D}$ of $\mathbf{V}$ composed of maps for which the root-face is a digon. The set of maps in $\mathbf{D}$ for which the root-vertex has degree k will be denoted by $\mathbf{D}_k$. We write $\mathbf{D}_{\geq k} = \bigcup_{j\geq k} \mathbf{D}_j$. Lastly, if $\mathbf{E}$ is one of the set $\mathbf{D}$, $\mathbf{D}_k$ or $\mathbf{D}_{\geq k}$, the symbol $\mathbb{E}$ denotes the corresponding (univariate) generating function, where the contribution of a map of size n is z^n. As observed before, $\mathbb{D} = \mathbb{V}(0)$.

We can now embark on the decomposition of $\mathbf{V}_4$. We consider a map M in $\mathbf{V}_4$ with root-vertex v. By definition, v has degree 4. Let $e_1,\ e_2,\ e_3,\ e_4$ be the edges incident to v in counterclockwise order starting from the root-edge e_1. We denote by $v_i,\ i = 1\ldots 4$ the endpoint of e_i distinct from v. Since M is non-separable and its root-face has degree at least 3, the vertices v_1 and v_4 are distinct. Moreover, since M has no loop we have $v_1 \neq v_2$, $v_2 \neq v_3$ and $v_3 \neq v_4$. Therefore, only three configurations are possible: either $v_1 = v_3$, the two other vertices being distinct, or symmetrically, $v_2 = v_4$, the other vertices being distinct, or v_1, v_2, v_3, v_4 are all distinct. The three cases are illustrated in Figure 37.

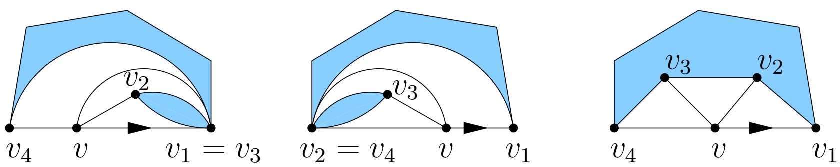


Figure 37: Three configurations for a map in $\mathbf{V}_4$.

In the case $v_1 = v_3$, the map can be decomposed into an ordered pair of maps in $\mathbf{V} \times \mathbf{D}_{\geq 4}$ (see Figure 38). This decomposition is clearly bijective. The symmetric case $v_2 = v_4$ admits a similar treatment. In the last case (v_1, v_2, v_3, v_4 all distinct) the map obtained from M by deleting e_1, e_2, e_3, e_4 is in $\mathbf{V}_{\geq 4,\geq 4}$ (see Figure 39). Note that this case contains several subcases depending on whether v_2 and v_3 are separating points or not. But again the correspondence is clearly bijective.

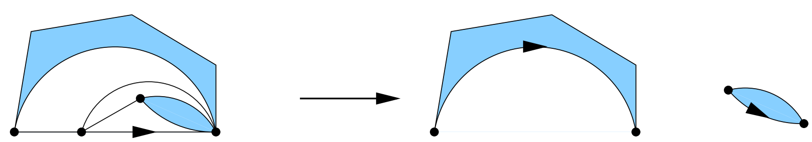


Figure 38: A bijection between maps of the first type in $\mathbf{V}_4$ and $\mathbf{V} \times \mathbf{D}_{\geq 4}$.

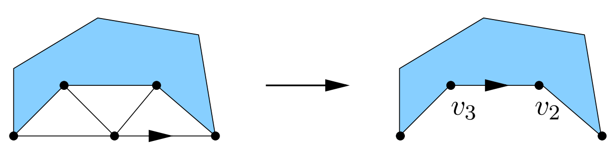


Figure 39: A bijection between maps of the third type in $\mathbf{V}_4$ and $\mathbf{V}_{\geq 4,\geq 4}$.

This correspondence gives

$$\mathbb{V}_4(x) \;=\; 2xz^4\mathbb{V}(x)\mathbb{D}_{\geq 4} \;+\; \frac{z^4}{x}\mathbb{V}_{\geq 4,\geq 4}(x)\ . \tag{47}$$

It remains to express the generating functions $\mathbb{D}_{\geq 4}$ and $\mathbb{V}_{\geq 4,\geq 4}(x)$ in terms of $\mathbb{V}(x)$. We start with $\mathbb{D}_{\geq 4}$.

We have $\mathbb{D}_{\geq 4} = \mathbb{D} - \mathbb{D}_1 - \mathbb{D}_2 - \mathbb{D}_3$. We know that $\mathbb{D} = \mathbb{V}(0)$. Moreover, the set $\mathbf{D}_1$ only contains the link-map and $\mathbf{D}_2$ is empty. Hence $\mathbb{D}_1 = z$ and $\mathbb{D}_2 = 0$. Lastly, the set $\mathbf{D}_3$ is in correspondence with $\mathbf{D}_{\geq 4}$ by the bijection represented in Figure 40. This gives $\mathbb{D}_3 = z^3 \mathbb{D}_{\geq 4}$.

Figure 40: A bijection between $\mathbf{D}_3$ and $\mathbf{D}_{\geq 4}$.

Putting these results together and solving for $\mathbb{D}_{\geq 4}$, we get

$$\mathbb{D}_{\geq 4} \;=\; \frac{\mathbb{V}(0) - z}{1 + z^3}\;. \tag{48}$$

We now want to express the generating function $\mathbb{V}_{\geq 4, \geq 4}(x)$. We first divide our problem as follows (the equation uses the trivial bijections between the sets $\mathbf{V}_{\alpha,\beta}$ and $\mathbf{V}_{\beta,\alpha}$) :

$$\mathbb{V}_{\geq 4,\geq 4}(x) = \mathbb{V}_{\geq 2,\geq 2}(x) - \mathbb{V}_{2,2}(x) - 2\mathbb{V}_{2,\geq 3}(x) - \mathbb{V}_{3,3}(x) - 2\mathbb{V}_{3,\geq 4}(x)\;. \tag{49}$$

We now treat separately the different summands in the right-hand-side of this equation.

• $\mathbf{V}_{\geq 2,\geq 2}$: It follows easily from the definitions that :

$$\mathbb{V}_{\geq 2,\geq 2}(x) = \sum_{k\geq 2, l\geq 2} \mathbb{V}_{k,l}(x) + 2\sum_{k\geq 2} \mathbb{V}_{\infty,k}(x) + \mathbb{V}_{\infty,\infty}(x).$$

- The set $\bigcup_{k\geq 2,l\geq 2} \mathbf{V}_{k,l}$ is the set of maps in $\mathbf{V}$ for which the root-face has degree at least 4. Thus,

$$\sum_{k\geq 2,l\geq 2} \mathbb{V}_{k,l}(x) \;=\; \mathbb{V}(x) - \mathbb{V}(0) - x[x]\mathbb{V}(x),$$

where $[x]\mathbb{V}(x)$ is the coefficient of x in $\mathbb{V}(x)$.

- By definition, the set $\bigcup_{k\geq 2} \mathbf{V}_{\infty,k}$ is in bijection with $\mathbf{V} \times \bigcup_{k\geq 2} \mathbf{V}_k$. Moreover, the set $\bigcup_{k\geq 2} \mathbf{V}_k$ is the set of maps in $\mathbf{V}$ for which the root-face has degree at least 3. This gives

$$\sum_{k\geq 2} \mathbb{V}_{\infty,k}(x) \;=\; x^2\mathbb{V}(x)\left(\mathbb{V}(x) - \mathbb{V}(0)\right).$$

- By definition, the set $\mathbf{V}_{\infty,\infty}$ is in bijection with $\mathbf{V}^3$, which gives

$$\mathbb{V}_{\infty,\infty}(x) \;=\; x^4\mathbb{V}(x)^3.$$

Summing these contributions we get

$$\mathbb{V}_{\geq 2,\geq 2}(x) = \mathbb{V}(x) - \mathbb{V}(0) - x[x]\mathbb{V}(x) + 2x^2\mathbb{V}(x)\left(\mathbb{V}(x) - \mathbb{V}(0)\right) + x^4\mathbb{V}(x)^3. \tag{50}$$

• $\mathbf{V}_{2,2}$: The set $\mathbf{V}_{2,2}$ is empty (the face at the left of the root-edge would be of degree at least 4), hence

$$\mathbb{V}_{2,2}(x) \;=\; 0\;. \tag{51}$$

• $\mathbf{V}_{2,\geq 3}$: The set $\mathbf{V}_{2,\geq 3}$ is in bijection with $\mathbf{V}_{\geq 2}$ by the mapping illustrated in Figure 41. This gives $\mathbb{V}_{2,\geq 3}(x) \;=\; xz^2\mathbb{V}_{\geq 2}(x)$. From this, using Equation (39), we obtain

$$\mathbb{V}_{2,\geq 3}(x) \;=\; xz^2(\mathbb{V}(x) - \mathbb{V}(0) + x^2\mathbb{V}(x)^2). \tag{52}$$

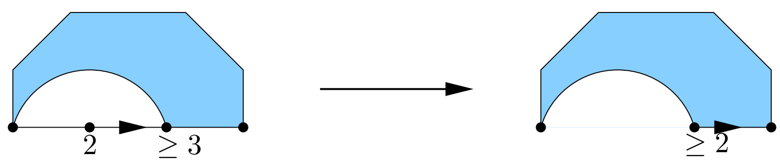


Figure 41: A bijection between $\mathbf{V}_{2,\geq 3}$ and $\mathbf{V}_{\geq 2}$.

• $\mathbf{V}_{3,3}$: We consider a map M in $\mathbf{V}_{3,3}$. We denote by v_1 the root-edge's origin, v_2 the root-edge's end, v_0 the vertex preceding v_1 on the root-face and v_3 the vertex following v_2 (see Figure 42). Since M is non-separable and its root-face has degree at least 4, the vertices $v_i,\ i = 1\ldots 4$ are all distinct. The third vertex v adjacent with v_1 is also the third vertex adjacent with v_2 (or the face at the left of the root-edge would not be a triangle). Since M has no loop, v is distinct from $v_i,\ i = 1\ldots 4$. From these considerations, it is easily seen that the set $\mathbf{V}_{3,3}$ is in bijection with the set $\mathbf{V}_{\geq 3}$ by the mapping illustrated in Figure 42. (Note that this correspondence includes two subcases depending on v becoming a separating point of not). We obtain

$$\mathbb{V}_{3,3}(x) \;=\; xz^5\mathbb{V}_{\geq 3}(x) \;=\; xz^5\left(\mathbb{V}_{\geq 2}(x) - \mathbb{V}_2(x)\right).$$

From this, using Equations (39) and (40), we get

$$\mathbb{V}_{3,3}(x) \;=\; xz^5(\mathbb{V}(x) - \mathbb{V}(0) + x^2\mathbb{V}(x)^2 - xz^2\mathbb{V}(x)). \tag{53}$$

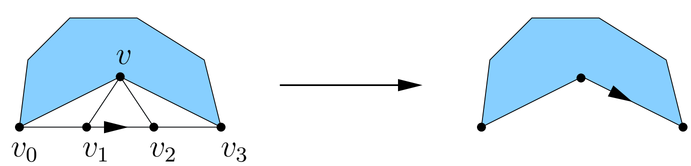


Figure 42: A bijection between $\mathbf{V}_{3,3}$ and $\mathbf{V}_{\geq 3}$.

• $\mathbf{V}_{3,\geq 4}$: Let M be a map in $\mathbf{V}_{3,\geq 4}$. We denote by v_1 the root-edge's origin, v_2 the root-edge's end, v_0 the vertex preceding v_1 on the root-face and v_3 the vertex following v_2 (see Figure 43). Since M is non-separable and its root-face has degree at least 4, the vertices $v_i,\ i = 1\ldots 4$ are all distinct. Let v be the third vertex adjacent to v_1. Two cases can occur: either $v = v_3$

in which case the map decomposes into an ordered pair of maps in $\mathbf{V} \times \mathbf{D}_{\geq 3}$, or v is distinct from $v_i,\ i = 1 \ldots 4$ in which case the map is in correspondence with a map in $\mathbf{V}_{\geq 4, \geq 3}$ (this includes two subcases depending on v becoming a separating point of not). In both cases the correspondence is clearly bijective. This gives

$$\mathbb{V}_{3,\geq 4}(x) \ = \ x^2 z^3 \mathbb{V}(x)\mathbb{D}_{\geq 3} \ + \ z^3 \mathbb{V}_{\geq 4, \geq 3}(x).$$

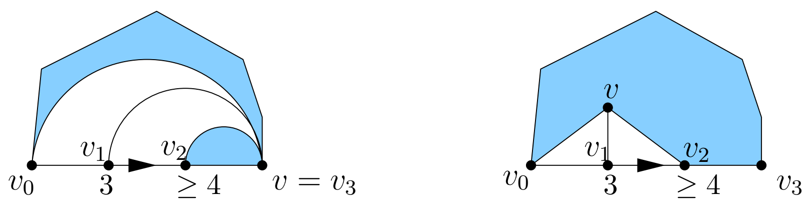


Figure 43: Two configurations for a map in $\mathbf{V}_{3,\geq 4}$.

Given that $\mathbb{D}_{\geq 3} \ = \ \mathbb{D} - \mathbb{D}_1 - \mathbb{D}_2 \ = \ \mathbb{V}(0) - z$, we obtain

$$\mathbb{V}_{3,\geq 4}(x) \ = \ x^2 z^3 \mathbb{V}(x)(\mathbb{V}(0) - z) \ + \ z^3(\mathbb{V}_{\geq 4,\geq 4}(x) + \mathbb{V}_{3,\geq 4}(x)),$$

and solving for $\mathbb{V}_{3,\geq 4}(x)$ we get

$$\mathbb{V}_{3,\geq 4}(x) \ = \ \frac{z^3}{1-z^3}\left(x^2\mathbb{V}(x)(\mathbb{V}(0) - z) \ + \ \mathbb{V}_{\geq 4,\geq 4}(x)\right). \tag{54}$$

We report Equations (50 - 54) in Equation (49) and solve for $\mathbb{V}_{\geq 4,\geq 4}$. We get

$$\begin{aligned}\mathbb{V}_{\geq 4,\geq 4}(x) = \frac{1-z^3}{1+z^3}\Big(\quad & \mathbb{V}(x) - \mathbb{V}(0) - x[x]\mathbb{V}(x) + 2x^2\mathbb{V}(x)\left(\mathbb{V}(x) - \mathbb{V}(0)\right) \\ & + x^4\mathbb{V}(x)^3 - xz^5(\mathbb{V}(x) - \mathbb{V}(0) + x^2\mathbb{V}(x)^2 - xz^2\mathbb{V}(x)) \\ & - 2xz^2(\mathbb{V}(x) - \mathbb{V}(0) + x^2\mathbb{V}(x)^2) - 2\frac{x^2z^3}{1-z^3}\mathbb{V}(x)(\mathbb{V}(0) - z) \ \Big).\end{aligned} \tag{55}$$

Now, using Equations (39) (40) (46) (47) (48) and (55) we can replace $\mathbb{V}_{\geq 2}$, $\mathbb{V}_2$, $\mathbb{V}_3$ and $\mathbb{V}_4$ by their expression in Equation (45). This establishes the following proposition.

Proposition 1.3 *The generating function $\mathbb{V}(x) = \mathbb{V}(x, z)$ of non-separable near-triangulations in which all internal vertices have degree at least 5 satisfies:*

$$\begin{aligned}\mathbb{V}(x) = \ & z + \frac{1}{1+z^3}\left(xz\mathbb{V}(x)^2 + z\frac{\mathbb{V}(x) - \mathbb{V}_0}{x} - z^3\mathbb{V}(x)\right) \\ & - \frac{z^5(1-z^3)}{1+z^3}\Bigg(\frac{\mathbb{V}(x) - \mathbb{V}_0 - x\mathbb{V}_1}{x^2} - z^2(2+z^3)\frac{\mathbb{V}(x) - \mathbb{V}_0}{x} - 2\mathbb{V}(x)(\mathbb{V}_0 - z) \\ & \qquad + x^2\mathbb{V}(x)^3 - xz^2(2+z^3)\mathbb{V}(x)^2 + 2\mathbb{V}(x)\left(\mathbb{V}(x) - \mathbb{V}_0\right) + z^7\mathbb{V}(x)\Bigg)\end{aligned} \tag{56}$$

where $\mathbb{V}_0 = \mathbb{V}(0)$ and $\mathbb{V}_1 = [x]\mathbb{V}(x)$ is the coefficient of x in $\mathbb{V}(x)$.

1.5 Algebraic equations for triangulations with high degree

In the previous section, we have exhibited functional equations concerning the families of *near-triangulations* $\mathbf{T}, \mathbf{U}, \mathbf{V}$. By definition, the generating functions $\mathbb{T}(t), \mathbb{U}(t), \mathbb{V}(t)$ are power series in the main variable z with polynomial coefficients in the secondary variable x. We now solve these equations and establish algebraic equations for the families of *triangulations* $\mathbf{F}, \mathbf{G}, \mathbf{H}$ in which vertices not incident to the root-edge have degree at least $3, 4, 5$ respectively. As observed in Section 1.2, the generating functions $\mathbb{F}(t), \mathbb{G}(t), \mathbb{H}(t)$ are closely related to the series $\mathbb{T}(0), \mathbb{U}(0), \mathbb{V}(0)$ (see Equation (31)).

Let us look at Equations (41), (44) and (56) satisfied by the series $\mathbb{T}(x)$, $\mathbb{U}(x)$ and $\mathbb{V}(x)$ respectively. We begin with Equation (41). This equation is (after multiplication by x) a polynomial equation in the main unknown series $\mathbb{T}(x)$, the secondary unknown $\mathbb{T}(0)$ and the variables x, z. It is easily seen that this equation allows us to compute the coefficients of $\mathbb{T}(x)$ (hence those of $\mathbb{T}(0)$) iteratively. Moreover, we see by induction that the coefficients of this power series are polynomials in the secondary variable x. The same property holds for Equation (44) (resp. (56)): it defines the series $\mathbb{U}(0)$ (resp. $\mathbb{V}(0)$) uniquely as a power series in z with polynomial coefficients in x.

In some sense, Equations (41), (44) and (56) answer our enumeration problems. However, we want to *solve* these equations, that is, to derive from them some equations for the series $\mathbb{T}(0)$, $\mathbb{U}(0)$ and $\mathbb{V}(0)$. Certain techniques for performing such manipulations appear in the combinatorics literature. In the cases of Equation (41) and (44) which are quadratic in the main unknown series $\mathbb{T}(x)$ and $\mathbb{U}(x)$ we can routinely apply the so-called *quadratic method* [Goul 83, Section 2.9]. This method allows one to solve polynomial equations which are quadratic in the bivariate unknown series and have one unknown univariate series. This method also applies to Equation (33) concerning $\mathbb{S}(x)$ and allows to prove Equation (34). However, Equation (56) concerning $\mathbb{V}(x)$ is cubic in this series and involves two unknown univariate series ($\mathbb{V}(0)$ and $[x]\mathbb{V}(x)$). Very recently, Bousquet-Mélou and Jehanne proposed a general method to solve polynomial equations of any degree in the bivariate unknown series and involving any number of unknown univariate series [Bous 05b]. We present their formalism.

Let us begin with Equation (41) concerning $\mathbb{T}(0)$. We define the polynomial

$$P(T, T_0, X, Z) = XZ + X^2ZT^2 + ZT - ZT_0 - XZ^3T - XT \ .$$

Equation (41) can be written as

$$P(\mathbb{T}(x), \mathbb{T}(0), x, z) = 0 \ . \tag{57}$$

Let us consider the equation $P_1'(\mathbb{T}(x), \mathbb{T}(0), x, z) = 0$, where P_1' denotes the derivative of P with respect to its first variable. This equation can be written as

$$2x^2 z\mathbb{T}(x) + z - xz^3 - x \ = \ 0 \ .$$

This equation is not satisfied for a generic x. However, considered as an equation in x, it is straightforward to see that it admits a unique power series solution $X(z)$.
Taking the derivative of Equation (57) with respect to x one obtains

$$\frac{\partial \mathbb{T}(x)}{\partial x} \cdot P_1'(\mathbb{T}(x), \mathbb{T}(0), x, z) + P_3'(\mathbb{T}(x), \mathbb{T}(0), x, z) = 0,$$

where P_3' denotes the derivative of P with respect to its third variable. Substituting the series $X(z)$ for x in that equation, we see that the series $X(z)$ is also a solution of the equation $P_3'(\mathbb{T}(x), \mathbb{T}(0), x, z) = 0$. Hence, we have a system of three equations

$$\begin{aligned} P(\mathbb{T}(X(z)), \mathbb{T}(0), X(z), z) &= 0 \ , \\ P_1'(\mathbb{T}(X(z)), \mathbb{T}(0), X(z), z) &= 0 \ , \\ P_3'(\mathbb{T}(X(z)), \mathbb{T}(0), X(z), z) &= 0 \ , \end{aligned}$$

for the three unknown series $\mathbb{T}(X(z))$, $\mathbb{T}(0)$ and $X(z)$. This polynomial system can be solved by elimination techniques using either resultant calculations or Gröbner bases. Performing these eliminations one obtains an algebraic equation for $\mathbb{T}(0)$:

$$\mathbb{T}(0) = z - 24z^4 + 3z^7 + z^{10} + (32z^3 + 30z^6 - 4z^9 - z^{12})\mathbb{T}(0) - 8z^2(1+z^3)^2\mathbb{T}(0)^2 - 16z^4\mathbb{T}(0)^3.$$

Using the fact that $\mathbb{T}(0) = z + z\mathbb{G}(z^3)$ we get the following theorem.

Theorem 1.4 *Let $\boldsymbol{G}$ be the set of non-separable triangulations in which any vertex not incident to the root-edge has degree at least 3, and let $\mathbb{G}(t)$ be its generating function. The series $\mathbb{G}(t)$ is uniquely defined as a power series in t by the algebraic equation:*

$$\begin{aligned} &16t^2\mathbb{G}(t)^3 + 8t(t^2+8t+1)\mathbb{G}(t)^2 \\ &+(t^4+20t^3+50t^2-16t+1)\mathbb{G}(t) + t^2(t^2+11t-1) = 0. \end{aligned} \tag{58}$$

Similar manipulations lead to a cubic equation for the set $\mathbf{H}$.

Theorem 1.5 *Let $\boldsymbol{H}$ be the set of non-separable triangulations in which any vertex not incident to the root-edge has degree at least 4, and let $\mathbb{H}(t)$ be its generating function. The series $\mathbb{H}(t)$ is uniquely defined as a power series in t by the algebraic equation:*

$$\begin{aligned} &16t^2(t-1)^4\mathbb{H}(t)^3 + (t^8+12t^7-14t^6-84t^5+207t^4-192t^3+86t^2-16t+1)\mathbb{H}(t) \\ &+8t(t-1)^2(t^4+4t^3-13t^2+8t+1)\mathbb{H}(t)^2 + t^4(t-1)(t^3+5t^2-8t+1) = 0. \end{aligned} \tag{59}$$

For Equation (56) concerning $\mathbb{V}(0)$ the method is almost identical. We see that there is a polynomial $Q(V, V_0, V_1, x, z)$ such that Equation (56) can be written as $Q(\mathbb{V}(x), \mathbb{V}(0), [x]\mathbb{V}(x), x, z) = 0$. But we can show that there are exactly *two* series $X_1(z),\ X_2(z)$ such that $Q'_1(\mathbb{V}(X(z)), \mathbb{V}(0), [x]\mathbb{V}(x), X(z), z) = 0$. Thus, we obtain a system of 6 equations

$$\begin{array}{rcll} Q(\mathbb{V}(X_i(z)), \mathbb{V}(0), [x]\mathbb{V}(x), X_i(z), z) & = & 0 & \\ Q'_1(\mathbb{V}(X_i(z)), \mathbb{V}(0), [x]\mathbb{V}(x), X_i(z), z) & = & 0 & \qquad i = 1,\ 2 \\ Q'_3(\mathbb{V}(X_i(z)), \mathbb{V}(0), [x]\mathbb{V}(x), X_i(z), z) & = & 0 & \end{array}$$

for the 6 unknown series $\mathbb{V}(X_1(z)),\ \mathbb{V}(X_2(z)),\ X_1(z),\ X_2(z),\ \mathbb{V}(0)$ and $[x]\mathbb{V}(x)$. This system can be solved via elimination techniques though the calculations involved are heavy. We obtain the following theorem.

Theorem 1.6 *Let* $\boldsymbol{K}$ *be the set of non-separable triangulations in which any vertex not incident to the root-edge has degree at least 5, and let* $\mathbb{K}(t)$ *be its generating function. The series* $\mathbb{K}(t)$ *is uniquely defined as a power series in* t *by the algebraic equation:*

$$\sum_{i=0}^{6} P_i(t)\mathbb{K}(t)^i, \tag{60}$$

where the polynomials $P_i(t), i = 0 \ldots 6$ *are given in Appendix 1.9.1.*

1.6 Constraining the vertices incident to the root-edge

So far, we have established algebraic equations for the generating functions $\mathbb{G}(t), \mathbb{H}(t), \mathbb{K}(t)$ of triangulations in which *any vertex not incident to the root-edge* has degree at least 3, 4, 5. The following theorems provide equations concerning the generating functions $\mathbb{G}^*(t), \mathbb{H}^*(t)$ of triangulations in which *any vertex* has degree at least 3, 4.

Theorem 1.7 *Let* $\boldsymbol{G}^*$ *be the set of non-separable triangulations in which* any *vertex has degree at least 3 and let* $\mathbb{G}^*(t)$ *be its generating function. The series* $\mathbb{G}^*$ *is related to the series* $\mathbb{G}$ *of Theorem 1.4 by*

$$\mathbb{G}^*(t) = (1 - 2t)\mathbb{G}(t)\ . \tag{61}$$

Theorem 1.8 *Let* $\boldsymbol{H}^*$ *be the set of non-separable triangulations in which* any *vertex has degree at least 4 and let* $\mathbb{H}^*(t)$ *be its generating function. The series* $\mathbb{H}^*$ *is related to the series* $\mathbb{H}$ *of Theorem 1.5 by*

$$\mathbb{H}^*(t) = \frac{1 - 5t + 5t^2 - 3t^3}{1 - t}\mathbb{H}(t)\ . \tag{62}$$

Let us make a few comments before proving these two theorems. First, observe that we can deduce from Theorems 1.4 and 1.7 (resp. 1.5 and 1.8) an algebraic equation for the generating function $\mathbb{G}^*$ (resp. $\mathbb{H}^*$) of triangulations in which *any* vertex has degree at least 3 (resp. 4). The algebraic equation obtained for $\mathbb{G}^*$ coincides with the result of Gao and Wormald [Gao 02, Theorem 2]. From the algebraic equations we can routinely compute the first coefficients of our series:

$$\mathbb{G}^*(t) = t^2 + 3t^3 + 19t^4 + 128t^5 + 909t^6 + 6737t^7 + 51683t^8 + 407802t^9 + o(t^9),$$
$$\mathbb{H}^*(t) = t^4 + 3t^5 + 12t^6 + 59t^7 + 325t^8 + 1875t^9 + 11029t^{10} + 65607t^{11} + o(t^{11}).$$

Recall that the coefficient of t^n in the series $\mathbb{G}^*(t)$, $\mathbb{H}^*(t)$ is the number of triangulations with $3n$ edges ($2n$ triangles, $n+2$ vertices) satisfying the required degree constraint. In the expansion of $\mathbb{G}^*(t)$, the smallest non-zero coefficient t^2 corresponds to the tetrahedron. In the expansion of $\mathbb{H}^*(t)$, the smallest non-zero coefficient t^4 corresponds to the octahedron (see Figure 44).

We were unable to find an equation that would permit to count non-separable triangulations in which *any vertex* has degree at least 5. However, we can use the algebraic equation (60) to compute the first coefficients of the series $\mathbb{K}(t)$:

$$\mathbb{K}(t) = t^{10} + 8t^{11} + 45t^{12} + 209t^{13} + 890t^{14} + 3600t^{15} + 14115t^{16} + 54306t^{17} + o(t^{18}).$$

The first non-zero coefficient t^{10} corresponds to the icosahedron (see Figure 44).

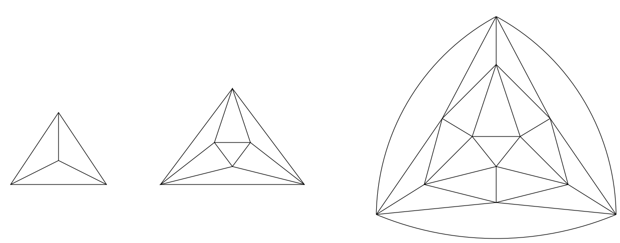


Figure 44: The platonic solids: tetrahedron, octahedron, icosahedron.

In order to prove Theorems 1.7 and 1.8 we need some new notations. The set $\mathbf{G}_{i,j,k}$ (resp. $\mathbf{H}_{i,j,k}$) is the set of triangulations such that the root-edge's origin has degree i, the root-edge's end has degree j, the third vertex of the root-face has degree k and all internal vertices have degree at least 3 (resp. 4). For $\mathbf{L} = \mathbf{G}, \mathbf{H}$ we define $\mathbf{L}_{\geq i,j,k} = \bigcup_{l \geq i} \mathbf{L}_{l,j,k}$ and with similar notation, $\mathbf{L}_{\geq i, \geq j,k}$ etc. If $\mathbf{L}$ is any of these sets, $\mathbb{L}(t)$ is the corresponding generating function, where a map with $3n$ edges has contribution t^n.

Proof of Theorem 1.7: By definition, $\mathbf{G} = \mathbf{G}_{\geq 2, \geq 2, \geq 3}$ and $\mathbf{G}^* = \mathbf{G}_{\geq 3, \geq 3, \geq 3}$. Hence,

$$\mathbb{G}^*(t) = \mathbb{G}(t) - \mathbb{G}_{2,2,\geq 3}(t) - 2\mathbb{G}_{2,\geq 3,\geq 3}(t). \tag{63}$$

• The set $\mathbf{G}_{2,2,\geq 3}$ is empty, hence $\mathbb{G}_{2,2,\geq 3}(t) = 0$.
• The set $\mathbf{G}_{2,\geq 3,\geq 3}$ is in bijection with $\mathbf{G}_{\geq 1,\geq 1,\geq 3} = \mathbf{G}$ by the mapping represented in Figure 45. This gives $\mathbb{G}_{2,\geq 3,\geq 3}(t) = t\mathbb{G}(t)$.
Plugging these results in (63) proves the theorem. □

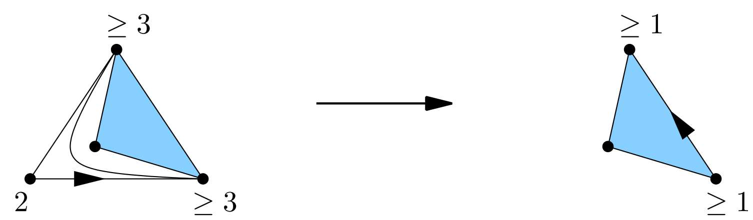


Figure 45: A bijection between $\mathbf{G}_{2,\geq 3,\geq 3}$ and $\mathbf{G}$ (resp. $\mathbf{H}_{2,\geq 3,\geq 3}$ and $\mathbf{H}$).

Proof of Theorem 1.8: By definition, $\mathbf{H}^* = \mathbf{H}_{\geq 4,\geq 4,\geq 4}$. Hence,

$$\mathbb{H}^*(t) = \mathbb{H}_{\geq 3,\geq 3,\geq 4}(t) - \mathbb{H}_{3,3,\geq 4}(t) - 2\mathbb{H}_{3,\geq 4,\geq 4}(t). \tag{64}$$

Recall that $\mathbf{H} = \mathbf{H}_{\geq 1,\geq 1,\geq 4} = \mathbf{H}_{\geq 2,\geq 2,\geq 4}$.
• Clearly, $\mathbb{H}_{\geq 3,\geq 3,\geq 4}(t) = \mathbf{H}_{\geq 2,\geq 2,\geq 4}(t) - \mathbf{H}_{2,2,\geq 4}(t) - 2\mathbf{H}_{2,\geq 3,\geq 3}(t)$.
• The set $\mathbb{H}_{2,2,\geq 4}(t)$ is empty, hence $\mathbf{H}_{2,2,\geq 4}(t) = 0$.
• The set $\mathbf{H}_{2,\geq 3,\geq 3}$ is in bijection with $\mathbf{H}_{\geq 1,\geq 1,\geq 4} = \mathbf{H}$ by the mapping represented in Figure 45, hence $\mathbf{H}_{2,\geq 3,\geq 3}(t) = t\mathbb{H}(t)$.
This gives

$$\mathbb{H}_{\geq 3,\geq 3,\geq 4}(t) = (1 - 2t)\mathbb{H}(t). \tag{65}$$

• The set $\mathbf{H}_{3,3,\geq 4}$ is in bijection with $\mathbf{H}_{\geq 1,\geq 1,\geq 4} = \mathbf{H}$ by the mapping represented in Figure 46. This gives

$$\mathbb{H}_{3,3,\geq 4}(t) = t^2\mathbb{H}(t). \tag{66}$$

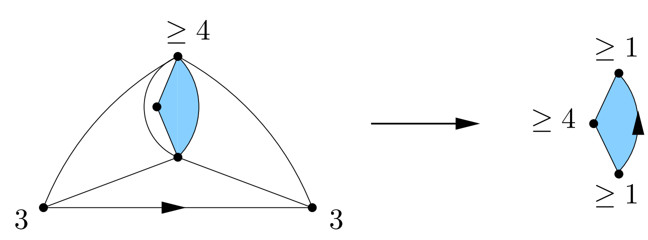


Figure 46: A bijection between $\mathbf{H}_{3,3,\geq 4}$ and $\mathbf{H}$.

• For any integer k greater than 2, the set $\mathbf{H}_{\geq k,\geq k,3}$ is in bijection with the set $\mathbf{H}_{\geq k-1,\geq k-1,\geq 3}$ by the mapping represented in Figure 47. This gives

$$\mathbb{H}_{\geq k,\geq k,3}(t) = t\mathbb{H}_{\geq k-1,\geq k-1,\geq 3}(t) \quad \text{for all } k \geq 2. \tag{67}$$

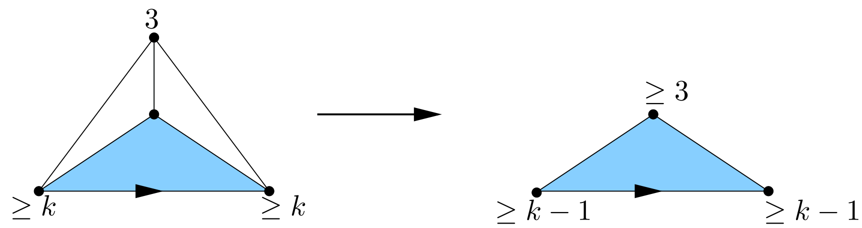


Figure 47: A bijection between $\mathbf{H}_{\geq k,\geq k,3}$ and $\mathbf{H}_{\geq k-1,\geq k-1,\geq 3}$.

Using Equation (67) for $k=4$ and then for $k=3$ (and trivial symmetry properties), we get

$$\begin{aligned}\mathbb{H}_{3,\geq 4,\geq 4}(t) &= \mathbb{H}_{\geq 4,\geq 4,3}(t) = t\mathbb{H}_{\geq 3,\geq 3,\geq 3}(t) = t\mathbb{H}_{\geq 3,\geq 3,\geq 4}(t) + t\mathbb{H}_{\geq 3,\geq 3,3}(t)\\ &= t\mathbb{H}_{\geq 3,\geq 3,\geq 4}(t) + t^2\mathbb{H}_{\geq 2,\geq 2,\geq 3}(t).\end{aligned}$$

- By Equation (65), we have $\mathbb{H}_{\geq 3,\geq 3,\geq 4}(t) = (1-2t)\mathbb{H}(t)$.
- Using Equation (67) for $k=2$ gives

$$\mathbb{H}_{\geq 2,\geq 2,\geq 3}(t) = \mathbb{H}_{\geq 2,\geq 2,\geq 4}(t) + \mathbb{H}_{\geq 2,\geq 2,3}(t) = \mathbb{H}(t) + t\mathbb{H}_{\geq 1,\geq 1,\geq 3}(t).$$

Given that $\mathbf{H}_{\geq 1,\geq 1,\geq 3} = \mathbf{H}_{\geq 2,\geq 2,\geq 3}$, we get $\mathbb{H}_{\geq 2,\geq 2,\geq 3}(t) = \dfrac{1}{1-t}\mathbb{H}(t)$.
Thus, we obtain

$$\mathbb{H}_{3,\geq 4,\geq 4}(t) = \frac{t(1-2t+2t^2)}{1-t}\mathbb{H}(t). \tag{68}$$

Plugging Equations (65), (66) and (68) in Equation (64) proves the theorem. □

1.7 Asymptotics

In Section 1.5, we established algebraic equations for the generating functions $\mathbb{L} = \mathbb{F}, \mathbb{G}, \mathbb{H}, \mathbb{K}$ of non-separable triangulations in which any vertex not incident to the root-edge has degree at least $d = 2, 3, 4, 5$ (Equations (34), (58), (59) and (60)). We will now derive the asymptotic form of the number $l_n = f_n, g_n, h_n, k_n$ of maps with $3n$ edges in each family by analyzing the singularities of the generating function $\mathbb{L} = \mathbb{F}, \mathbb{G}, \mathbb{H}, \mathbb{K}$ (l_n is the coefficient of t^n in $\mathbb{L}$). The principle of this method is a general correspondence between the expansion of a generating function at its dominant singularities and the asymptotic form of its coefficients [Flaj 90, Flaj].

Lemma 1.9 *Each of the generating functions $\mathbb{L} = \mathbb{F}, \mathbb{G}, \mathbb{H}, \mathbb{K}$ has a unique dominant singularity $\rho_L > 0$ and a singular expansion with singular exponent $\frac{3}{2}$ at ρ_L, in the sense that*

$$\mathbb{L}(t) = \alpha_L + \beta_L(1-\frac{t}{\rho_L}) + \gamma_L(1-\frac{t}{\rho_L})^{3/2} + O((1-\frac{t}{\rho_L})^2), \tag{69}$$

with $\gamma_L \neq 0$. The dominant singularities of the series $\mathbb{F}$ and $\mathbb{G}$ are respectively $\rho_F = \frac{2}{27}$ and $\rho_G = \frac{3\sqrt{3}-5}{2}$. The dominant singularities ρ_H and ρ_K of the series $\mathbb{H}$ and $\mathbb{K}$ are defined by algebraic equations given in Appendix 1.9.2.

Proof (sketch): The (systematic) method we follow is described in [Flaj, Chapter VII.4]). Calculations were performed using the Maple package *gfun* [Salv 94].
Let us denote generically by ρ_L the radius of convergence of the series $\mathbb{L}$ and by $Q(\mathbb{L}, t)$ the algebraic equation satisfied by $\mathbb{L}$ (Equations (34), (58), (59) and (60)). It is known that the singular points of the series $\mathbb{L}$ are among the roots of the polynomial $R(t) = D(t)\Delta(t)$ where $D(t)$ is the dominant coefficient of $Q(y, t)$ and $\Delta(t)$ is the discriminant of $Q(y, t)$ considered as a polynomial in y. Moreover, since the series $\mathbb{L}$ has non-negative coefficients, we know (by Pringsheim's Theorem) that the point $t = \rho_L$ is singular. In our cases, the smallest positive root of $R(t)$ is found to be indeed a singular point of the series $\mathbb{L}$. (This requires to solve some connection problems that we do not detail.) Moreover, no other root of $R(t)$ has the same modulus. This proves that the series $\mathbb{L}$ has a unique dominant singularity.
The second step is to expand the series $\mathbb{L}$ near its singularity ρ_L. This calculation can be performed using *Newton's polygon method* (see [Flaj, Chapter VII.4]) which is implemented in the *algeqtoseries* Maple command [Salv 94]. □

From Lemma 1.9, we can deduce the asymptotic form of the number $l_n = f_n, g_n, h_n, k_n$ of non-separable triangulations of size n in each family.

Theorem 1.10 *The number $l_n = f_n, g_n, h_n, k_n$ of non-separable triangulations of size n ($3n$ edges) in which any vertex not incident to the root-edge has degree at least $d = 2, 3, 4, 5$ has asymptotic form*

$$l_n \sim \lambda_L n^{-5/2} \left(\frac{1}{\rho_L}\right)^n .$$

The growth constants $\rho_F, \rho_G, \rho_H, \rho_K$ are given in Lemma 1.9. Numerically,

$$\frac{1}{\rho_F} = 13.5, \quad \frac{1}{\rho_G} \approx 10.20, \quad \frac{1}{\rho_H} \approx 7.03, \quad \frac{1}{\rho_K} \approx 4.06 .$$

Remark: The subexponential factor $n^{-5/2}$ is typical of planar maps families (see for instance [Band 01] where 15 classical families of maps are listed all displaying this subexponential factor $n^{-5/2}$).

Remark: Using Theorems 1.7 and 1.8, it is easily seen that the series $\mathbb{L}^* = \mathbb{G}^*, \mathbb{H}^*$ has dominant singularity $\rho_L = \rho_G, \rho_H$ with singular exponent $\frac{3}{2}$ at ρ_L:

$$\mathbb{L}(t) = \alpha_L^* + \beta_L^*(1 - \frac{t}{\rho_L}) + \gamma_L^*(1 - \frac{t}{\rho_L})^{3/2} + O((1 - \frac{t}{\rho_L})^2).$$

Therefore, we obtain the asymptotic form

$$l_n^* \sim \lambda_L^* n^{-5/2} \left(\frac{1}{\rho_L}\right)^n$$

for the number $l^*_n = g^*_n, h^*_n$ of non-separable triangulations of size n with vertex degree at least $d = 3, 4$. Hence, the numbers l^*_n and l_n are equivalent up to a (known) constant multiplicative factor $\frac{\lambda^*_L}{\lambda_L}$:

$$\begin{aligned}\frac{\lambda^*_G}{\lambda_G} &= \frac{\gamma^*_G}{\gamma_G} = 1 - 2\rho_G = 6 - 3\sqrt{3},\\ \frac{\lambda^*_H}{\lambda_H} &= \frac{\gamma^*_H}{\gamma_H} = \frac{1 - 5\rho_H + 5{\rho_H}^2 - 3{\rho_H}^3}{1 - \rho_H}.\end{aligned}$$

We do not have such precise information about the asymptotic form of the number k^*_n of non-separable triangulations of size n ($3n$ edges) with vertex degree at least 5. However, we do know that $k^*_n = \Theta(k_n) = \Theta(n^{-5/2}{\rho_K}^{-n})$. Indeed, we clearly have $k^*_n \leq k_n$ and, in addition, $k^*_n \geq k_{n-9} \sim {\rho_K}^9 k_n$. The latter inequality is proved by observing that the operation of replacing the root-face of a triangulation by an icosahedron is an injection from the set of triangulations of size n in which any vertex *not incident to the root-edge* has degree 5 to the set of triangulations of size $n + 9$ in which *any* vertex has degree at least 5.

1.8 Concluding remarks

We have established algebraic equations for the generating functions of loopless triangulations (i.e. non-separable triangulations) in which *any vertex not incident to the root-edge* has degree at least $d = 3, 4, 5$. We have also established algebraic equations for loopless triangulations in which *any vertex* has degree at least $d = 3, 4$. However, have not found a similar result for $d = 5$. The algebraic equations we have obtained can be converted into differential equations (using for instance the *algeqtodiffeq* Maple command available in the *gfun* package [Salv 94]) from which one can compute the coefficients of the series in a linear number of operations. Moreover, the asymptotic form of their coefficients can also be found routinely from the algebraic equations.

The approach we have adopted is based on a classic decomposition scheme allied with a generating function approach. Alternatively, it is possible to obtain some of our results by a compositional approach. This is precisely the method followed by Gao and Wormald to obtain the algebraic equation concerning loopless triangulations in which any vertex has degree at least 3 [Gao 02]. This substitution approach can also be extended to obtain the algebraic equation concerning loopless triangulations in which any vertex has degree at least 4. However, we do not see how to apply this method to loopless triangulations in which vertices not incident to the root-edge have degree at least 5.

Recently, Poulalhon and Schaeffer gave a bijective proof based on the *conjugacy classes of tree* for the number of loopless triangulations [Poul 03a]. However, it is dubious that this approach should apply for the families $\mathbb{H}$, $\mathbb{K}$ of loopless triangulations in which vertices have degree at least $d = 4, 5$. Indeed, for a large number of families of maps $\mathbf{L}$, the generating function $\mathbb{L}(t)$ is *Lagrangean*, that is, there exists a series $\mathbb{X}(t)$ and two rational functions Ψ, Φ satisfying

$$\mathbb{L}(t) = \Psi(\mathbb{X}(t)) \quad \text{and} \quad \mathbb{X}(t) = t\Phi(\mathbb{X}(t))$$

(see for instance [Band 01] where 15 classical families are listed together with a Lagrangean parametrization). Often, a parametrization can be found such that the series $\mathbb{X}(t)$ looks like the generating function of a family of trees (i.e. $\Phi(x)$ is a series with non-negative coefficients) suggesting that a bijective approach exists based on the enumeration of certain trees [Bous 03b, Bout 02, Bout 05]. However, it is known that an algebraic series is Lagrangean if and only if the genus of the algebraic equation is 0 [Abhy 90, Chapter 15]. In our case, the algebraic equations defining the series $\mathbb{F}$, $\mathbb{G}$, $\mathbb{H}$ and $\mathbb{K}$ have respective genus 0, 0, 2 and 25. (The genus can be computed using the Maple command *genus*.) Thus, whereas the series $\mathbb{F}$, $\mathbb{G}$ are Lagrangean (with a parametrization given in Appendix 1.9.3), the series $\mathbb{H}$, $\mathbb{K}$ are not.

Lastly, we claim some generality to our approach. Here, we have focused on loopless triangulations, but it is possible to practice the same kind of manipulations for *general* triangulations and for 3-connected ones. The method should also apply to some other families of maps, like quadrangulations. Thus, a whole new class of map families is expected to have algebraic generating functions.

1.9 Appendix

1.9.1 Coefficients of the algebraic equation (60)

The coefficients $P_i(t), i = 0..6$ in the algebraic equation (60) are:

$P_0(t) = t^{10}(-1 + 82552t^{11} - 163081t^{12} + 277796t^{13} - 308156t^{14} - 443851t^{16} + t^{34} + 13t + 32t^{31} + 454t^5 - 2434t^6 - 5762t^8 + 4373t^7 - 53961t^{10} + 23037t^9 + 354387t^{15} + 163964t^{20} - 28454t^{21} - 38408t^{22} + 36713t^{23} - 11737t^{24} + t^{33} + 2t^{32} - 278t^{25} + 242t^{28} - 1678t^{27} + 2714t^{26} + 36t^{29} - 64t^{30} - 70t^2 + 180t^3 - 195t^4 - 273662t^{19} + 122688t^{18} + 262614t^{17}),$

$P_1(t) = (1 - 594873t^{11} + 1078572t^{12} - 1457943t^{13} + 1921912t^{14} + 1327736t^{16} + 1462t^{38} - 3168t^{37} - 611t^{39} + 25956t^{35} - 56515t^{34} - 3826t^{36} - 21t - 467567t^{31} - 4545t^5 + 3916t^6 + 60304t^8 - 13364t^7 + 275068t^{10} - 142715t^9 - 2t^{42} + 9t^{43} + t^{44} - 2338117t^{15} - 4673450t^{20} + 5167054t^{21} - 1145738t^{22} - 2425736t^{23} + 2298353t^{24} + 66635t^{33} + 90827t^{32} + 559893t^{25} - 874518t^{28} + 2995671t^{27} - 3225500t^{26} - 526335t^{29} + 763474t^{30} + 68t^{41} + 75t^{40} + 193t^2 - 988t^3 + 2913t^4 + 1719643t^{19} - 945302t^{18} + 541155t^{17}),$

$P_2(t) = t(8 + 2011979t^{11} - 1422607t^{12} + 2174211t^{13} - 4910332t^{14} - 9095603t^{16} - 814t^{38} + 688t^{37} + 306t^{39} - 16997t^{35} + 43703t^{34} + 1292t^{36} - 4t + 370239t^{31} - 3000t^5 + 20421t^6 - 268574t^8 + 72382t^7 - 1309172t^{10} + 527412t^9 + 8t^{42} + 5383141t^{15} + 31153077t^{20} - 16211612t^{21} - 2143067t^{22} + 7886923t^{23} - 2902691t^{24} - 50536t^{33} - 26161t^{32} - 4609909t^{25} + 156674t^{28} - 3199107t^{27} + 6488106t^{26} + 970079t^{29} - 902321t^{30} + 12t^{41} + 4t^{40} - 556t^2 + 3851t^3 - 8840t^4 - 18494688t^{19} - 9439987t^{18} + 17752182t^{17}),$

$P_3(t) = t^2(16 + 1278321t^{11} - 2978655t^{12} + 1697247t^{13} + 5975715t^{14} + 54631824t^{16} + 166t^{38} - 90t^{37} - 32t^{39} + 3984t^{35} - 13104t^{34} - 868t^{36} - 192t - 105251t^{31} + 17247t^5 - 36981t^6 + 521925t^8 - 74982t^7 + 835782t^{10} - 1142394t^9 - 29427957t^{15} - 39935486t^{20} + 7773505t^{21} + 6824437t^{22} - 5541795t^{23} - 1619262t^{24} + 18648t^{33} + 4941t^{32} + 5146785t^{25} + 349680t^{28} + 880004t^{27} - 3411645t^{26} - 600239t^{29} + 358687t^{30} + 16t^{40} + 1046t^2 - 2554t^3 - 397t^4 + 60017232t^{19} - 26467945t^{18} - 34977363t^{17}),$

$P_4(t) = 9t^5(t-1)^2(8 + 722739t^{11} - 1888278t^{12} + 1483343t^{13} + 679876t^{14} + 1099122t^{16} - 84t - 20t^{31} + 9250t^5 - 17908t^6 + 144652t^8 - 22565t^7 + 87721t^{10} - 234335t^9 - 1820089t^{15} - 5409t^{20} - 64607t^{21} + 41918t^{22} - 12628t^{23} - 1362t^{24} + 8t^{32} + 6200t^{25} - 189t^{28} + 1127t^{27} - 3809t^{26} - 103t^{29} + 84t^{30} + 368t^2 - 583t^3 - 2069t^4 + 110521t^{19} - 69119t^{18} - 243772t^{17}),$

$P_5(t) = 81t^8(t-1)^4(1 + 25926t^{11} - 14080t^{12} + 2973t^{13} - 369t^{14} + 348t^{16} - 9t + 2118t^5 - 2936t^6 + 23913t^8 - 4134t^7 - 6330t^{10} - 25946t^9 - 970t^{15} + 12t^{20} - 22t^{21} + 12t^{22} - 3t^{23} + t^{24} + 30t^2 - 15t^3 - 747t^4 + 42t^{19} - 219t^{18} + 405t^{17}),$

$P_6(t) = 59049t^{15}(t+1)(t-1)^9$.

1.9.2 Algebraic equations for the dominant singularity of the series $\mathbb{H}(t)$ and $\mathbb{K}(t)$

The dominant singularity ρ_H (resp. ρ_K) of the generating function $\mathbb{H}(t)$ (resp. $\mathbb{K}(t)$) is the smallest positive root of the polynomial $r_H(t)$ (resp. $r_K(t)$) where
$r_H(t) = 2 - 17t + 22t^2 - 10t^3 + 2t^4$,
and
$r_K(t) = 256 - 5504t + 51744t^2 - 265664t^3 + 755040t^4 - 1069751t^5 + 1411392t^6 - 9094370t^7 + 30208920t^8 - 14854607t^9 - 106655904t^{10} + 169679596t^{11} + 1693392t^{12} + 58535932t^{13} - 263701752t^{14} - 751005332t^{15} + 2215033200t^{16} - 2276240390t^{17} + 2301677920t^{18} - 1558097344t^{19} - 2448410184t^{20} + 6223947236t^{21} - 7440131352t^{22} + 6100648148t^{23} + 1602052848t^{24} - 9604816702t^{25} + 6144202392t^{26} + 996698032t^{27} + 551560496t^{28} - 3299013583t^{29} - 728097928t^{30} + 4881643814t^{31} - 3845803168t^{32} + 494467523t^{33} + 1677669800t^{34} - 1787552140t^{35} + 825330824t^{36} + 1529759t^{37} - 340280968t^{38} + 301075034t^{39} - 121555768t^{40} - 1710967t^{41} + 37850432t^{42} - 27659392t^{43} + 9430688t^{44} - 152352t^{45} - 1901664t^{46} + 1245152t^{47} - 400416t^{48} + 47744t^{49} + 30720t^{50} - 22528t^{51} + 7680t^{52} - 1792t^{53} + 256t^{54}$.

1.9.3 Lagrangean parametrization for the series $\mathbb{F}(t)$, $\mathbb{G}(t)$ and $\mathbb{G}^*(t)$

The series $\mathbb{F}(t)$ has the following Lagrangean parametrization:

$$\mathbb{F}(t) = \frac{\mathbb{X}(1+\mathbb{X})}{2},$$

where

$$\mathbb{X} \equiv \mathbb{X}(t) = 2t(1+\mathbb{X}(t))^3.$$

The series $\mathbb{G}(t)$ and $\mathbb{G}^*(t)$ have the following Lagrangean parametrization:

$$\begin{aligned}\mathbb{G}(t) &= 2t\mathbb{Y}(1+\mathbb{Y})(1-\mathbb{Y}-\mathbb{Y}^2),\\ \mathbb{G}^*(t) &= 4t^2(1+\mathbb{Y})(1-\mathbb{Y}-\mathbb{Y}^2)(1+3\mathbb{Y}+6\mathbb{Y}^2+2\mathbb{Y}^3),\end{aligned}$$

where

$$\mathbb{Y} \equiv \mathbb{Y}(t) = 2t(1+\mathbb{Y}(t))(1+4\mathbb{Y}(t)+2\mathbb{Y}(t)^2).$$

Part II

Bijective counting of maps

Chapter 2

Kreweras walks and loopless triangulations

Abstract: We consider lattice walks in the plane starting at the origin, remaining in the first quadrant $i, j \geq 0$ and made of West, South and North-East steps. In 1965, Germain Kreweras discovered a remarkably simple formula giving the number of these walks (with prescribed length and endpoint). Kreweras' proof was very involved and several alternative derivations have been proposed since then. But the elegant simplicity of the counting formula remained unexplained. We give the first purely combinatorial explanation of this formula. Our approach is based on a bijection between Kreweras walks and triangulations with a distinguished spanning tree. We obtain simultaneously a bijective way of counting loopless triangulations.

Résumé : On considère les chemins planaires constitués de pas Sud, Ouest et Nord-Est partant de l'origine et restant dans le quart de plan. En 1965, Germain Kreweras démontra une formule remarquablement simple donnant le nombre de ces chemins (à taille et point d'arrivée fixés). La preuve originale de Kreweras est particulièrement complexe et plusieurs démonstrations alternatives ont été proposées depuis lors. Mais l'élégante simplicité de la formule de comptage resta inexpliquée. Nous apportons la première preuve entièrement bijective de cette formule. Notre approche est basée sur une bijection entre les chemins de Kreweras et les triangulations dont un arbre couvrant est distingué. Nous obtenons simultanément une preuve bijective pour le comptage des triangulations sans boucle.

2.1 Introduction

We consider lattice walks in the plane starting from the origin (0,0), remaining in the first quadrant $i, j \geq 0$ and made of three kind of steps: West, South and North-East. These walks were first studied by Kreweras [Krew 65] and inherited his name. A Kreweras walk ending at the origin is represented in Figure 48.

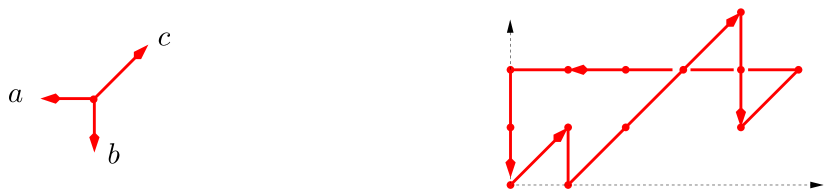


Figure 48: The Kreweras walk *cbcccbbcaaaaabb*.

These walks have remarkable enumerative properties. Kreweras proved in 1965 that the number of walks of length $3n$ ending at the origin is:

$$k_n \quad = \quad \frac{4^n}{(n+1)(2n+1)}\binom{3n}{n}. \tag{70}$$

The original proof of this result is complicated and somewhat unsatisfactory. It was performed by *guessing* the number of walks of size n ending at a generic point (i, j). The conjectured formulas were then checked using the recurrence relations between these numbers. The *checking part* involved several hypergeometric identities which were later simplified by Niederhausen [Nied 83]. In 1986, Gessel gave a different proof in which the *guessing part* was reduced [Gess 86]. More recently, Bousquet-Mélou proposed a constructive proof (that is, without guessing) of these results and some extensions [Bous 05a]. Still, the simple looking formula (70) remained without a direct combinatorial explanation. The problem of finding a combinatorial explanation was mentioned by Stanley in [Stan 05]. Our main goal in this chapter is to provide such an explanation.

Formula (70) for the number of Kreweras walks is to be compared to another formula proved the same year. In 1965, Mullin, following the seminal steps of Tutte, proved via a generating function approach [Mull 65] that the number of loopless triangulations of size n (see below for precise definitions) is

$$t_n \quad = \quad \frac{2^n}{(n+1)(2n+1)}\binom{3n}{n}. \tag{71}$$

A bijective proof of (71) was outlined by Schaeffer in his Ph.D thesis [Scha 98]. See also [Poul 03a] for a more general construction concerning loopless triangulations of a k-gon. We will give an alternative bijective proof for the number of loopless triangulations. Technically speaking, we will work instead on *bridgeless cubic maps* which are the dual of loopless triangulations.

It is interesting to observe that both (70) and (71) admit a nice generalization. Indeed, the number $k_{n,i}$ of Kreweras walks of size n ending at point $(i,0)$ and the number $c_{n,i}$ of loopless triangulations of size n of an $(i+2)$-gon both admit a closed formula (see (77) and (78)). Moreover, the numbers $k_{n,i}$ and $c_{n,i}$ are related by the equation $k_{n,i} = 2^n c_{n,i}$. This relation is explained in Section 2.8. Alas, we have found no way of proving these formulas by our approach.

2.2 How the proofs work

We begin with an account of this chapter's content in order to underline the (slightly unusual) logical structure of our proofs.

• In Section 2.3, we first recall some definitions on planar maps. We also define a special class of spanning trees called *depth-first search trees* or *dfs-trees* for short. Dfs-trees are closely related to the trees that can be obtained by a *depth-first search* algorithm.

Then, we consider a larger family of walks containing the Kreweras walks. These walks are made of West, South and North-East steps, start from the origin and remain in the half-plane $i + j \geq 0$. We borrow a terminology from probability theory and call these walks *meanders*. We call *excursion* a meander ending on the second diagonal (i.e. the line $i + j = 0$). An excursion is represented in Figure 49.

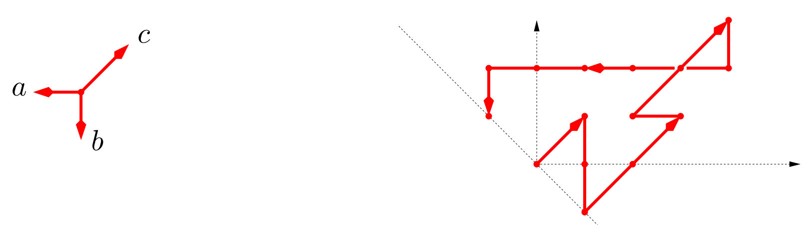


Figure 49: An excursion.

Unlike Kreweras walks, excursions are easy to count. By applying the cycle lemma (see [Stan 99, Section 5.3]), we prove that the number of excursions of size n (length $3n$) is

$$e_n = \frac{4^n}{2n+1}\binom{3n}{n}.$$

• In Section 2.4, we define a mapping Φ between excursions and cubic maps with a distinguished dfs-tree. In Section 2.5 we prove that the mapping Φ is a $(n+1)$-to-1 correspondence

Φ between excursions (of size n) and bridgeless cubic maps (of size n) with a distinguished dfs-tree. As a consequence, the number of bridgeless cubic maps of size n with a distinguished dfs-tree is found to be:

$$d_n = \frac{e_n}{n+1} = \frac{4^n}{(n+1)(2n+1)}\binom{3n}{n}.$$

• In Section 2.6, we prove that the correspondence Φ, restricted to Kreweras walks, induces a *bijection* between Kreweras walks (of size n) ending at the origin and bridgeless cubic maps (of size n) with a distinguished dfs-tree. As a consequence, we obtain:

$$k_n = d_n = \frac{4^n}{(n+1)(2n+1)}\binom{3n}{n},$$

where k_n is the number of Kreweras walks of size n ending at the origin. This gives a combinatorial proof of (70).

• In Section 2.7, we enumerate dfs-trees on cubic maps. We prove that the number of such trees for a cubic map of size n is 2^n. As a consequence, the number of cubic maps of size n is

$$c_n = \frac{d_n}{2^n} = \frac{2^n}{(n+1)(2n+1)}\binom{3n}{n}.$$

This gives a combinatorial proof of (71).

• In Section 2.8, we extend the mapping Φ to Kreweras walks ending at $(i,0)$ and discuss some open problems.

2.3 Preliminaries

2.3.1 Planar maps and dfs-trees

Planar maps. A *planar map*, or *map* for short, is an embedding of a connected planar graph in the sphere without intersecting edges, defined up to orientation preserving homeomorphisms of the sphere. Loops and multiple edges are allowed. The *faces* are the connected components of the complement of the graph. By removing the midpoint of an edge we obtain two *half-edges*, that is, one-dimensional cells incident to one vertex. We say that each edge has two *half-edges*, each of them incident to one of the endpoints.

A map is *rooted* if one of its half-edges is distinguished as the *root*. The edge containing the root is the *root-edge* and its endpoint is the *root-vertex*. Graphically, the root is indicated by an arrow pointing on the root-vertex (see Figure 50). All the maps considered in this chapter are rooted and we shall not further precise it.

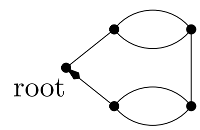


Figure 50: A rooted map.

Growing maps. Our constructions lead us to consider maps with some *legs*, that is, half-edges that are not part of a complete edge. A *growing map* is a (rooted) map together with some legs, one of them being distinguished as the *head*. We require the legs to be (all) in the same face called *head-face*. The endpoint of the head is the *head-vertex*. Graphically, the head is indicated by an arrow pointing away from the head-vertex. The root of a growing map can be the head, another leg or a regular half-edge. For instance, the growing map in Figure 51 has 2 legs beside the head, and its root is not a leg.

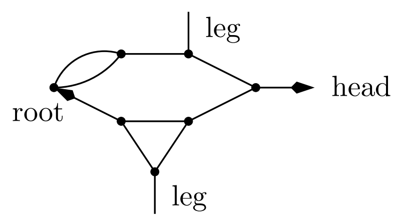


Figure 51: A growing map.

Cubic maps. A map (or growing map) is *cubic* if every vertex has degree 3. It is k-near-cubic if the root-vertex has degree k and any other vertex has degree 3. For instance, the map in Figure 50 is 2-near-cubic and the growing map in Figure 51 is cubic. Observe that cubic maps are in bijection with 2-near-cubic maps not reduced to a loop by the mapping illustrated in Figure 52.

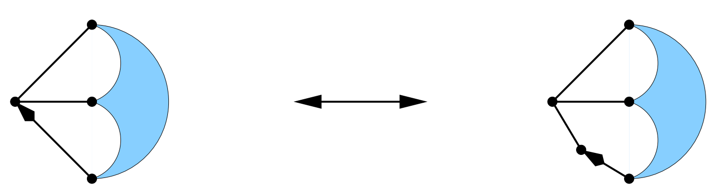


Figure 52: Bijection between cubic maps and 2-near-cubic maps.

The incidence relation between vertices and edges in cubic maps shows that the number of edges is always a multiple of 3. More generally, if M is a k-near-cubic map with e edges and v vertices, the incidence relation reads: $3(v-1)+k = 2e$. Equivalently, $3(v-k+1) = 2(e-2k+3)$.

The number $v - k + 1$ is non-negative for *non-separable* k-near-cubic maps (see definition below). (This property can be shown by induction on the number of edges by contracting the root-edge.) Hence, the number of edges has the form $e = 3n + 2k - 3$, where n is a non-negative integer. We say that a k-near-cubic map has *size* n if it has $e = 3n + 2k - 3$ edges (and $v = 2n + k - 1$ vertices). In particular, the mapping of Figure 52 is a bijection between cubic maps of size n ($3n+3$ edges) and 2-near-cubic maps of size $n+1$ ($3n+4$ edges).

Non-separable maps. A map is *non-separable* if its edge set cannot be partitioned into two non-empty parts such that only one vertex is incident to some edges in both parts. In particular, a non-separable map not reduced to an edge has no loop nor bridge (a *bridge* or *isthmus* is an edge whose deletion disconnects the map). For cubic maps and 2-near-cubic maps it is equivalent to be non-separable or bridgeless. The mapping illustrated in Figure 52 establishes a bijection between bridgeless cubic maps and bridgeless 2-near-cubic maps not reduced to a loop.

Bridgeless cubic maps are interesting because their *dual* are the loopless triangulations. Recall that the dual M^* of a map M is the map obtained by putting a vertex of M^* in each face of M and an edge of M^* across each edge of M. See Figure 53 for an example.

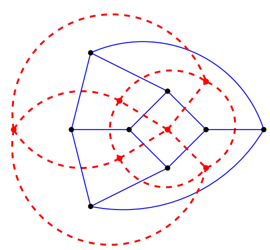


Figure 53: A cubic map and the dual triangulation (dashed lines).

Dfs-trees. A *tree* is a connected graph without cycle. A subgraph T of a graph G is a *spanning tree* if it is a tree containing every vertex of G. An edge of the graph G is said to be *internal* if it is in the spanning tree T and *external* otherwise. For any pair of vertices u, v of the graph G, there is a unique path between u and v in the spanning tree T. We call it the *T-path* between u and v. A map (or growing map) M with a distinguished spanning tree T will be denoted by M_T. Graphically, we shall indicate the spanning tree by thick lines as in Figure 54. A vertex u of M_T is an *ancestor* of another vertex v if it is on the T-path between the root-vertex and v. In this case, v is a *descendant* of u. Two vertices are *comparable* if one is the ancestor of the other. For instance, in Figure 54, the vertices u_1 and v_1 are comparable whereas u_2 and v_2 are not.

A *dfs-tree* is a spanning tree such that any external edge joins comparable vertices. Moreover, we require the edge containing the root to be external. In Figure 54, the tree on the left side is a dfs-tree but the tree on the right side is not a dfs-tree since the edge (u_2, v_2) breaks the rule. The dfs-trees are strongly related to the *depth-first search* algorithm (see Section 2.7) and are also known as the *Trémaux trees* [Fray 82, Fray 85]. A *dfs-map* is a map with a distinguished dfs-tree. A *marked-dfs-map* is a dfs-map with a marked external edge.

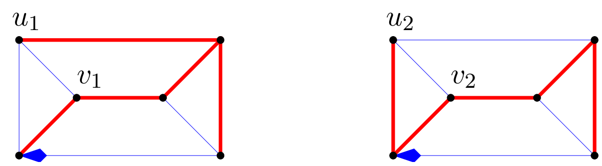


Figure 54: A dfs-tree (left) and a non-dfs-tree (right).

2.3.2 Kreweras walks and meanders

In what follows, Kreweras walks are considered as words on the alphabet $\{a, b, c\}$. The letter a (resp. b, c) corresponds to a West (resp. South, North-East) step. For instance, the walk in Figure 48 is *cbcccbbcaaaaabb*. The length of a word w is denoted by $|w|$ and the number of occurrences of a given letter α is denoted by $|w|_\alpha$. Kreweras walks are the words w on the alphabet $\{a, b, c\}$ such that any prefix w' of w satisfies

$$|w'|_a \leq |w'|_c \quad \text{and} \quad |w'|_b \leq |w'|_c \ . \tag{72}$$

Kreweras walks ending at the origin satisfy the additional constraint

$$|w|_a \ = \ |w|_b \ = \ |w|_c. \tag{73}$$

These conditions can be interpreted as a ballot problem with three candidates. This is why Kreweras walks sometimes appear under this formulation in the literature [Nied 83].

Similarly, the meanders, that is, the walks remaining in the half-plane $i + j \geq 0$, are the words w on $\{a, b, c\}$ such that any prefix w' of w satisfies

$$|w'|_a \ + \ |w'|_b \ \leq \ 2|w'|_c \ . \tag{74}$$

Excursions, that is, meanders ending on the second diagonal, satisfy the additional constraint

$$|w|_a \ + \ |w|_b \ = \ 2|w|_c \ . \tag{75}$$

Note that the length of any walk ending on the second diagonal is a multiple of 3. The *size* of such a walk of length $3n$ is n. Note also that a walk ending at point $(i, 0)$ has a length of the form $l = 3n + 2i$ where n is a non-negative integer. A Kreweras walk of length $l = 3n + 2i$ ending at $(i, 0)$ has *size* n.

Unlike Kreweras walks, the excursions are easy to count.

Proposition 2.1 *There are*

$$e_n = \frac{4^n}{2n+1}\binom{3n}{n} \tag{76}$$

excursions of size n.

Proof: We consider *projected walks*, that is, one-dimensional lattice walks starting and ending at 0, remaining non-negative and made of steps $+2$ and -1. (They correspond to projections of excursions on the first diagonal.) A projected walk is represented in Figure 55. Projected walks can be seen as words w on the alphabet $\{\alpha, c\}$ with $|w|_\alpha = 2|w|_c$ and such that any prefix w' of w satisfies $|w'|_\alpha \leq 2|w'|_c$. The projected walks can be counted bijectively by applying the cycle lemma (see Section 5.3 of [Stan 99]): there are

$$p_n = \frac{1}{3n+1}\binom{3n+1}{2n+1} = \frac{1}{2n+1}\binom{3n}{n}$$

projected walks of size n (length $3n$).
Given an excursion, we obtain a projected walk by replacing the occurrences of a and b by α. Conversely, taking a projected walk of length $3n$ and replacing the $2n$ letters α by a sequence of letters in $\{a, b\}$ one obtains an excursion. This establishes a 4^n-to-1 correspondence between excursions (of size n) and projected walks (of size n). Thus, there are $4^n p_n$ excursions of size n.

$\square$

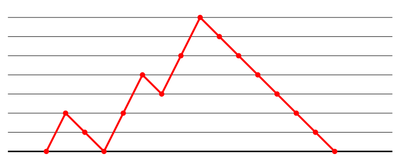


Figure 55: The projected walk associated to the excursion of Figure 49.

2.4 A bijection between excursions and cubic marked-dfs-maps

In this section we define a mapping Φ between excursions and bridgeless 2-near-cubic *marked-dfs-maps* (2-near-cubic maps with a distinguished dfs-tree and a marked external edge). We shall prove in Section 2.5 that the mapping Φ is a bijection between excursions and bridgeless 2-near-cubic marked-dfs-maps. The general principle of the mapping Φ is to read the excursion from right to left and interpret each letter as an operation for constructing the map and the tree. This step-by-step construction is illustrated in Figure 57. The intermediate steps are *tree-growing maps*, that is, growing maps together with a distinguished spanning tree (indicated by thick lines).

• We start with the tree-growing map $M_\bullet^0$ consisting of one vertex and two legs. One of the legs is the root, the other is the head (see Figure 56). The spanning tree is reduced to the unique vertex.

• We apply successively certain elementary mappings φ_a, φ_b, φ_c (Definition 2.2) corresponding to the letters a, b, c of the excursion read *from right to left*.

• When the whole excursion is read, there is only one leg remaining beside the head. At this stage, we *close* the tree-growing map, that is, we glue the head and the remaining leg into a *marked* external edge as shown in Figure 58.

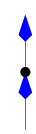


Figure 56: The tree-growing map $M_\bullet^0$.

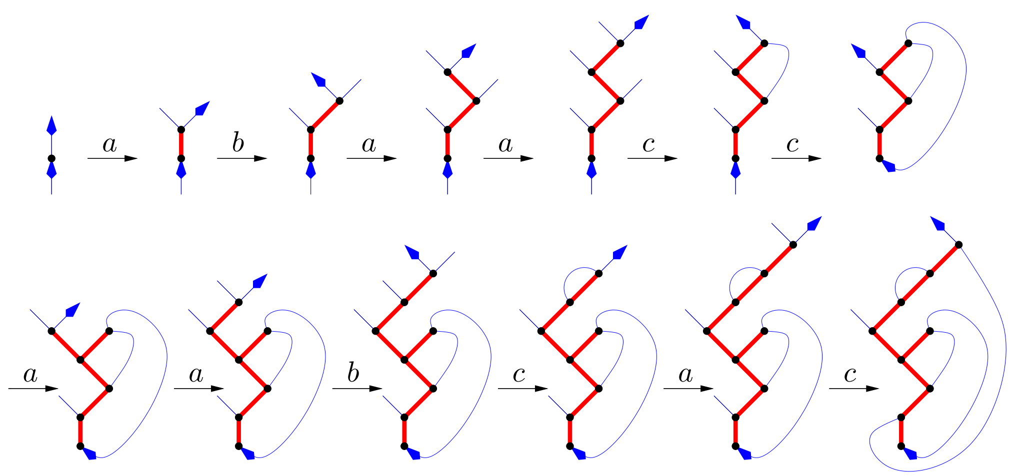


Figure 57: Successive applications of the mappings φ_a, φ_b, φ_c for the walk *cacbaaccaaba* (read from right to left).

Let us enter in the details and define the mapping Φ. Consider a growing map M. We *make a tour* of the head-face if we follow its border in counterclockwise direction (i.e. the

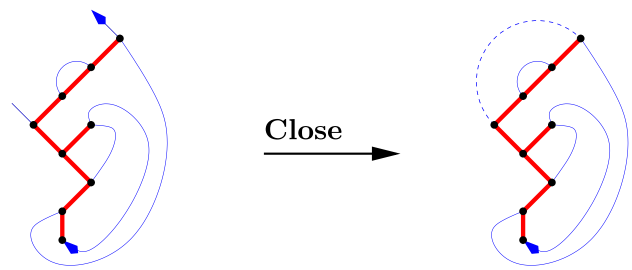


Figure 58: Closing the map (the marked edge is dashed).

border of the head-face stays on our left-hand side) starting from the head (see Figure 59). This journey induces a linear order on the legs of M. We shall talk about the *first* and *last* legs of M.

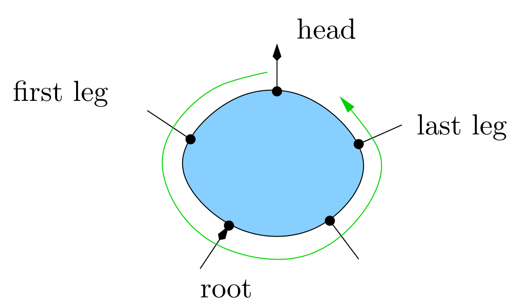


Figure 59: Making the tour of the head-face.

We define three mappings φ_a, φ_b, φ_c on tree-growing maps.

Definition 2.2 *Let M_T be a tree-growing map (the map is M and the distinguished tree is T).*

- *The mappings φ_a and φ_b are represented in Figure 60. The tree-growing map $M'_{T'} = \varphi_a(M_T)$ (resp. $\varphi_b(M_T)$) is obtained from M_T by replacing the head by an edge e together with a new vertex v incident to the new head and another leg at its left (resp. right). The tree T' is obtained from T by adding the edge e and the vertex v.*

- *The tree-growing map $\varphi_c(M_T)$ is only defined if the first and last legs exist (that is, if the head-face contains some legs beside the head) and have distinct and comparable endpoints. We call these legs s and t* with the convention that the endpoint of s is an ancestor of the endpoint of t.
In this case, the tree-growing map $M'_T = \varphi_c(M_T)$ is obtained from M_T by gluing together the head and the leg s while the leg t becomes the new head (see Figure 61). The spanning tree T is unchanged.

• *For a word $w = a_1a_2 \dots a_n$ on the alphabet $\{a, b, c\}$, we denote by φ_w the mapping $\varphi_{a_1} \circ \varphi_{a_2} \circ \cdots \circ \varphi_{a_n}$.*

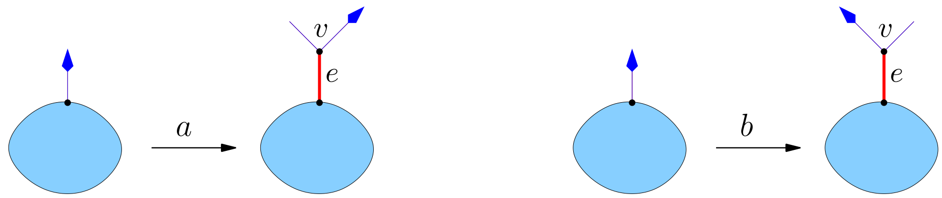


Figure 60: The mappings φ_a and φ_b.

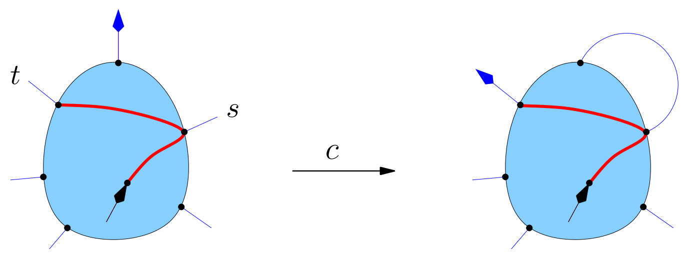


Figure 61: The mapping φ_c.

Definition 2.3 *The image of an excursion w by the mapping Φ is the map with a distinguished spanning tree and a marked external edge obtained by closing the tree-growing map $\varphi_w(M_\bullet^0)$, that is, by gluing the head and the unique remaining leg into a marked edge.*

The mapping Φ has been applied to the excursion *cacbaaccaaba* in Figure 57 and 58. Of course, we still need to prove that the mapping Φ is well defined.

Proposition 2.4 *The mapping Φ is well defined on any excursion w:*
• *It is always possible to apply the mapping φ_c when required.*
• *The tree-growing map $\varphi_w(M_\bullet^0)$ has exactly one leg beside the head. This leg and the head are both in the head-face, hence can be glued together.*

Before proving Proposition 2.4, we need three technical results.

Lemma 2.5 *Let w be a word on the alphabet $\{a, b, c\}$ such that $\varphi_w(M_\bullet^0)$ is well defined. Then, $\varphi_w(M_\bullet^0)$ is a tree-growing map.*

Proof: Let $M_T = \varphi_w(M_\bullet^0)$. It is clear by induction that T is a spanning tree. The only point to prove is that the legs of $\varphi_w(M_\bullet^0)$ are in the head-face. We proceed by induction on the length of w. This property holds for the empty word. If the property holds for $M_T = \varphi_w(M_\bullet^0)$ it clearly holds for $\varphi_a(M_T)$ and $\varphi_b(M_T)$. If φ_c can be applied, the head is glued either to the first or to the last leg of M_T. Thus, all the remaining legs (including the head of $\varphi_c(M_T)$) are in the same face. □

We shall see shortly (Lemma 2.7) that whenever the tree-growing map $\varphi_w(M_\bullet^0)$ is well defined, the endpoints of any leg is an ancestor of the head-vertex. Observe that in this case the endpoints of the legs are comparable.

Lemma 2.6 *Let M_T be a tree-growing map. Suppose that the endpoint of any leg is an ancestor of the head-vertex. Suppose also that the first and last legs exist and have distinct endpoints. We call these endpoints u and v with the convention that u is an ancestor of v. Then, v is the last vertex incident to a leg on the T-path from the root-vertex to the head-vertex.*

Proof: The situation is represented in Figure 62. We make an induction on the number of edges that are not in the T-path P from the root-vertex to the head-vertex. The property is clearly true if the tree-growing map is reduced to the path P plus some legs. If not, the deletion of a edge not in P does not change the order of appearance of the legs around the head-face. In particular, the first and last legs are unchanged. □

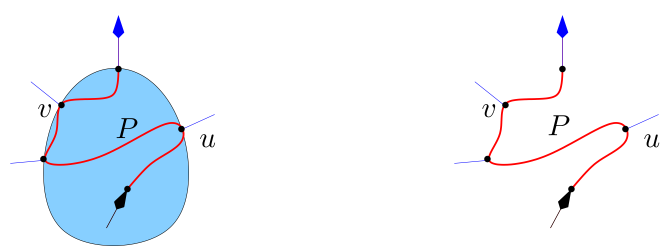


Figure 62: The last vertex incident to a leg on the T-path from the root-vertex to the head-vertex is v.

Lemma 2.7 *Let w be a word on the alphabet $\{a, b, c\}$ such that $\varphi_w(M_\bullet^0)$ is defined. Then the endpoint of any leg of $\varphi_w(M_\bullet^0)$ is an ancestor of the head-vertex.*

Proof: We proceed by induction on the length of w. The property holds for the empty word. We suppose that it holds for $M_T = \varphi_w(M_\bullet^0)$. It is clear that the property holds for the tree-growing maps $\varphi_a(M_T)$ and $\varphi_b(M_T)$. If φ_c can be applied, the endpoints of the first and last leg are distinct and comparable. We call these endpoints u and v with the convention that u is an ancestor of v. By the induction hypothesis, the conditions of Lemma 2.6 are satisfied by M_T. Therefore, the vertex v is the last vertex incident to a leg on the T-path from the root-vertex to the head-vertex. Hence, any endpoint of a leg of $\varphi_c(M_T)$ is an ancestor of v which is the head-vertex of $\varphi_c(M_T)$. □

Proof of Proposition 2.4: Let w be an excursion. We consider a suffix w' of w and denote by $M'_T = \varphi_{w'}(M^0_\bullet)$ the corresponding tree-growing map (if it is well defined).

• *If M'_T is well defined, it has $|w'|_a + |w'|_b - 2|w'|_c + 1$ legs besides the head.* (Observe that, by (74) and (75), the quantity $|w'|_a + |w'|_b - 2|w'|_c$ is non-negative.)
We proceed by induction on the length of w'. The property holds for the empty word. Moreover, applying φ_a or φ_b increases by 1 the number of legs whereas applying φ_c decreases this number by 2. Thus, the property follows easily by induction.

• *The tree-growing map M'_T is well defined.*
We proceed by induction on the length of w'. The property holds for the empty word. We write $w' = \alpha w''$ and suppose that $M''_T = \varphi_{w''}(M^0_\bullet)$ is well defined. If $\alpha = a$ or b the tree-growing map $M'_T = \varphi_\alpha(M''_T)$ is well defined. We suppose now that $\alpha = c$. The tree-growing map M''_T has $|w''|_a + |w''|_b - 2|w''|_c + 1 = |w'|_a + |w'|_b - 2|w'|_c + 3 > 2$ legs besides the head. It is clear by induction that all these legs have distinct endpoints. Moreover, by Lemma 2.7, all the endpoints of these legs are ancestors of the head-vertex. Thus the endpoints of the legs are comparable. In particular, the endpoints of the first and last legs are comparable. Hence, the mapping φ_c can be applied.

• *The tree-growing map $M_T = \varphi_w(M^0_\bullet)$ is well defined and has exactly one leg beside the head.*
This property follows from the preceding points since $|w|_a + |w|_b - 2|w|_c = 0$. □

We now state the key result of this chapter.

Theorem 2.8 *The mapping Φ is a bijection between excursions of size n and bridgeless 2-near-cubic marked-dfs-maps of size n.*

The proof of Theorem 2.8 is postponed to the next section. For the time being we explore its enumerative consequences. We denote by d_n the number of bridgeless 2-near-cubic dfs-maps of size n. Consider a 2-near-cubic map M of size n ($3n+1$ edges, $2n+1$ vertices) and a spanning tree T. Since T has $2n+1$ vertices, M_T has $2n$ internal edges and $n+1$ external edges. Hence, there are $(n+1)d_n$ bridgeless 2-near-cubic marked-dfs-maps. By Theorem 2.8, this number is equal to the number e_n of excursions of size n. Using Proposition 2.1, we obtain the following result.

Corollary 2.9 *There are* $d_n = \dfrac{e_n}{n+1} = \dfrac{4^n}{(n+1)(2n+1)}\dbinom{3n}{n}$ *bridgeless 2-near-cubic dfs-maps of size n.*

Observe that d_n is also the number of bridgeless cubic dfs-maps of size $n-1$ since the bijection between cubic maps and 2-near-cubic maps represented in Figure 52 can be turned into a bijection between cubic dfs-maps and 2-near-cubic dfs-maps.

2.5 Why the mapping Φ is a bijection

In this section, we prove that the mapping Φ is a bijection between excursions and bridgeless 2-near-cubic marked-dfs-maps. We first prove that the image of any excursion by the mapping Φ is a bridgeless 2-near-cubic marked-dfs-map (Proposition 2.10). Then we define a mapping Ψ from bridgeless 2-near-cubic marked-dfs-maps to excursions (Definition 2.13) and prove that Φ and Ψ are inverse mappings (Propositions 2.16 and 2.18).

Proposition 2.10 *The image $\Phi(w)$ of any excursion w is a bridgeless 2-near-cubic marked-dfs-map.*

Proof: Let w' be a suffix of w and let $M'_T = \varphi_{w'}(M^0_\bullet)$ be the corresponding tree-growing map.

• *The tree-growing map M'_T is 2-near-cubic.*
Applying φ_a or φ_b creates a new vertex of degree 3 and does not change the degree of the other vertices. Applying φ_c does not affect the degree of the vertices. The property follows by induction.

• *The head and the root of M'_T are distinct half-edges.*
The property holds for the empty word. We now write $w' = \alpha w''$. If $\alpha = a$ or b the property clearly holds for w'. Suppose now that $\alpha = c$. Let u and v be the vertices incident to the first and last legs of $M''_T = \varphi_{w''}(M^0_\bullet)$ with the convention that u is an ancestor of v. By definition, v is the head-vertex of $M'_T = \varphi_c(M''_T)$ and is a proper descendant of u. Hence, the head-vertex v and the root-vertex of M'_T are distinct.

• *The tree T is a dfs-tree of M'_T.*
The external edges are created by applying the mapping φ_c, that is, by gluing the head to another leg. By Lemma 2.7, any vertex incident to a leg is an ancestor of the head-vertex. Hence, any external edge joins comparable vertices. Moreover, by the preceding point, if the root is part of a complete edge, then this edge is external (internal edges are created by the mappings φ_a or φ_b which replace the head by a complete edge).

• Let u_0 be the first vertex of M'_T incident to a leg on the T-path from the root-vertex to the head-vertex. *Any isthmus of M'_T is in the T-path between u_0 and the head-vertex.*
We proceed by induction on the length of w'. The property holds for the empty word. We write $w' = \alpha w''$ and suppose that it holds for $M''_T = \varphi_{w''}(M^0_\bullet)$. If $\alpha = a$ or b the property clearly holds for $M'_T = \varphi_\alpha(M''_T)$. We suppose now that $\alpha = c$. We denote by u_1 the first vertex of M''_T incident to a leg on the T-path from the root-vertex to the head-vertex. Let u and v be the vertices incident to the first and last legs of M''_T with the convention that u is an ancestor of v. By Lemma 2.7, the vertices u_1, u and v are all ancestors of the head-vertex v_1 of M''_T. Hence, u and v are on the T-path between u_1 and v_1. This situation is represented in Figure 63. By definition, the tree-growing map M'_T is obtained from M''_T by creating an

edge e_1 between u and v_1 while v becomes the new head-vertex. We denote by P_1 (resp. P_2) the T-path between u_1 and u (resp. u and v_1). We consider an isthmus e of M'_T. The edge e is an isthmus of M''_T (since M''_T is obtained from M_T by deleting an edge). By the induction hypothesis, the isthmus e is either in P_1 or in P_2. The edge e is not in the path P_2 since the new edge e_1 creates a cycle with P_2. The isthmus e is in P_1, therefore the vertices u_1 and u are distinct. Hence $u_1 = u_0$ is the first vertex of M'_T incident to a leg on the T-path from the root-vertex to the head-vertex. Thus, the isthmus e is in the T-path from u_0 to the head-vertex v of M'_T.

• *The dfs-map* $\Phi(w)$ *has no isthmus.*
By the preceding points, any isthmus of $M_T = \varphi_w(M^0_\bullet)$ is on the T-path between the head-vertex and the endpoint of the only remaining leg. Hence, no isthmus remains once the map closed.

□

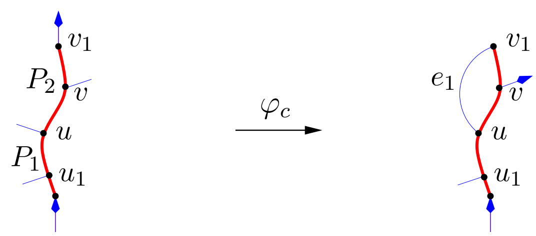


Figure 63: Isthmuses are in the T-path between u_0 and the head-vertex.

We will now define a mapping Ψ (Definition 2.13) that we shall prove to be the inverse of Φ. The mapping Ψ destructs the tree-growing map that Φ constructs and recovers the walk. Looking at Figure 57 from bottom-to-top and right-to-left we see how Ψ works.

We first define three mappings ψ_a, ψ_b, ψ_c on tree-growing maps that we shall prove to be the inverse of φ_a, φ_b and φ_c respectively. We consider the following conditions for a tree-growing map M_T:
(a) The head-vertex has degree 3 and is incident to an edge and a leg at the left of the head.
(b) The head-vertex has degree 3 and is incident to an edge and a leg at the right of the head.
(c) The head-vertex has degree 3 and is incident to 2 edges which are not isthmuses. Furthermore, the tree T is a dfs-tree.

The conditions (a), (b), (c) are the domain of definition of ψ_a, ψ_b, ψ_c respectively. Before defining these mappings we need a technical lemma.

Lemma 2.11 *If Condition* (c) *holds for the tree-growing map* M_T*, then there exists a unique external edge* e_0 *incident to the head-face with one endpoint* u *ancestor of the head-vertex and* *one endpoint v_0 descendant of the head-vertex.*

Lemma 2.11 is illustrated by Figure 64.

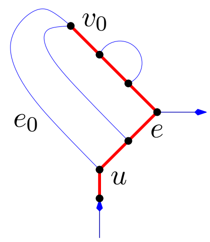


Figure 64: The unique edge e_0 satisfying the conditions of Lemma 2.11.

Proof: We suppose that M_T satisfies Condition (c). One of the two edges incident to the head-vertex is in the T-path from the root-vertex to the head-vertex. Denote it e. The edge e separates the tree T in two subtrees T_1 and T_2. We consider the set E_0 of external edges having one endpoint in T_1 and the other in T_2. Any edge satisfying the conditions of Lemma 2.11 is in E_0. Since e is not an isthmus, the set E_0 is non-empty. Moreover, any edge in E_0 has one endpoint that is a descendant of the head-vertex. Since T is a dfs-tree, the other endpoint is an ancestor of the head-vertex. It remains to show that there is a unique edge e_0 in E_0 incident to the head-face. By contracting every edge in T_1 and T_2 we obtain a map with 2 vertices. The edges incident to both vertices are precisely the edges in $E_0 \cup \{e\}$. It is clear that exactly 2 of these edges are incident to the head-face. One is the internal edge e and the other is an external edge $e_0 \in E_0$. This edge e_0 is the only external edge satisfying the conditions of Lemma 2.11.

□

We are now ready to define the mappings ψ_a, ψ_b and ψ_c.

Definition 2.12 *Let M_T be a tree-growing map.*

- *The tree-growing map $M'_{T'} = \psi_a(M_T)$ (resp. $\psi_b(M_T)$) is defined if Condition (a) (resp. (b)) holds. In this case, the tree-growing map $M'_{T'}$ is obtained by suppressing the head-vertex v and the 3 incident half-edges. The other half of the edge incident to v becomes the new head.*

- *The tree-growing map $M'_{T'} = \psi_c(M_T)$ is defined if Condition (c) holds. In this case, we consider the unique external edge e_0 with endpoints u, v_0 satisfying the conditions of Lemma 2.11. The edge e_0 is broken into two legs. The leg incident to v_0 becomes the new head (the former head becomes an anonymous leg).*

- *For a word $w = a_1 a_2 \dots a_n$ on the alphabet $\{a, b, c\}$, we denote by ψ_w the mapping $\psi_{a_n} \circ \psi_{a_{n-1}} \circ \cdots \circ \psi_{a_1}$. Moreover, we say that the word w is* readable *on a tree-growing map*

M_T if the mapping ψ_w is well defined on M_T.

Remarks:

- Applying one of the mappings ψ_a, ψ_b or ψ_c to a 2-near-cubic map cannot delete the root (only half-edges incident to a vertex of degree 3 can disappear by application of ψ_a or ψ_b).
- The conditions (a), (b), (c) are incompatible. Thus, for any tree-growing map M_T, there is at most one readable word of a given length.
- Applying the mapping ψ_a, ψ_b or ψ_c decreases by one the number of edges. Therefore, the length of any readable word on a tree-growing map M_T is less than or equal to the number of edges in M_T.

We now define the mapping Ψ on bridgeless 2-near-cubic marked-dfs-maps. Let M_T be such a map and let e be the marked (external) edge. Observe first that, unless M_T is reduced to a loop, the edge e has two distinct endpoints (or the endpoint of e would be incident to an isthmus). We denote by u and v the endpoints of e with the convention that u is an ancestor of v. We *open* this map if we disconnect the edge e into two legs and choose the leg incident to v to be the head. We denote by $M_T^{\dashv\vdash}$ the tree-growing map obtained by opening M_T. By convention, opening the 2-near-cubic marked-dfs-map reduced to a loop gives $M_{\bullet}^0$. Note that we obtain M_T by *closing* $M_T^{\dashv\vdash}$. We now define the mapping Ψ.

Definition 2.13 *Let M_T be a bridgeless 2-near-cubic marked-dfs-map. The word $\Psi(M_T)$ is the longest word readable on $M_T^{\dashv\vdash}$.*

We want to prove that Φ and Ψ are inverse mappings. We begin by proving that the mapping ψ_α is the inverse of φ_α for $\alpha = a, b, c$.

We say that a tree-growing map satisfies Condition (c') if it satisfies Condition (c) and is such that the endpoint of every leg is an ancestor of the head-vertex.

Lemma 2.14

- *For $\alpha = a$ or b, the mapping $\psi_\alpha \circ \varphi_\alpha$ is the identity on all tree-growing maps and the mapping $\varphi_\alpha \circ \psi_\alpha$ is the identity on tree-growing maps satisfying Condition (α).*
- *The mapping $\psi_c \circ \varphi_c$ is the identity on tree-growing maps such that the endpoints of the first and last legs exist and are distinct ancestors of the head-vertex. The mapping $\varphi_c \circ \psi_c$ is the identity on tree-growing maps satisfying Condition (c').*

Before proving Lemma 2.14, we need the following technical result.

Lemma 2.15 *Let M_T be a tree-growing map satisfying Condition (c') and let e_0 be the edge with endpoints u, v_0 satisfying the conditions of Lemma 2.11. By definition, the tree-growing* *map $\psi_c(M_T)$ is obtained by breaking e_0 into two legs s and h incident to u and v_0 respectively while h becomes the new head. The pair of first and last legs of $\psi_c(M_T)$ is the pair $\{s, t\}$, where t is the head of M_T.*

Lemma 2.15 is illustrated by Figure 65.

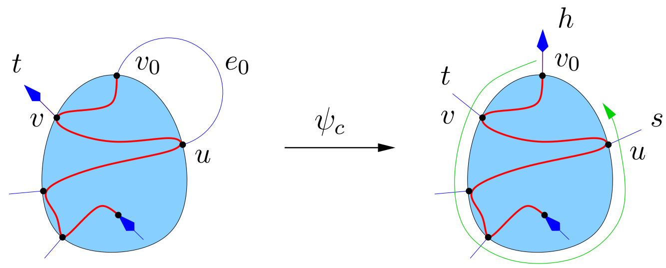


Figure 65: The pair of first and last legs of the tree-growing map $\psi_c(M_T)$ is the pair $\{s, t\}$.

Proof of Lemma 2.15:

• Let v be the head-vertex of M_T (i.e. the endpoint of t). By Condition (c'), the endpoint of any leg of M_T is an ancestor of v. Therefore, in the tree-growing map $\psi_c(M_T)$, the vertex v is the last vertex incident to a leg on the T-path from the root-vertex to the head-vertex v_0. Hence, by Lemma 2.6, the leg t is either the first or the last leg of $\psi_c(M_T)$.

• No leg lies between s and h on the tour of the head-face of $\psi_c(M_T)$ since this leg would have been inside a non-head face of M_T. Thus the leg s is either the first or the last leg of $\psi_c(M_T)$.

□

Proof of Lemma 2.14:

• For $\alpha = a$ or b, it is clear from the definitions that $\varphi_\alpha \circ \psi_\alpha$ is the identity mapping on all tree-growing maps and that $\varphi_\alpha \circ \psi_\alpha$ is the identity on tree-growing maps satisfying Condition (α).

• Consider a tree-growing map M_T such that the endpoints of the first and last legs exist and are distinct ancestors of the head-vertex v_0. We call these legs s and t with the convention that the endpoint u of s is an ancestor of the endpoint v of t. By definition, $\varphi_c(M_T)$ is obtained by gluing the head of M_T to s while t becomes the new head. Let e_0 be the external edge created by gluing the head to s. The head-vertex v of the tree-growing map $\varphi_c(M_T)$ is on the cycle made of e_0 and the T-path between its two endpoints u and v_0, thus $\varphi_c(M_T)$ satisfies Condition (c). Moreover, the external edge e_0 satisfies the conditions of Lemma 2.11. Thus, $\psi_c \circ \varphi_c(M_T) = M_T$.

• We consider a tree-growing map M_T satisfying Condition (c'). We consider the edge e_0 with endpoints $u,\ v_0$ satisfying the conditions of Lemma 2.11. By definition, $\psi_c(M_T)$ is obtained by breaking e_0 into two legs s and h incident to u and v_0 respectively while h becomes the new head. By Lemma 2.15, the pair of first and last legs of $\psi_c(M_T)$ is $\{s,t\}$. Moreover, the endpoint u of s is an ancestor of the endpoint v of t (by definition of e_0, u, v_0 in Lemma 2.11). Therefore, the identity $\varphi_c \circ \psi_c(M_T) = M_T$ follows from the definitions. $\square$

Proposition 2.16 *The mapping $\Psi \circ \Phi$ is the identity on excursions.*

Proof:
• *For any word w on the alphabet $\{a,b,c\}$ such that the tree-growing map $\varphi_w(M_\bullet^0)$ is well defined, the word w is readable on $\varphi_w(M_\bullet^0)$ and $\psi_w \circ \varphi_w(M_\bullet^0) = M_\bullet^0$.*
We proceed by induction on the length of w. The property holds for the empty word. We write $w = \alpha w'$ with $\alpha = a, b$ or c and suppose that it holds for w'. Let $M'_T = \varphi_{w'}(M_\bullet^0)$. If $\alpha = c$, the endpoints of the first and last legs of M'_T are distinct and comparable (since φ_c is defined on M'_T). Moreover, we know by Lemma 2.7 that these endpoints are ancestors of the head-vertex. Thus, for $\alpha = a, b$ or c, Lemma 2.14 ensures that $\psi_\alpha \circ \varphi_\alpha(M'_T) = M'_T$. Therefore,

$$\psi_{\alpha w'} \circ \varphi_{\alpha w'}(M_\bullet^0) = \psi_{w'} \circ \psi_\alpha \circ \varphi_\alpha \circ \varphi_{w'}(M_\bullet^0) = \psi_{w'} \circ \psi_\alpha \circ \varphi_\alpha(M'_T) = \psi_{w'}(M'_T),$$

and $\psi_{w'}(M'_T) = M_\bullet^0$ by the induction hypothesis.

• *For any excursion w, we have $\Psi \circ \Phi(w) = w$.*
By definition, the map $M_T = \Phi(w)$ is obtained by closing $\varphi_w(M_\bullet^0)$. In order to conclude that $M_T^{\dashv\vdash} = \varphi_w(M_\bullet^0)$, we only need to check that the head of $M_T^{\dashv\vdash}$ is the head of $\varphi_w(M_\bullet^0)$ (and the non-head leg of $M_T^{\dashv\vdash}$ is the non-head leg of $\varphi_w(M_\bullet^0)$). This is true since the endpoint of the non-head leg of $\varphi_w(M_\bullet^0)$ is an ancestor of the head-vertex by Lemma 2.7. By the preceding point, the word w is readable on $M_T^{\dashv\vdash} = \varphi_w(M_\bullet^0)$ and $\psi_w(M_T) = \psi_w \circ \varphi_w(M_\bullet^0) = M_\bullet^0$. Since no letter is readable on $M_\bullet^0$, the longest word readable on $M_T^{\dashv\vdash}$ is w. Thus, $\Psi \circ \Phi(w) = \Psi(M_T) = w$. $\square$

It remains to show that $\Phi \circ \Psi$ is the identity mapping on bridgeless 2-near-cubic marked-dfs-maps. We first prove that the image of bridgeless 2-near-cubic marked-dfs-maps by Ψ are excursion.

Proposition 2.17 *For any bridgeless 2-near-cubic marked-dfs-map M_T, the longest word w readable on $M_T^{\dashv\vdash}$ is an excursion. Moreover, the tree-growing map $\psi_w(M_T^{\dashv\vdash})$ is $M_\bullet^0$.*

Proof: If M_T is the map reduced to a loop the result is trivial. We exclude this case in what follows. Let w be a word readable on $M_T^{\dashv\vdash}$ and let $N_T = \psi_w(M_T^{\dashv\vdash})$. We denote by u_0 the first vertex of N_T incident to a leg on the T-path from the root-vertex to the head-vertex.

• *Any isthmus of N_T is in the T-path between u_0 and the head-vertex.*
We proceed by induction on the length of w. Suppose first that w is the empty word. Let e_0

be the marked edge of M_T. By definition, the tree-growing map $N_T = M_T^{\dashv\vdash}$ is obtained from M_T by breaking e_0 into two legs: the head and another leg incident to u_0. Let e be an isthmus of N_T and let $N_1,\ N_2$ be the two connected submaps obtained by deleting e. Since e is not an isthmus of M_T, the edge e_0 joins N_1 and N_2. Therefore, the root-vertex and head-vertex are not in the same submap. Thus, the isthmus e is in any path between u_0 and the head-vertex, in particular it is in the T-path.

We now write $w = \alpha w'$ with $\alpha = a, b$ or c and suppose, by the induction hypothesis, that the property holds for w'. We denote by u'_0 the first vertex of $N'_T = \psi_{w'}(M_T^{\dashv\vdash})$ incident to a leg on the T-path from the root-vertex to the head-vertex. Suppose first that $\alpha = a$ or b. The edge incident to the head-vertex of N'_T is an isthmus hence, by the induction hypothesis, it is in the T-path between u'_0 and the head-vertex v'_0 of N'_T. Hence, $u'_0 \neq v'_0$. Thus, $u_0 = u'_0$ and every isthmus of $N_T = \psi_\alpha(N'_T)$ is in the T-path between u_0 and the head-vertex. Suppose now that $\alpha = c$. Since w is readable on $M_T^{\dashv\vdash}$, the tree-growing map $N'_T = \psi_{w'}(M_T^{\dashv\vdash})$ satisfies Condition (c). We consider the edge e_0 with endpoints $u,\ v_0$ satisfying the conditions of Lemma 2.11. The map $N_T = \psi_c(N'_T)$ is obtained from N_T by breaking e_0 into two legs. By definition, the head-vertex of N_T is v_0. Moreover, the vertex u_0 is either u'_0 or u if u is an ancestor of u'_0. We consider an isthmus e of N_T. If e is an isthmus of N'_T, it is in the T-path between u'_0 to the head-vertex of N'_T which is included in the T-path between u_0 and v_0. If e is not an isthmus of N'_T, we consider the two connected submaps $N_1,\ N_2$ obtained from N_T by deleting the isthmus e. Since e is not an isthmus of N'_T, the edge e_0 joins N_1 and N_2. Hence, the endpoints u and v_0 of e_0 are not in the same submap. Thus, the isthmus e is in every path of N_T between u and the head-vertex v_0, in particular, it is in the T-path between u_0 and v_0.

• *The tree-growing map N_T has at least one leg beside the head.*

We proceed by induction. The property holds for the empty word. We now write $w = \alpha w'$ with $\alpha = a, b$ or c and suppose that the property holds for w'. Suppose first that $\alpha = a$ or b. Since Condition (α) holds, the edge incident to the head-vertex v'_0 of the tree-growing map $N'_T = \psi_{w'}(M_T^{\dashv\vdash})$ is an isthmus. By the preceding point, this edge is on the T-path between u'_0 and v'_0, where u'_0 be the first vertex of N'_T incident to a leg on the T-path from the root-vertex to the head-vertex. Thus $u'_0 \neq v'_0$ and $N_T = \psi_\alpha(N'_T)$ has at least one leg (the one incident to u'_0) beside the head. In the case $\alpha = c$, the tree-growing map $N_T = \psi_c(N'_T)$ has one more legs than N'_T, hence it has at least one leg beside the head.

• *The head and root of N_T are distinct half-edges.*

By definition, the map $M_T^{\dashv\vdash}$ has one leg beside the head whose endpoint is a proper ancestor of the head-vertex. Hence, the head-vertex and root-vertex are distinct. We suppose now that $w = \alpha w'$ with $\alpha = a, b$ or c. If $\alpha = a$ or b the head of N_T is an half-edge of $N'_T = \psi_{w'}(M_T^{\dashv\vdash})$ which is part of an internal edge. Hence it is not the root. If $\alpha = c$, the head of N_T is part of an external edge e of $N'_T = \psi_{w'}(M_T^{\dashv\vdash})$. The edge is broken into the head of N_T and another leg whose endpoint is a proper ancestor of the head-vertex. Hence, the head-vertex and root-vertex of N_T are distinct.

• *If w is the longest readable word, then $N_T = M^0_\bullet$.*
We first prove that the root-vertex and the head-vertex of N_T are the same. Suppose they are distinct. In this case, the head-vertex has degree 3 and is incident to at least one edge. If it is incident to one edge, then one of the conditions (a) or (b) holds and w is not the longest readable word. Hence the head-vertex is incident to two edges e_1 and e_2. One of these edges, say e_1, is in the T-path from the root-vertex to the head-vertex and the other e_2 is not. By a preceding point, the edge e_2 is not an isthmus . Therefore, e_1 is not an isthmus either (e_1 and e_2 have the same ability to disconnect the map). In this case, Condition (c) holds (since T is a dfs-tree) and w is not the longest readable word. Thus, the root-vertex and the head-vertex of N_T are the same. Therefore, the root-vertex has degree 2 and is incident to the head and the root. The head and the root are distinct (by the preceding point). Moreover the root is a leg. Indeed, if the root was not a leg it would be part of an external edge which is an isthmus (which is impossible since the tree T is spanning). Hence the root-vertex is incident to two legs: the root and the head. Thus, $N_T = M^0_\bullet$.

• *The tree-growing map N_T has $2|w|_c - |w|_a - |w|_b + 1$ legs beside the head.*
The tree-growing map $M_T^{\dashv\vdash}$ has one leg beside the head. Moreover, applying mapping ψ_a or ψ_b decreases by one the number of legs whereas applying mapping ψ_c increases this number by two. Hence the property follows easily by induction.

• *The longest word w readable on $M_T^{\dashv\vdash}$ is an excursion.*
By the preceding points, any prefix w' of w satisfies $2|w'|_c - |w'|_a - |w'|_b + 1 \geq 1$ (since this quantity is the number of non-head legs of $\psi_{w'}(M_T^{\dashv\vdash})$). Moreover, since $\psi_w(M_T^{\dashv\vdash}) = M^0_\bullet$ has one leg beside the root, we have $2|w|_c - |w|_a - |w|_b + 1 = 1$. These properties are equivalent to (74) and (75), hence w is an excursion.

□

Proposition 2.18 *The mapping $\Phi \circ \Psi$ is the identity on bridgeless 2-near-cubic marked-dfs-maps.*

Proof: Let M_T be a bridgeless 2-near-cubic marked-dfs-map.

• *For any word w readable on $M_T^{\dashv\vdash}$, the endpoints of any leg of $\psi_w(M_T^{\dashv\vdash})$ is an ancestor of the head-vertex.*
We proceed by induction on the length of w. The property holds for the empty word. We now write $w = \alpha w'$ with $\alpha = a, b$ or c and suppose that it holds for w'. For $\alpha = a$ or b, the property clearly holds for w. Suppose now that $\alpha = c$. Since w is readable, the tree-growing map $N'_T = \psi_{w'}(M_T^{\dashv\vdash})$ satisfies Condition (c). We consider the edge e_0 with endpoints u, v_0 satisfying the conditions of Lemma 2.11. By definition, the head-vertex v_0 of $N_T = \psi_c(N'_T)$

is a descendant of the head-vertex v of N'_T. By the induction hypothesis, the endpoint of any leg of N'_T is an ancestor of v. Hence, the endpoint of any leg of N_T is an ancestor of the head-vertex v_0.

• *For any word w readable on $M_T^{\dashv\vdash}$, we have $\varphi_w \circ \psi_w(M_T^{\dashv\vdash}) = M_T^{\dashv\vdash}$.*
We proceed by induction. The property holds for the empty word. We now write $w = \alpha w'$ with $\alpha = a, b$ or c and suppose that the property holds for w'. If $\alpha = a$ or b the induction step is given directly by Lemma 2.14 (since Condition (α) holds for $M'_T = \psi_{w'}(M_T^{\dashv\vdash})$). If $\alpha = c$, that is, Condition (c) holds for $M'_T = \psi_{w'}(M_T^{\dashv\vdash})$, we must prove that Condition (c') holds (in order to apply Lemma 2.14). But we are ensured that Condition (c') holds by the preceding point. Thus, for $\alpha = a, b$ or c, Lemma 2.14 ensures that $\varphi_\alpha \circ \psi_\alpha(M'_T) = M'_T$. Therefore,

$$\varphi_{\alpha w'} \circ \psi_{\alpha w'}(M_T^{\dashv\vdash}) = \varphi_{w'} \circ \varphi_\alpha \circ \psi_\alpha \circ \psi_{w'}(M_T^{\dashv\vdash}) = \varphi_{w'} \circ \varphi_\alpha \circ \psi_\alpha(M'_T) = \varphi_{w'}(M'_T),$$

and $\varphi_{w'}(M'_T) = M_T^{\dashv\vdash}$ by the induction hypothesis.

• *$\Phi \circ \Psi(M_T) = M_T$.*
By definition, the word $w = \Psi(M_T)$ is the longest readable word on $M_T^{\dashv\vdash}$. Hence, by Proposition 2.17, $\psi_w(M_T^{\dashv\vdash}) = M_\bullet^0$. By the preceding point, $\varphi_w(M_\bullet^0) = \varphi_w \circ \psi_w(M_T^{\dashv\vdash}) = M_T^{\dashv\vdash}$. By definition, the map $\Phi(w)$ is obtained by closing $\varphi_w(M_\bullet^0) = M_T^{\dashv\vdash}$, hence $\Phi(w) = M_T$. Thus, $\Phi \circ \Psi(M_T) = \Phi(w) = M_T$.
□

By Proposition 2.10, the mapping Φ associates a bridgeless 2-near-cubic marked-dfs-map with any excursion. Conversely, by Proposition 2.17, the mapping Ψ associates an excursion with any bridgeless 2-near-cubic marked-dfs-map. The mappings Φ and Ψ are inverse mappings by Propositions 2.16 and 2.18. Thus, the mapping Φ is a bijection between excursions and bridgeless 2-near-cubic marked-dfs-maps. Moreover, if an excursion w has size n (length $3n$), the 2-near-cubic dfs-map $\Phi(w)$ has size n ($3n + 1$ edges). This concludes the proof of Theorem 2.19.
□

2.6 A bijection between Kreweras walks and cubic dfs-maps

In this section, we prove that the mapping Φ establishes a bijection between Kreweras walks ending at the origin and 2-near-cubic dfs-maps. This result is stated more precisely in the following theorem.

Theorem 2.19 *Let w be an excursion. The marked edge of the 2-near-cubic dfs-map $\Phi(w)$ is the root-edge if and only if the excursion w is a Kreweras walk ending at the origin.*
Thus, the mapping Φ induces a bijection between Kreweras walks of size n (length $3n$) ending at the origin and bridgeless 2-near-cubic dfs-maps of size n ($3n + 1$ edges).

Figure 66 illustrates an instance of Theorem 2.19. Before proving this theorem we explore its enumerative consequences. From Theorem 2.19, the number k_n of Kreweras walks of size n is equal to the number d_n of bridgeless 2-near-cubic dfs-maps of size n. The number d_n is given by Corollary 2.9. We obtain the following result.

Theorem 2.20 *There are* $k_n = \dfrac{4^n}{(n+1)(2n+1)}\dbinom{3n}{n}$ *Kreweras walks of size n (length $3n$) ending at the origin.*

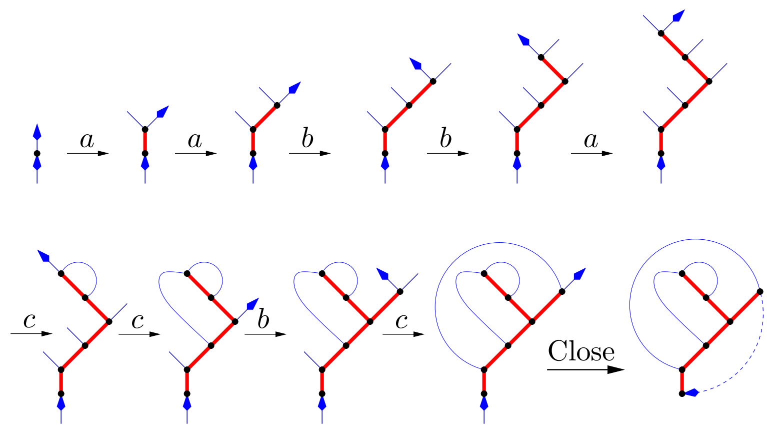


Figure 66: The image of a Kreweras walk by Φ: the root-edge is marked.

The rest of this section is devoted to the proof of Theorem 2.19.

Consider a growing map M such that the root is a leg. Recall that *making the tour of the head-face* means following its border in counterclockwise direction starting from the head (see Figure 59). We call *left* (resp. *right*) the legs encountered before (resp. after) the root during the tour of the head-face. For instance, the growing map in Figure 59 has one left leg and two right legs.

Lemma 2.21 *For any Kreweras walk w ending at the origin, the marked edge of $\Phi(w)$ is the root-edge.*

Proof: Let w' be a suffix of w and let $M'_T = \varphi_{w'}(M^0_\bullet)$ be the corresponding tree-growing map.

• *The root of M'_T is a leg and M'_T has $|w'|_a - |w'|_c$ left legs and $|w'|_b - |w'|_c$ right legs.* (Observe that, these quantities are non-negative by (72) and (73).)
We proceed by induction on the length of w'. The property holds for the empty word. We now write $w' = \alpha w''$ with $\alpha = a, b$ or c and suppose that the property holds for w''. If $\alpha = a$ or b the property holds for w' since applying φ_a (resp. φ_b) increases by one the number of left (resp.

right) legs. We now suppose that $\alpha = c$. We know that $|w''|_a - |w''|_c = |w'|_a - |w'|_c + 1 \geq 1$. Hence, by the induction hypothesis, the tree-growing-map $M_T'' = \varphi_{w''}(M_\bullet^0)$ has at least one left leg. Similarly, M_T'' has at least one right leg. Therefore, the first (resp. last) leg of M_T'' is a left (resp. right) leg. Hence, applying φ_c to M_T'' decreases by one the number of left (resp. right) legs. Thus, the property holds for w'.

• For $w' = w$, the preceding point shows that $\varphi_w(M_\bullet^0)$ has only one leg beside the head and that this leg is the root. Thus, the marked edge of $\Phi(w)$ is the root-edge. □

Lemma 2.22 *For any bridgeless 2-near-cubic dfs-map M_T marked on the root-edge, the word $w = \Psi(M_T) = \Phi^{-1}(M_T)$ is a Kreweras walk ending at the origin.*

Proof: Let w be a word readable on $M_T^{\dashv\vdash}$ and let $N_T = \psi_w(M_T^{\dashv\vdash})$. Observe that the root of N_T is a leg (since it is the case in $M_T^{\dashv\vdash}$ and the root never disappears).

• *The tree-growing map N_T has $|w|_c - |w|_a$ left legs and $|w|_c - |w|_b$ right legs.*
We proceed by induction on the length of w. The property holds for the empty word. We now write $w = \alpha w'$ with $\alpha = a, b$ or c and suppose that the property holds for w'. If $\alpha = a$ or b the property holds for w since applying ψ_a (resp. ψ_b) decreases by one the number of left (resp. right) legs. We now suppose that $\alpha = c$. The map $N_T' = \psi_{w'}(M_T^{\dashv\vdash})$ satisfies Condition (c). We have already proved (see the first point in the proof of Lemma 2.17) that the endpoint of every leg is an ancestor of the head-vertex. Hence N_T' satisfies Condition (c'). Therefore, Lemma 2.15 holds for N_T'. We adopt the notations h, s, t of this lemma which is illustrated in Figure 65. By Lemma 2.15, the pair of first and last head of $N_T = \psi_c(N_T')$ is the pair $\{s, t\}$. Hence, in the pair $\{s, t\}$ one is a left leg and the other is a right leg of N_T. Moreover, the other left and right legs of N_T are the same as in N_T'. Thus, applying ψ_c to N_T' increases by one the number of left (resp. right) legs. Hence, the property holds for w.

• *The word $w = \Psi(M_T)$ is a Kreweras walk ending at the origin.*
By definition, w is the longest word readable on $M_T^{\dashv\vdash}$. By Proposition 2.17, $\psi_w(M_T^{\dashv\vdash}) = M_\bullet^0$. By the preceding point, we get $|w|_c - |w|_a = 0$ and $|w|_c - |w|_b = 0$ (since $M_\bullet^0$ has no left nor right leg). Moreover, for any suffix w' of w, the preceding point proves that $|w'|_c - |w'|_a \geq 0$ and $|w'|_c - |w'|_b \geq 0$. These properties are equivalent to (72) and (73), hence w is a Kreweras walk ending at the origin. □

2.7 Enumerating dfs-trees and cubic maps

In Section 2.4, we exhibited a bijection Φ between excursions and bridgeless 2-near-cubic marked-dfs-maps. As a corollary we obtained the number of bridgeless 2-near-cubic dfs-maps of size n: $d_n = \frac{4^n}{(n+1)(2n+1)}\binom{3n}{n}$. In this section, we prove that any bridgeless 2-near-cubic map of size n has 2^n dfs-trees (Corollary 2.27). Hence, the number of bridgeless 2-near-cubic maps of size n is $c_n = \frac{d_n}{2^n} = \frac{2^n}{(n+1)(2n+1)}\binom{3n}{n}$. Given the bijection between 2-near-cubic maps and cubic maps (see Figure 52), we obtain the following theorem.

Theorem 2.23 *There are* $c_n = \dfrac{2^n}{(n+1)(2n+1)}\dbinom{3n}{n}$ *bridgeless cubic maps with* $3n$ *edges.*

By duality, c_n is also the number of loopless triangulations with $3n$ edges. Hence, we recover Equation (71) announced in the introduction. As mentioned above, an alternative bijective proof of Theorem 2.23 was given in [Poul 03a].

The rest of this section is devoted to the counting of dfs-trees on cubic maps and, more generally, on cubic (potentially non-planar) graphs. We first give an alternative characterization of dfs-trees. This characterization is based on the depth-first search (DFS) algorithm (see Section 23.3 of [Corm 90]). We consider the DFS algorithm as an algorithm for constructing a spanning tree of a graph.

Consider a graph G with a distinguished vertex v_0. If the DFS algorithm starts at v_0, the subgraph T (see below) constructed by the algorithm remains a tree containing v_0. We call *visited* the vertices in T and *unvisited* the other vertices. The distinguished vertex v_0 is considered as the root-vertex of the tree. Hence, any vertex in T distinct from v_0 has a *father* in T.

Definition 2.24 *Depth-first search (DFS) algorithm.*

Initialization: *Set the* current vertex *to be* v_0 *and the tree* T *to be reduced to* v_0.

Core: *While the current vertex* v *is adjacent to some unvisited vertices or is distinct from* v_0 *do:*

If there are some edges linking the current vertex v *to an unvisited vertex, then choose one of them. Add the chosen edge* e *and its unvisited endpoint* v' *to the tree* T. *Set the current vertex to be* v'.

Else, backtrack, that is, set the current vertex to be the father of v *in* T.

End: *Return the tree* T.

It is well known that the DFS algorithm returns a spanning tree. It is also known [Corm 90] that the two following properties are equivalent for a spanning tree T of a graph G having a distinguished vertex v_0:

(i) Any external edge joins comparable vertices.

(ii) The tree T can be obtained by a DFS algorithm on the graph G starting at v_0.

Before stating the main result of this section, we need an easy preliminary lemma.

Lemma 2.25 *Let G be a connected graph with a distinguished vertex v_0 whose deletion does not disconnect the graph. Then, any spanning tree T of G satisfying conditions **(i)**-**(ii)** has exactly one edge incident to v_0.*

Proof: Let e_0 be an edge of T incident to v_0 and let v_1 be the other endpoint of e_0. We partition the vertex set V of G into $\{v_0\} \cup V_0 \cup V_1$, where V_1 is the set of descendants of v_1. There is no internal edge joining a vertex in V_0 and a vertex in V_1. There is no external edge either or it would join two non-comparable vertices. Thus $V_0 = \emptyset$ or the deletion of v_0 would disconnect the graph.

□

Theorem 2.26 *Let G be a loopless connected graph with a distinguished vertex v_0 whose deletion does not disconnect the graph. Let e_0 be an edge incident to v_0. If G is a k-near-cubic graph (v_0 has degree k and the other vertices have degree 3) of size n ($3n+2k-3$ edges), then there are 2^n trees containing e_0 and satisfying conditions **(i)**-**(ii)**.*

Given that the dfs-trees are the spanning trees satisfying conditions **(i)**-**(ii)** and not containing the root, the following corollary is immediate.

Corollary 2.27 *Any bridgeless 2-near-cubic map of size n ($3n + 1$ edges) has 2^n dfs-trees.*

Remark: Theorem 2.26 implies that any k-near-cubic loopless graph of size n has $k2^n$ trees satisfying the conditions **(i)**-**(ii)**.

The rest of this section is devoted to the proof of Theorem 2.26. The proof relies on the intuition that exactly n *real* binary choices have to be made during the execution of a DFS algorithm on a k-near-cubic map of size n.

Given a graph G and a subset of vertices U, we say that two vertices u and v are *U-connected* if there is a path between u and v containing only vertices in $U \cup \{u, v\}$.

Lemma 2.28 *Let v be the current vertex and let U be the set of unvisited vertices at a given time of the DFS algorithm. The vertices that will be visited before the last visit to v are the vertices in U that are U-connected to v.*

Proof: Let S be the set of vertices in U that are U-connected to v. We make an induction on the cardinality of S. If the set S is empty, there is no edge linking v to an unvisited vertex. Hence, the next step in the algorithm is to backtrack and the vertex v will never be visited again. In other words, it is the last visit to v, hence the property holds. Suppose now that S is non-empty. In this case, there are some edges linking the current vertex v to an unvisited vertex. Let e be the edge chosen by the DFS algorithm and let $v' \in U$ be the corresponding endpoint. Let S_1 be the set of vertices in U that are U-connected to v' and let $S_2 = S \setminus S_1$. Observe that no edge joins a vertex in S_1 and a vertex in S_2. This situation is represented in Figure 67. The set of vertices in $U' = U \setminus \{v'\}$ that are U'-connected to v is $S_1' = S_1 \setminus \{v'\}$ (since a vertex is U-connected to v' if and only if it is U'-connected to v'). By the induction hypothesis, S_1' is the set of vertices visited between the first and last visit to v'. Hence S_1 is the set of vertices visited before the algorithm returns to v. Since no edge joins a vertex in S_1 and a vertex in S_2, the vertices in S_2 are the vertices in $U \setminus S_1$ that are $(U \setminus S_1)$-connected to v. By the induction hypothesis, S_2 is the set of vertices visited before the last visit to v. Thus, the property holds.

□

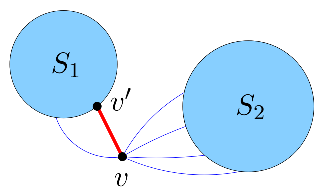


Figure 67: Partition of the vertices in S.

Proof of Theorem 2.26: Clearly, the spanning trees containing e_0 and satisfying the conditions **(i)**-**(ii)** are the spanning trees obtained by a DFS algorithm for which the first *core step* is to choose e_0. We want to prove that there are 2^n such spanning trees.

We consider an execution of the DFS algorithm for which the first *core step* is to choose e_0 and denote by $\mathcal{T}$ the spanning tree returned by the DFS algorithm (in order to distinguish it from the evolving tree T). After the first *core step*, the tree T is reduced to e_0 and its two endpoints v_0 and v_0'. Let V be the vertex set of G and let $V' = V \setminus \{v_0, v_0'\}$. Since the deletion of v_0 does not disconnect the graph, every vertex in V' is V'-connected to v_0'. Hence, by Lemma 2.28, every vertex will be visited before the algorithm returns to v_0. Thus, from this stage on, the current vertex v is incident to 3 edges $e,\ e_1,\ e_2$, where $e \in T$ links v to its father.

• We denote by v_1 and v_2 the endpoints of e_1 and e_2 respectively (these endpoints are not necessarily distinct) and we denote by U the set of unvisited vertices. We distinguish three cases:

(α) at least one of the vertices $v_1,\ v_2$ is not in U,

(β) the two vertices $v_1,\ v_2$ are in U and are U-connected with each other,

(γ) the two vertices $v_1,\ v_2$ are in U and are not U-connected with each other.

The three cases are illustrated by Figure 68. We prove successively the following properties:

- *In case* (α)*, no choice has to be done by the algorithm.*

Indeed, there is at most one edge (e_1 or e_2) linking the current vertex v to an unvisited vertex.

- *In case (β), the algorithm has to choose between* e_1 *and* e_2. *This choice* necessarily *leads to two different spanning trees* $\mathcal{T}$. *Indeed the edge* e_1 *(resp.* e_2*) is in* $\mathcal{T}$ *if and only if the choice of* e_1 *(resp.* e_2*) is made.*

Suppose (without loss of generality), that the choice of e_1 is made. The vertex v_2 is $(U \cup \{v_1\})$-connected to v_1 (a vertex is $(U \cup \{v_1\})$-connected to v_1 if and only if it is U-connected to v_1). Hence, by Lemma 2.28, the vertex v_2 will be visited before the last visit to v_1, that is, before the algorithm returns to the vertex v. Therefore, the edge e_2 will not be in the spanning tree $\mathcal{T}$.

- *In case (γ), the algorithm has to choose between* e_1 *and* e_2. *Moreover, any tree* $\mathcal{T}$ *obtained by choosing* e_1 *can be also obtained by choosing* e_2.

Let S_1 and S_2 be the set of vertices in U that are U-connected to v_1 and v_2 respectively. Observe that the sets S_1 and S_2 are disjoint and no edge links a vertex in S_1 and a vertex in S_2 (otherwise the vertices v_1 and v_2 would be U-connected). Suppose that the choice of e_1 is made. The set of vertices in $U \setminus \{v_1\}$ that are $U \setminus \{v_1\}$-connected to v_1 is $S_1 \setminus \{v_1\}$. Hence, by Lemma 2.28, the set of vertices visited before the last visit to v_1, that is, before the algorithm returns to v is S_1. Since v_2 is not in S_1 the next step of the algorithm is to choose e_2. Let $U_2 = U \setminus S_1$ be the set of unvisited vertices at this stage. Since no vertex in S_1 is adjacent to a vertex in S_2, the set of vertices in $U_2 \setminus \{v_2\}$ that are $(U_2 \setminus \{v_2\})$-connected to v_2 is $S_2 \setminus \{v_2\}$. Hence, by Lemma 2.28, the set of vertices visited before the last visit to v_2, that is, before the algorithm returns to v is S_2. Let T_1 (resp. T_2) be the subtree constructed by the algorithm between the first and last visit to v_1 (resp. v_2). Since no vertex in S_1 is adjacent to a vertex in S_2, the subtree T_1 could have been constructed exactly the same way if the algorithm had chosen e_2 (instead of e_1) at the beginning. Similarly, the subtree T_2 could have been constructed exactly in the same way if the algorithm had chosen e_2 at the beginning. Therefore, the tree $\mathcal{T}$ returned by the algorithm could have been constructed if the algorithm had chosen e_2 (instead of e_1) at the beginning.

• *During any execution of the DFS algorithm we are exactly n times in case (β).*

The k-near-cubic graph G has $3n+2k-3$ edges and $2n+2k-1$ vertices. Hence, the spanning tree $\mathcal{T}$ has $2n+2k-2$ edges. Thus, there are $n+k-1$ external edges among which $k-1$ are incident to v_0. Let E_β be the set of the n external edges not incident to v_0. Since G is loopless and the spanning tree $\mathcal{T}$ satisfies **(i)-(ii)**, the edges in E_β have distinct and comparable endpoints. For any edge e in E_β, we denote by v_e the endpoint of e which is the ancestor of the other endpoint. The vertex v_e is incident to e, to the edge of $\mathcal{T}$ linking v to its father and to another edge in $\mathcal{T}$ linking v_e to its son (otherwise v_e has no descendant). In particular, if e and e' are distinct edges in E_β, then the vertices v_e and $v_{e'}$ are distinct. Thus, the set of vertices $V_\beta = \{v_e / e \in E_\beta\}$ has size n.

We want to prove that the case (β) occurs when the algorithm visit a vertex in V_β for the first time (and not otherwise). Let v be a vertex in V_β. The vertex v is incident to an edge

e_1 in E_β, an edge e in $\mathcal{T}$ linking v to its father and another edge e_2 in $\mathcal{T}$ linkink v to its son. Let T be the tree constructed by the algorithm at the time of the first visit to v and let U be the set of unvisited vertices. Any descendant of v is in U. In particular, the endpoints v_1 and v_2 of e_1 and e_2 are in U and are U-connected with each other (take the $\mathcal{T}$-path between v_1 and v_2). Thus, we are in case (β). Conversely, if we are in case (β) during the algorithm, the current vertex v is visited for the first time (or one of the vertices v_1, v_2 would already be in U). Moreover, by the preceding point, one of the edges (e_1 or e_2) incident to v is not in $\mathcal{T}$ and joins v to one of its descendants. Hence, the current vertex v is in V_β.

• During the DFS algorithm we have to make n binary choices that will affect the outcome of the algorithm (case (β)). The other choices (case (γ)) do not affect the outcome of the algorithm. Therefore, there are 2^n possible outcomes.

□

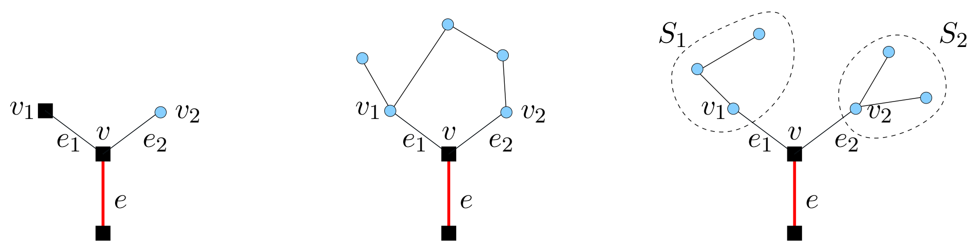


Figure 68: Case (α) (left), case (β) (middle) and case (γ) (right). The visited vertices are indicated by a square while unvisited ones are indicated by a circle.

2.8 Applications, extensions and open problems

2.8.1 Random generation of triangulations

The random generation of excursions of length $3n$ (with uniform distribution) reduces to the random generation of 1-dimensional walks of length $3n$ with steps +2, -1 starting and ending at 0 and remaining non-negative. The random generation of these walks is known to be feasible in linear time. (One just needs to generate a word of length $3n+1$ containing n letters c and $2n+1$ letters α and to apply the cycle lemma.) Given an excursion w, the construction of the 2-near-cubic marked-dfs-map $\Phi(w)$ can be performed in linear time. Therefore, we have a linear time algorithm for the random generation (with uniform distribution) of bridgeless 2-near-cubic marked-dfs-maps. For any bridgeless 2-near-cubic map there are 2^n dfs-trees and $(n+1)$ possible marking. Therefore, if we drop the marking and the dfs-tree at the end of the process, we obtain a uniform distribution on bridgeless 2-near-cubic maps. This allows us to generate uniformly bridgeless cubic maps or, dually, loopless triangulations, in linear time.

2.8.2 Kreweras walks ending at $(i, 0)$ and $(i + 2)$-near-cubic maps

The Kreweras walks ending at $(i, 0)$ are the words w on the alphabet $\{a, b, c\}$ with $|w|_a + i = |w|_b = |w|_c$ such that any suffix w' of w satisfies $|w'|_a + i \geq |w'|_c$ and $|w'|_b \geq |w'|_c$. There is a very nice formula [Krew 65] giving the number of Kreweras walks of size n (length $3n + 2i$) ending at $(i, 0)$:

$$k_{n,i} \quad = \quad \frac{4^n(2i+1)}{(n+i+1)(2n+2i+1)}\binom{2i}{i}\binom{3n+2i}{n}. \tag{77}$$

There is also a similar formula [Mull 65] for non-separable $(i + 2)$-near-cubic maps of size n ($3n + 2i + 1$ edges):

$$c_{n,i} \quad = \quad \frac{2^n(2i+1)}{(n+i+1)(2n+2i+1)}\binom{2i}{i}\binom{3n+2i}{n}. \tag{78}$$

In this subsection, we show that the bijection Φ (Definition 2.3) can be extended to Kreweras walks ending at $(i, 0)$. This gives a bijective correspondence explaining why $k_{n,i} = 2^n c_{n,i}$.

Consider the tree-growing map $M^i_\bullet$ reduced to a vertex, a root, a head and i left legs (Figure 69). We define the image of a Kreweras walk w ending at $(i, 0)$ as the map obtained by closing $\varphi_w(M^i_\bullet)$. We get the following extension of Theorem 2.19.

Theorem 2.29 *The mapping Φ is a bijection between Kreweras walks of size n (length $3n+2i$) ending at $(i, 0)$ and non-separable $(i+2)$-near-cubic maps of size n ($3n+2i+1$ edges) marked on the root-edge with a dfs-tree that contains the edge following the root in counterclockwise order around the root-vertex.*

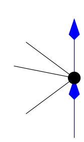


Figure 69: The tree-growing map $M^i_\bullet$ when $i = 3$.

By Theorem 2.26, there are 2^n such dfs-trees. Consequently, we obtain the following corollary:

Corollary 2.30 *The number $k_{n,i}$ of Kreweras walks of size n ending at $(i, 0)$ and the number $c_{n,i}$ of non-separable $(i+2)$-near-cubic maps of size n are related by the equation $k_{n,i} = 2^n c_{n,i}$.*

One can define the counterpart of excursions for Kreweras walks ending at $(i, 0)$. These are the walks obtained when one chooses an external edge in a non-separable $(i+2)$-near-cubic dfs-map such that the edge following the root is in the tree and applies the mapping $\Psi = \Phi^{-1}$. Alas, we have found no simple characterization of this set of walks nor any bijective proof explaining why this set has cardinality $\dfrac{4^n(2i+1)}{(2n+2i+1)}\dbinom{2i}{i}\dbinom{3n+2i}{n}$.

Chapter 3

Bijective decomposition of tree-rooted maps

Abstract: The number of tree-rooted maps, that is, rooted planar maps with a distinguished spanning tree, of size n is $\mathcal{C}_n\mathcal{C}_{n+1}$ where $\mathcal{C}_n = \frac{1}{n+1}\binom{2n}{n}$ is the n^{th} Catalan number. We present a (long awaited) simple bijection which explains this result. We prove that our bijection is isomorphic to a former recursive construction on shuffles of parenthesis systems due to Cori, Dulucq and Viennot.

Résumé : On considère les *cartes boisées*, c'est-à-dire les cartes planaires enracinées dont un arbre couvrant est distingué. Le nombre de cartes boisées de taille n est donné par le produit $\mathcal{C}_n\mathcal{C}_{n+1}$ où $\mathcal{C}_n = \frac{1}{n+1}\binom{2n}{n}$ est le $n^{\text{ème}}$ nombre de Catalan. Nous présentons une bijection simple (et longtemps attendue) qui explique ce résultat. Nous montrons ensuite que notre bijection est isomorphe à une construction récursive antérieure due à Cori, Dulucq et Viennot et définie sur les mélanges de mots de parenthèses.

3.1 Introduction

In the late sixties, Mullin published an enumerative result concerning planar maps on which a spanning tree is distinguished [Mull 67]. He proved that the number of rooted planar maps with a distinguished spanning tree, or *tree-rooted maps* for short, of size n is $\mathcal{C}_n\mathcal{C}_{n+1}$ where $\mathcal{C}_n = \frac{1}{n+1}\binom{2n}{n}$ is the n^{th} Catalan number. This means that tree-rooted maps of size n are in one-to-one correspondence with pairs of plane trees of size n and $n+1$ respectively. But although Mullin asked for a bijective explanation of this result, no natural mapping was found between tree-rooted maps and pairs of trees. Twenty years later, Cori, Dulucq and Viennot exhibited one such mapping while working on Baxter permutations [Cori 86]. More precisely, they established a bijection between pairs of trees and *shuffles of two parenthesis systems*, that is, words on the alphabet $a, \overline{a}, b, \overline{b}$, such that the subword consisting of the letters $a, \overline{a}$ and the subword consisting of the letters $b, \overline{b}$ are parenthesis systems. It is known that tree-rooted maps are in one-to-one correspondence with shuffles of two parenthesis systems [Mull 67, Lehm 72], hence the bijection of Cori *et al.* somehow answers Mullin's question. But this answer is quite unsatisfying in the world of maps. Indeed, the bijection of Cori *et al.* is recursively defined on the set of prefixes of shuffles of parenthesis systems and it was not understood how this bijection could be interpreted on maps. We fill this gap by defining a natural, non-recursive, bijection between tree-rooted maps and pairs made of a tree and a non-crossing partition. Then, we show that our construction is isomorphic to the construction of Cori *et al.* via the encoding of tree-rooted maps by shuffles of parenthesis systems.

Tree-rooted maps, or alternatively shuffles of parenthesis systems, are in one-to-one correspondence with square lattice walks confined in the quarter plane (we explicit this correspondence in the next section). Therefore, our bijection can also be seen as a way of counting these walks. Some years ago, Guy, Krattenthaler and Sagan worked on walks in the plane [Guy 92] and exhibited a number of nice bijections. However, they advertised the result of Cori *et al.* as being *considerably harder* to prove bijectively. We believe that the encoding in terms of tree-rooted maps makes this result more natural.

The outline of this chapter is as follows. In Section 3.2, we recall some definitions and preliminary results on tree-rooted maps. In Section 2.6, we present our bijection between tree-rooted maps of size n and pairs consisting of a tree and a non-crossing partition of size n and $n+1$ respectively. This simple bijection explains why the number of tree-rooted maps of size n is $\mathcal{C}_n\mathcal{C}_{n+1}$. In Section 3.4, we prove that our bijection is isomorphic to the construction of Cori *et al.*

Our study requires to introduce a large number of mappings; we refer the reader to Figure 87 which summarizes our notations.

3.2 Preliminary results

We begin by some preliminary definitions on planar maps. A *planar map*, or *map* for short, is a two-cell embedding of a connected planar graph into the oriented sphere considered up to orientation preserving homeomorphisms of the sphere. Loops and multiple edges are allowed. A *rooted* map is a map together with a half-edge called the *root*. A rooted map is represented in Figure 70. The vertex (resp. the face) incident to the *root* is called the *root-vertex* (resp. *root-face*). When representing maps in the plane, the root-face is usually taken as the infinite face and the root is represented as an arrow pointing on the root-vertex (see Figure 70). Unless explicitly mentioned, all the maps considered in this chapter are rooted.

A *planted plane tree*, or *tree* for short, is a rooted map with a single face. A vertex v is an *ancestor* of another vertex v' in a tree T if v is on the (unique) path in T from v' to the root-vertex of T. When v is the first vertex encountered on that path, it is the *father* of v'. A *leaf* is a vertex which is not a father. Given a rooted map M, a submap of M is a *spanning tree* if it is a tree containing all vertices of M. (The spanning tree inherit its root from the map.) We now define the main object of this study, namely tree-rooted maps. A *tree-rooted map* is a rooted map together with a distinguished spanning tree. Tree-rooted maps shall be denoted by symbols like M_T where it is implicitly assumed that M is the underlying map and T the spanning tree. Graphically, the distinguished spanning tree will be represented by thick lines (see Figure 74). The *size* of a map, a tree, a tree-rooted map, is the number of edges.

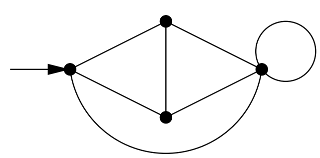


Figure 70: A rooted map.

A number of classical bijections on trees are defined by following the border of the tree. Doing the *tour of the tree* means following its border in counterclockwise direction starting and finishing at the root (see Figure 73). Observe that the tour of the tree induces a linear order, the order of appearance, on the vertex set and on the edge set of the tree. For tree-rooted maps, the tour of the spanning tree T also induces a linear order on half-edges not in T (any of them is encountered once during a tour of T). We shall say that a vertex, an edge, a half-edge *precedes* another one *around* T.

Our constructions lead us to consider *oriented maps*, that is, maps in which all edges are oriented. If an edge e is oriented from u to v, the vertex u is called the *origin* and v the *end*. The half-edge incident to the origin (resp. end) is called the *tail* (resp. *head*). The root of an oriented map will always be considered and represented as a head.

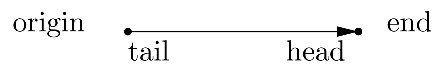


Figure 71: Half-edges and endpoints.

We now recall a well-known correspondence between tree-rooted maps and shuffles of two parenthesis systems [Mull 67, Lehm 72]. We derive from it the enumerative result mentioned above: the number of tree-rooted maps of size n (i.e. with n edges) is $\mathcal{C}_n\mathcal{C}_{n+1}$. For this purpose, we introduce some notations on words. A word w on a set A (called the alphabet) is a finite sequence of elements (letters) in A. The length of w (that is, the number of letters in w) is denoted $|w|$ and, for a in A, the number of occurrences of a in w is denoted $|w|_a$. A word w on the two-letter alphabet $\{a,\overline{a}\}$ is a *parenthesis system* if $|w|_a = |w|_{\overline{a}}$ and for all prefixes w', $|w'|_a \geq |w'|_{\overline{a}}$. For instance, $aa\overline{a}a\overline{a}\overline{a}$ is a parenthesis system. A *shuffle of two parenthesis systems*, or *parenthesis-shuffle* for short, is a word on the alphabet $\{a,\overline{a},b,\overline{b}\}$ such that the subword of w consisting of letters in $\{a,\overline{a}\}$ and the subword consisting of letters in $\{b,\overline{b}\}$ are parenthesis systems. For instance $abab\overline{b}\overline{a}ba\overline{a}\overline{b}\overline{a}$ is a parenthesis-shuffle.

Parenthesis-shuffles can also be seen as walks in the quarter plane. Consider walks made of steps *North, South, East, West,* confined in the quadrant $x \geq 0,\ y \geq 0$. The parenthesis-shuffles of size n are in one-to-one correspondence with walks of length $2n$ starting and returning at the origin. This correspondence is obtained by considering each letter a (resp. $\overline{a},b,\overline{b}$) as a *North* (resp. *South, East, West*) step. For instance, we represented the walk corresponding to $abb\overline{a}baa\overline{b}\overline{b}\overline{a}\overline{a}\overline{b}$ in Figure 72. The fact that the subword of w consisting of letters in $\{a,\overline{a}\}$ (resp. $\{b,\overline{b}\}$) is a parenthesis system implies that the walk stays in the half-plane $y \geq 0$ (resp. $x \geq 0$) and returns at $y = 0$ (resp. $x = 0$).

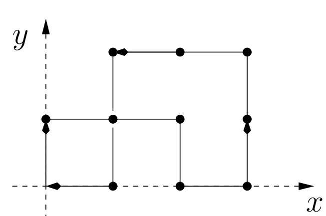


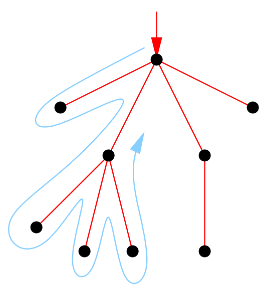
Figure 72: A walk in the quarter plane.

The size of a parenthesis system, a parenthesis-shuffle, is half its length. For instance, the parenthesis-shuffle $abab\overline{b}\overline{a}ba\overline{a}\overline{b}\overline{a}$ has size 5. It is well known that the number of parenthesis systems of size n is the n^{th} Catalan number $\mathcal{C}_n = \frac{1}{n+1}\binom{2n}{n}$. From this, a simple calculation proves that the number of parenthesis-shuffles of size n is $\mathcal{S}_n = \mathcal{C}_n\mathcal{C}_{n+1}$. Indeed, there are $\binom{2n}{2k}$ ways to shuffle a parenthesis system of size k (on $\{a, \overline{a}\}$) with a parenthesis system of size $n-k$ (on $\{b, \overline{b}\}$). And summing on k gives the result:

$$\begin{aligned}\mathcal{S}_n &= \sum_{k=0}^{n} \binom{2n}{2k} \mathcal{C}_k \mathcal{C}_{n-k} = \frac{(2n)!}{(n+1)!^2} \sum_{k=0}^{n} \binom{n+1}{k}\binom{n+1}{n-k} \\ &= \frac{(2n)!}{(n+1)!^2}\binom{2n+2}{n} = \mathcal{C}_n\mathcal{C}_{n+1}.\end{aligned}$$

Note, however, that this calculation involves the Chu-Vandermonde identity.

It remains to show that tree-rooted maps of size n are in one-to-one correspondence with parenthesis-shuffles of size n. We first recall a very classical bijection between trees and parenthesis systems. This correspondence is obtained by making the tour of the tree. Doing so and writing a the first time we follow an edge and $\overline{a}$ the second time we follow that edge (in the opposite direction) we obtain a parenthesis system. This parenthesis system is indicated for the tree of Figure 73. Conversely, any parenthesis system can be seen as a code for constructing a tree.



$a\overline{a}aa\overline{a}a\overline{a}a\overline{a}\overline{a}aa\overline{a}\overline{a}a\overline{a}$

Figure 73: A tree and the associated parenthesis system.

Now, consider a tree-rooted map. During the tour of the spanning tree we cross edges of the map that are not in the spanning tree. In fact, each edge not in the spanning tree will be crossed twice (once at each half-edge). Hence, making the tour of the spanning tree and writing a the first time we follow an edge of the tree, $\overline{a}$ the second time, b the first time we cross an edge not in the tree and $\overline{b}$ the second time, we obtain a parenthesis-shuffle. We shall denote by Ξ this mapping from tree-rooted maps to parenthesis-shuffles. We applied the mapping Ξ to the tree-rooted map of Figure 74.

The reverse mapping can be described as follows: given a parenthesis-shuffle w we first create the tree corresponding to the subword of w consisting of letters $a, \overline{a}$ (this will give the spanning tree) then we glue to this tree a head for each letter b and a tail for each letter $\overline{b}$. There is only one way to connect heads to tails so that the result is a planar map (that is, no

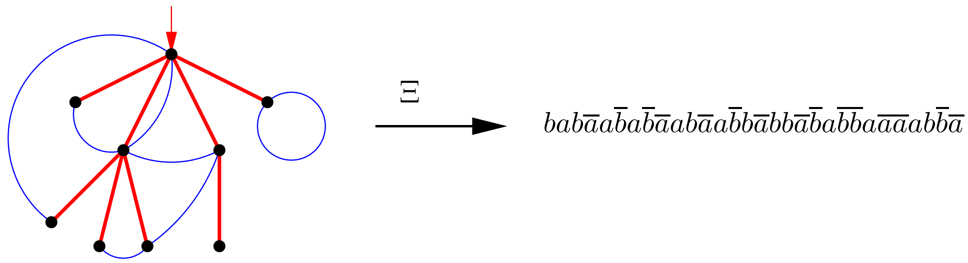


Figure 74: A tree-rooted map and the associated parenthesis-shuffle.

edges intersect). Note that, if the map M has size n, the corresponding parenthesis-shuffle w has size n since $|w|_a$ is the number of edges in the tree and $|w|_b$ is the number of edges not in the tree.
This encoding due to Walsh and Lehman [Lehm 72] establishes a one-to-one correspondence between tree-rooted maps of size n and parenthesis-shuffles of size n. Hence, there are $\mathcal{C}_n\mathcal{C}_{n+1}$ tree-rooted maps of size n.

Such an elegant enumerative result is intriguing for combinatorists since Catalan numbers have very nice combinatorial interpretations. We have just seen that these numbers count parenthesis systems and trees. In fact, Catalan numbers appear in many other contexts (see for instance Ex. 6.19 of [Stan 99] where 66 combinatorial interpretations are listed). We now give another classical combinatorial interpretation of Catalan numbers, namely *non-crossing partitions*. A non-crossing partition is an equivalence relation $\sim$ on a linearly ordered set S such that no elements $a < b < c < d$ of S satisfy $a \sim c$, $b \sim d$ and $a \nsim b$. The equivalence classes of non-crossing partitions are called *parts*. Non-crossing partitions have been extensively studied (see [Simi 00] and references therein).

Non-crossing partitions can be represented as cell decompositions of the half-plane. If the set S is $\{s_1, \ldots, s_n\}$ with $s_1 < s_2 < \cdots < s_n$, we associate with s_i the vertex of coordinates $(i, 0)$ and with each part we associate a connected region of the lower half-plane $y \leq 0$ incident to the vertices of that part. The existence of a cell decomposition with no intersection between cells is precisely the definition of non-crossing partitions. A non-crossing partition of size 8 is represented in Figure 75. The only non-trivial parts of this non-crossing partition are $\{1, 4, 5\}$ and $\{6, 8\}$.

Non-crossing partitions of size n (i.e. on a set of size n) are in one-to-one correspondence with trees of size n. One way of seeing this is to draw the dual of the cell-representation of the partition, that is, to draw a vertex in each part and each *anti-part* (connected cells complementary to parts in the half-plane decomposition) and connect vertices corresponding to adjacent cells by an edge. The root is chosen in the infinite cell as indicated in Figure 75. In the sequel, this mapping between non-crossing partitions and trees is denoted Υ. It is a bijection between non-crossing partitions of size n and trees of size n. It proves that the number of non-crossing partitions of size n is $\mathcal{C}_n$.

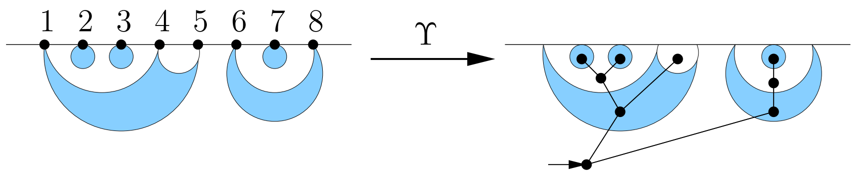


Figure 75: A non-crossing partition and the associated tree.

3.3 Bijective decomposition of tree-rooted maps

We begin with the presentation of our bijection between tree-rooted maps and pairs consisting of a tree and a non-crossing partition. This bijection has two steps: first we orient the edges of the map and then we disconnect properly the vertices.

Map orientation: Let M_T be a tree-rooted map. We denote by $\vec{M}^T$ the oriented map obtained by orienting the edges of M according to the following rules:

• edges in the tree T are oriented from the root to the leaves,

• edges not in the tree T are oriented in such a way that their head precedes their tail around T.

As always in this chapter, the root is considered as a head.

In the sequel, the mapping $M_T \mapsto \vec{M}^T$ is denoted δ. We applied this mapping to the tree-rooted map of Figure 76. Note that any vertex of $\vec{M}^T$ is incident to at least one head (since the spanning tree is oriented from the root to the leaves).

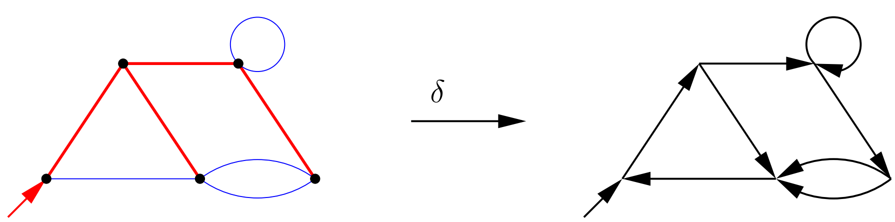


Figure 76: A tree-rooted map M_T and the corresponding oriented map $\vec{M}^T$.

Vertex explosion: We replace each vertex v of the oriented map $\vec{M}^T$ by as many vertices as heads incident to v and we suppress some adjacency relations between half-edges incident to v according to the rule represented in Figure 77. That is, each tail t becomes adjacent to exactly one head which is the first head encountered in counterclockwise direction around v

starting from t.

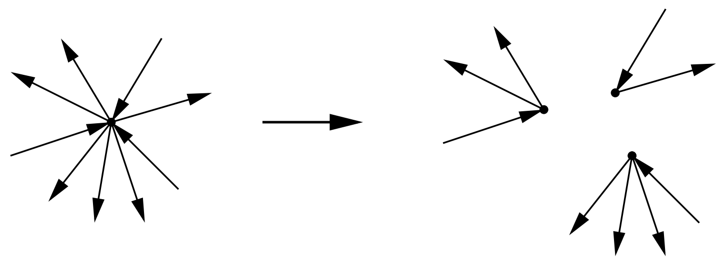


Figure 77: Local rule for suppressing the adjacency relations.

We shall prove (Lemma 3.11) that this suppression of some adjacency relations in $\vec{M}^T$ produces a tree denoted $\varphi_0(\vec{M}^T)$. Observe that this tree has the same number of edges, say n, as the original map M. Hence, its vertex set S has size $n+1$. This set is linearly ordered by the order of appearance around the tree $\varphi_0(\vec{M}^T)$. We define an equivalence relation $\varphi_1(\vec{M}^T)$ on S: two vertices are equivalent if they come from the same vertex of $\vec{M}^T$. We will prove (Lemma 3.12) that the equivalence relation $\varphi_1(\vec{M}^T)$ is a non-crossing partition on the set S. The mapping $\vec{M}^T \mapsto (\varphi_0(\vec{M}^T), \varphi_1(\vec{M}^T))$ is called the *vertex explosion process* and is denoted φ.

Therefore, with any tree-rooted map M_T of size n we associate a tree $\varphi_0(\vec{M}^T)$ of size n and a non-crossing partition $\varphi_1(\vec{M}^T)$ of size $n+1$. The following theorem states that this correspondence is one-to-one.

Theorem 3.1 *Let Φ be the mapping associating the ordered pair $(\varphi_0(\vec{M}^T), \varphi_1(\vec{M}^T))$ with the tree-rooted map M_T. This mapping is a bijection between the set of tree-rooted maps of size n and the Cartesian product of the set of trees of size n and the set of non-crossing partitions of size $n+1$.*
It follows that the number of tree-rooted maps of size n is $\mathcal{C}_n\mathcal{C}_{n+1}$.

Graphically, the bijection Φ is best represented by keeping track of the underlying non-crossing partition during the vertex explosion process. This is done by creating for each vertex of M a connected cell representing the corresponding part of the non-crossing partition. The graphical representation of the vertex explosion process φ becomes as indicated in Figure 78. For instance, we applied the mapping φ to the oriented map of Figure 79.

The rest of this section is devoted to the proof of Theorem 3.1. We first give a characterization of the set of oriented maps, called *tree-oriented maps*, associated to tree-rooted maps by the mapping δ. We also define the reverse mapping γ. Then we prove that the vertex explosion process φ is a bijection between tree-oriented maps (of

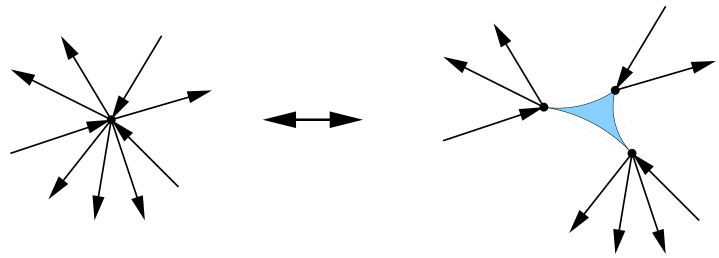


Figure 78: The vertex explosion process and a part of the non-crossing partition.

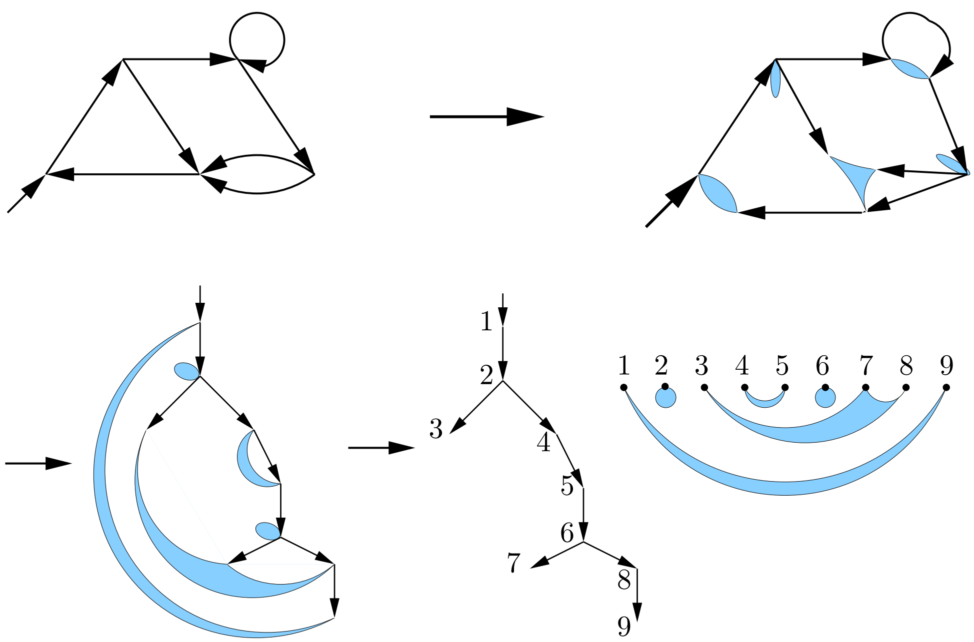


Figure 79: The vertex explosion process φ.

size n) and pairs made of a tree and a non-crossing partition (of size n and $n+1$ respectively).

3.3.1 Tree-rooted maps and tree-oriented maps

In this subsection, we consider certain orientations of maps called *tree-orientations* (Definition 3.2). We prove that the mapping $\delta : M_T \mapsto \vec{M}^T$ restricted to any given map M induces a bijection between spanning trees and tree-orientations of M. The key property explaining why the mapping δ is injective is that during a tour of a spanning tree T, the tails of edges in T are encountered before their heads whereas it is the contrary for the edges not in T. Using this property we will define a procedure γ for recovering spanning trees from tree-orientations of M (Definition 3.5). We will prove that δ and γ are reverse mappings that establish a one-to-one correspondence between tree-rooted maps and tree-oriented maps

(Proposition 3.3). The mapping $\delta M_T \mapsto \vec{M}^T$ shall appear again but in a more general setting in Chapter 6. In this Chapter, the mapping δ is extended into a bijection between subgraphs and orientations.

We begin with some definitions concerning cycles and paths in oriented maps. A simple cycle (resp. simple path) is *directed* if all its edges are oriented consistently. A simple cycle defines two regions of the sphere. The *interior region* (resp. *exterior region*) of a directed cycle is the region situated at its left (resp. right) as indicated in Figure 80. We call *positive cycle* a directed cycle having the root in its exterior region. Graphically, positive cycles appear as counterclockwise directed cycles when the map is projected on the plane with the root in the infinite face.

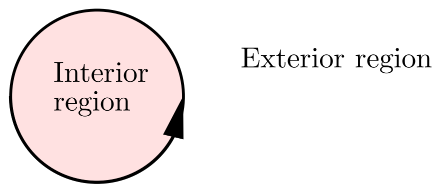


Figure 80: Interior and exterior regions of a directed cycle.

Definition 3.2 *A* tree-orientation *of a map is an orientation without positive cycle such that any vertex can be reached from the root by a directed path. A* tree-oriented map *is a map with a tree-orientation.*

We will prove that the images of tree-rooted maps by the mapping δ are tree-oriented maps. More precisely, we have the following proposition.

Proposition 3.3 *For any given map M, the mapping $\delta : M_T \mapsto \vec{M}^T$ induces a bijection between spanning trees and tree-orientations of M.*

We first prove the following lemma.

Lemma 3.4 *For all tree-rooted map M_T, the map $\vec{M}^T$ is tree-oriented.*

Proof: For any vertex v, there is a path in T from the root to v. This path is oriented from the root to v in $\vec{M}^T$. It remains to prove that there is no positive cycle. Suppose the contrary and consider a positive cycle C. By definition, the root is in the exterior region of C. Since C is a cycle there are edges of C which are not in T. Consider the first such edge e encountered during the tour of T. When we first cross e we enter for the first time the interior region of C. Given the orientation of C, the half-edge of e that we first cross is its tail (see Figure 81). But, by definition of $\vec{M}^T$, the half-edge of e that we first cross should be its head. This gives a contradiction.

$\square$

Figure 81: Entering the cycle C.

We now define a procedure γ constructing a spanning tree T on a tree-oriented map $\vec{M}$.

Algorithm 3.5

Procedure γ***:***

1. *At the beginning, the submap T is reduced to the root and the root-vertex.*

2. *We make the tour of T (starting from the root) and apply the following rule.*
   When the tail of an edge e is encountered and its head has not been encountered yet, we add e to T (together with its end).
   Then we continue the tour of T, that is, if e is in T we follow its border, otherwise we cross e.

3. *We stop when arriving at the root and return the submap T.*

We prove the correction of the procedure γ.

Lemma 3.6 *The mapping γ is well defined (terminates) on tree-oriented maps and returns a spanning tree.*

Proof:

• *At any stage of the procedure, the submap T is a tree.*

Suppose not, and consider the first time an edge e creating a cycle is added to T. We denote by T_0 the tree T just before that time. The edge e is added to T_0 when its tail t is encountered. At that time, its head h has not been encountered but is incident to T_0 (since adding e creates a cycle). We know that, when e is added, the border of T_0 from the root to t has been followed but not the border of T_0 from t to the root. Moreover, the head h lies after t around T_0 (since h has not been encountered yet). Observe that the right border of any edge of T_0 has been followed (just after this edge was added to T_0). Thus, the border of T_0 from t to h is made of the left borders of some edges $e_1, e_2, \ldots, e_k$. Hence, these edges form a directed path from h to t and $e, e_1, e_2, \ldots, e_k$ form a directed cycle C. Since h lies after t around T_0, the root is in the exterior region of C (see Figure 82). Therefore, the cycle C is positive which is impossible.

• *The procedure γ terminates.*

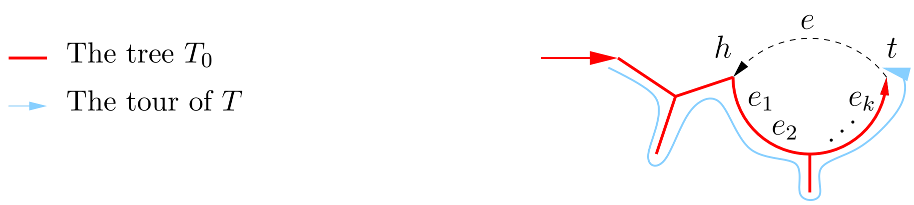


Figure 82: The submap T remains a tree.

The set T remains a tree connected to the root. Hence, it is impossible to follow the same border of the same edge twice without encountering the root.

• *At the end of the procedure γ, the tree T is spanning.*

At the end of the procedure, the whole border of T has been followed. Hence, any half-edge incident to T has been encountered. Now, suppose that a vertex v is not in T and consider a directed path from the root to v. (This path exists by definition of tree-orientations.) There is an edge of this path with its origin in T and its end out of T. Therefore, its tail is incident to T but not its head. Thus, it should have been added to T (with its end) when its tail was encountered. We reach a contradiction.

□

We continue the proof of Proposition 3.3. We proved that the mapping δ associates a tree-orientation of a map to any spanning tree of that map (Lemma 3.4). We proved that the mapping γ associates a spanning tree of a map to any tree-orientation of that map (Lemma 3.6). It remains to prove that $\delta \circ \gamma$ and $\gamma \circ \delta$ are identity mappings.

Lemma 3.7 *Let $\vec{M}$ be a tree-oriented map and T be the spanning tree constructed by the procedure γ. The edges in T are oriented from the root to the leaves and the edges not in T are oriented in such a way that their head precedes their tail around T.*

Proof:

• *Edges in T are oriented from the root to the leaves.* An edge e is added to T when its tail is encountered. At that time the end of e is not in T or adding e would create a cycle. The property follows by induction.

• *Edges not in T are oriented in such a way that their head precedes their tail around T.* If an edge breaks this rule it should have been added to T when its tail was encountered.

□

Corollary 3.8 *The mapping $\delta \circ \gamma$ is the identity mapping on tree-oriented maps.*

Proof: Let $\vec{M}$ be a tree-oriented map and T be the tree constructed by the procedure γ. By Lemma 3.7, the edges in T are oriented from the root to the leaves and the edges not in T

are oriented in such a way that their head precedes their tail around T. By definition of δ, this is also the case in $\delta \circ \gamma(\vec{M})$. Thus, $\delta \circ \gamma$ is the identity mapping on tree-oriented maps. □

Lemma 3.9 *The mapping $\gamma \circ \delta$ is the identity mapping on tree-rooted maps.*

Proof: Let M_T be a tree-rooted map. Suppose the spanning tree T' constructed by the procedure $\gamma(\delta(M_T))$ differs from T. We consider the order of edges induced by the tour of T. Let e be the smallest edge in the symmetric difference of T and T'. The tours of T and T' must coincide until a half-edge h of e is encountered. We distinguish the head and the tail of e according to its orientation in $\delta(M_T)$. If e is in T, its tail is encountered before its head around T (by definition of $\delta(M_T)$). In this case, h is a tail. If e is not in T', its head is encountered before its tail around T' (by Lemma 3.7). In this case, h is a head. Therefore, e cannot be in $T \setminus T'$. Similarly, e cannot be in $T' \setminus T$ since e being in T' implies that h is a head and e not being in T implies that h is a tail. We obtain a contradiction. □

This completes the proof of Proposition 3.3: tree-oriented maps are in one-to-one correspondence with tree-rooted maps. □

3.3.2 The vertex explosion process on tree-oriented maps

This subsection is devoted to the proof of the following proposition.

Proposition 3.10 *The mapping $\varphi : \vec{M} \mapsto (\varphi_0(\vec{M}), \varphi_1(\vec{M}))$ is a bijection between tree-oriented maps of size n and ordered pairs consisting of a tree of size n and a non-crossing partition of size $n+1$.*

We start with a lemma concerning the mapping φ_0.

Lemma 3.11 *The image of any tree-oriented map $\vec{M}$ by φ_0 is a tree (oriented from the root to the leaves).*

Proof: Let $\vec{M}$ be a tree-oriented map. Any vertex is incident to at least one head (there is a directed path from the root to any vertex), hence the mapping φ_0 is well defined. The image $\varphi_0(\vec{M})$ has the same number of edges, say n, as $\vec{M}$. The map $\vec{M}$ has $n+1$ heads (one per edge plus one for the root). Since any vertex in $\varphi_0(\vec{M})$ is incident to exactly one head, the image $\varphi_0(\vec{M})$ has $n+1$ vertices. Thus, it is sufficient to prove that $\varphi_0(\vec{M})$ has no cycle (it will imply the connectivity).
Suppose $\varphi_0(\vec{M})$ contains a simple cycle C. Since any vertex in C is incident to exactly one head, the edges of C are oriented consistently. We identify the edges of $\vec{M}$ and the edges of $\varphi_0(\vec{M})$. The edges of C form a cycle in $\vec{M}$ but this cycle might not be simple. We consider a directed path P in $\vec{M}$ from the root to a vertex v (of $\vec{M}$) incident with an edge of C.

We suppose (without loss of generality) that v is the only vertex of P incident with an edge of C. Let h be the head in P incident with v and t' be the first tail in C following h in counterclockwise direction around v. We can construct a directed simple cycle C' (in $\vec{M}$) made of edges in C and containing t' (see Figure 83). Let h' be the head of C' incident with v. Since C' is a directed cycle of the tree-oriented map $\vec{M}$, it contains the root in its interior region. Since v is the only vertex of P incident with an edge in C', the head h is in the interior region of C'. Therefore, in counterclockwise direction around v we have h, h' and t' (and possibly some other half-edges). We consider the tail t following h in the cycle C (considered as a directed simple cycle of $\varphi_0(\vec{M})$). By the choice of t' we know that t is between t' and h in counterclockwise direction around v (t and t' may be distinct or not). Hence, in counterclockwise direction around v we have h, h' and t. Hence, h' is not the first head encountered in counterclockwise direction around v starting from t. Therefore, by definition of the vertex explosion process, h' and t are not adjacent in $\varphi_0(\vec{M})$. We reach a contradiction.

□

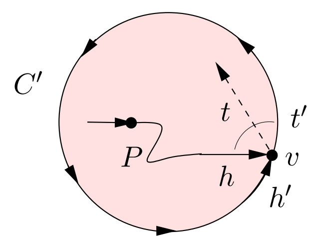


Figure 83: The cycle C' in $\vec{M}$.

We now study the properties of the mapping φ_1. Two consecutive half-edges around a vertex define a *corner*. A vertex has as many corners as incident half-edges. Let T be a tree and v be a vertex of T. The *first corner* of the vertex v is the first corner of v encountered around T. If the tree is oriented from the root to the leaves, the first corner of v is at the right of the head incident to v as shown in Figure 84.

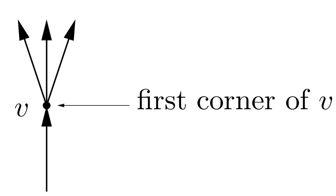


Figure 84: The first corner of a vertex.

We compare the vertices of the tree $\varphi_0(\vec{M})$ according to their order of appearance around this tree. We write $u < v$ if u precedes v (i.e. the first corner of u precedes the first corner of

v) around the tree.

Lemma 3.12 *For any tree-oriented map $\vec{M}$, the equivalence relation $\varphi_1(\vec{M})$ on the set of vertices of the tree $\varphi_0(\vec{M})$ ordered by their order of appearance around this tree is a non-crossing partition.*

Proof: The proof relies on the graphical representation of the equivalence relation $\sim = \varphi_1(\vec{M})$ given by Figure 78. During the vertex explosion process, we associate a connected cell C_v with each vertex v of $\vec{M}$, that is, with each equivalence class of the relation $\sim$. The cell C_v can be chosen to be incident only with the first corners of the vertices in its class but not otherwise incident with the tree. Moreover the cells can be chosen so that they do not intersect.
Suppose $v_1 < v_2 < v_3 < v_4$, $v_1 \sim v_3$ and $v_2 \sim v_4$. One can draw a path from the first corner of v_1 to the first corner of v_3 staying in a cell C and a path from the first corner of v_2 to the first corner of v_4 staying in a cell C'. It is clear that these two paths intersect (see Figure 85). Thus $C = C'$ and $v_1 \sim v_2$.

□

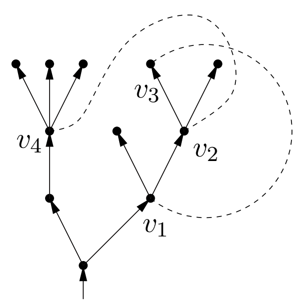


Figure 85: The two paths intersect.

We have proved that the application $\varphi : \vec{M} \mapsto (\varphi_0(\vec{M}), \varphi_1(\vec{M}))$ associates a tree of size n and a non-crossing partition of size $n+1$ with any tree-oriented map of size n. Conversely, we define the mapping ψ.

Definition 3.13 *Let T be a tree of size n and $\sim$ be a non-crossing partition on a linearly ordered set S of size $n+1$. We identify S with the set of vertices of T ordered by the order of appearance around T. We construct the oriented map $\psi(T,\sim)$ as follows. First we orient the tree T from the root to the leaves. With each part $\{v_1, v_2, \ldots, v_k\}$ of the partition, we associate a simply connected cell incident to the first corner of v_i, $i = 1 \ldots k$ but not otherwise incident with T. Since $\sim$ is a non-crossing partition, these cells can be chosen without intersections. Then we contract each cell into a vertex in such a way no edges of T intersect.*

We first prove the following lemma.

Lemma 3.14 *For any tree T of size n and any non-crossing partition $\sim$ of size $n+1$, the oriented map $\psi(T,\sim)$ is tree-oriented.*

Proof: Every vertex of $\vec{M} = \psi(T, \sim)$ is connected to the root by a directed path (since it is the case in T). It remains to show that there is no positive cycle.
Let C be a positive cycle of $\vec{M}$ and e an edge of C. We consider the directed path P of T from the root to e (the root and e included). By definition, the root is in the exterior region of C. Let h be the last head of P contained in the exterior region of C and t the tail following h in P (the tail t exists since the last edge e of P is in C). By definition, the tail t is either in C or in its interior region. Let v be the end of h (i.e the origin of t) in $\vec{M}$ and h' the head of C incident with v (see Figure 86). In counterclockwise direction around v, we have h, t and h' (and possibly some other half-edges). The vertex v is obtained by contracting a cell C_v of the partition $\sim$ corresponding to some vertices of T. Each of these vertices is incident to one head in T, hence h and h' were incident to two distinct vertices, say v_1 and v_2, of T. The cell C_v is incident to the first corner of v_1 which is situated between h and t in counterclockwise direction around v_1. Therefore, after the cell C_v is contracted, the half-edges of v_2 are situated between h and t in counterclockwise direction around v. Thus, in counterclockwise direction around v, we have h, h' and t (and possibly some other half-edges). We obtain a contradiction.

□

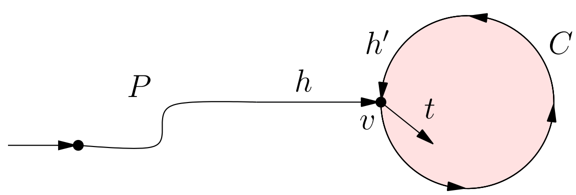


Figure 86: The map $\vec{M} = \psi(T, \sim)$ has no positive cycle.

We now conclude the proof of Theorem 3.1.

• Let $\vec{M}$ be a tree-oriented map. We know from Lemma 3.11 that $T = \varphi_0(\vec{M})$ is a tree oriented from the root to the leaves. Moreover, we know from Lemma 3.12 that the partition $\sim = \varphi_0(\vec{M})$ of the vertex set of T is non-crossing. Let u be a vertex of T. Let $\{v_1, \ldots, v_k\}$ be a part of the partition $\sim$ corresponding to a vertex v of $\vec{M}$. The cell C_v associated to v during the vertex explosion process is incident to the corner of v_i, $i = 1 \ldots k$ at the right of the head incident with v_i (see Figure 78). Since T is oriented from the root to the leaves, this corner is the first corner of v_i. Therefore, by definition of ψ, we have $\psi \circ \varphi(\vec{M}) = \vec{M}$. Thus, $\psi \circ \varphi$ is the identity mapping on tree-oriented maps.

• Let T be a tree of size n and $\sim$ be a non-crossing partition on a linearly ordered set S of size $n+1$. We know from Lemma 3.14 that $\vec{M} = \psi(T, \sim)$ is a tree-oriented map. We think to the tree T as being oriented from the root to the leaves and we identify the set S with the vertex set of T. Let v be a vertex of $\vec{M}$ corresponding to the part $\{v_1, \ldots, v_k\}$ of the partition $\sim$. The vertex v is obtained by contracting a cell C_v incident with the first corner of v_i,

$i = 1 \dots k$, that is, the corner at the right of the head h_i incident with v_i. Therefore, if t is a tail incident with v_i in T, then, h_i is the first head encountered in counterclockwise direction around v starting from t (in $\vec{M}$). Given the definition of the vertex explosion process, the adjacency relations between the half-edges incident with v that are preserved by the vertex explosion process are exactly the adjacency relations in the tree T. Thus, the trees $\varphi_0(\vec{M})$ and T are the same. Moreover, the part of the partition $\varphi_1(\vec{M})$ associated to the vertex v is $\{v_1, \dots, v_k\}$. Thus, the partitions $\varphi_1(\vec{M})$ and $\sim$ are the same. Hence, $\varphi \circ \psi$ is the identity mapping on pairs made of a tree of size n and a non-crossing partition of size $n+1$.
Thus, the mapping φ is a bijection between tree-oriented maps of size n and pairs made of a tree of size n and a non-crossing partition of size $n+1$. This completes the proof of Proposition 3.10 and Theorem 3.1.

□

3.4 Correspondence with a bijection due to Cori, Dulucq and Viennot

In this section, we prove that our bijection Φ is isomorphic to a former bijection due to Cori, Dulucq and Viennot defined on parenthesis-shuffles [Cori 86]. We know that tree-rooted maps are in one-to-one correspondence with parenthesis-shuffles by the mapping Ξ defined in Section 3.2. Our bijection $\Phi : M_T \mapsto (\varphi_0(\vec{M}^T), \varphi_1(\vec{M}^T))$ associates with any tree-rooted map M_T of size n, a tree $\varphi_0(\vec{M}^T)$ of size n and a non-crossing partition $\varphi_1(\vec{M}^T)$ of size $n+1$. The bijection $\Lambda : w \mapsto (\lambda'_0(w), \lambda'_1(w))$ of Cori *et al.* associates with any parenthesis-shuffle w of size n, a tree $\lambda'_0(w)$ of size n and a binary tree $\lambda'_1(w)$ of size $n+1$. We shall prove that these two bijections are isomorphic via the encoding of tree-rooted maps by parenthesis-shuffles. That is, we shall prove that there exist two independent bijections Ω and Θ such that, if $w = \Xi(M_T)$, then $\varphi_0(\vec{M}^T) = \Omega(\lambda'_0(w))$ and $\varphi_1(\vec{M}^T) = \Theta(\lambda'_1(w))$. In fact, we have adjusted some definitions from [Cori 86] so that Ω is the identity mapping on trees. This situation is represented in Figure 87.

$$\begin{tikzcd}
\text{Tree-rooted maps} \; M_T \arrow[rrr, bend left, "\Phi"] \arrow[r, shift left, "\delta"] \arrow[d, "\Xi"'] & \text{Tree-oriented maps} \; \vec{M}^T \arrow[l, shift left, "\gamma"] \arrow[r, shift left, "\varphi"] & \text{Trees} \times \text{Non-crossing partitions} \; \varphi_0(\vec{M}^T), \arrow[l, shift left, "\psi"] & \varphi_1(\vec{M}^T) \\
w \; \text{Parenthesis-shuffles} \arrow[rr, "\Lambda"] & & \text{Trees} \times \text{Binary trees} \; \lambda'_0(w), \arrow[u, "Id"] & \lambda'_1(w) \arrow[u, "\Theta"']
\end{tikzcd}$$

Figure 87: The bijection diagram.

3.4.1 The bijection Λ of Cori, Dulucq and Viennot

We begin with a presentation of the bijection Λ of Cori *et al.* For the sake of simplicity, the presentation given here is not *completely* identical to the one of the original article [Cori 86]. But, whenever our definitions differ there is an obvious equivalence via a composition with a simple, well-known bijection. The interested reader can look for more details in the original article. In this article, Cori *et al.* defined recursively two mappings λ_0 and λ_1 on the set of *prefix-shuffles*. A prefix-shuffle is a word w on the alphabet $\{a, \overline{a}, b, \overline{b}\}$ such that, for all prefixes w' of w, we have $|w'|_a \geq |w'|_{\overline{a}}$ and $|w'|_b \geq |w'|_{\overline{b}}$. Note that the set of prefix-shuffles is the set of prefixes of parenthesis-shuffles. The mappings λ_0 and λ_1 both eventually return trees. In the original article [Cori 86], the trees returned by λ_0 and λ_1 were called the *leaf code* and the *tree code* respectively.

We first define the mapping λ_0. It involves the mapping σ that associates the tree $\sigma(T_1, T_2)$ represented in Figure 88 with the ordered pair of trees (T_1, T_2).

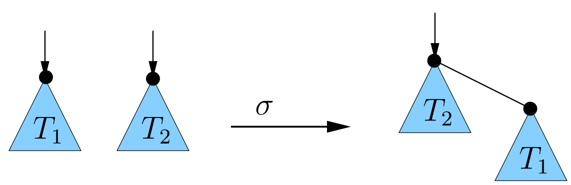


Figure 88: The mapping σ on ordered pairs of trees.

We consider the alphabet $\mathbf{U} = \{u, v\}$ and the infinite alphabet $\mathbf{T}$ consisting of all trees. A word s on the alphabet $\mathbf{U} \cup \mathbf{T}$ is a *tree-sequence* if $s = ut_1u \ldots t_{i-1}ut_ivt_{i+1} \ldots t_kv$ where $1 \leq i \leq k$ and $t_1, \ldots, t_k$ are trees. The mapping λ_0 associates tree-sequences with prefix-shuffles.

Definition 3.15 *The mapping λ_0 is recursively defined on prefix-shuffles by the following rules:*

- *If $w = \epsilon$ is the empty word, $\lambda_0(w)$ is the tree-sequence $u\tau v$ where τ is the tree reduced to a root and a vertex.*

  τ : 

- *If $w = w'a$, the tree-sequence $\lambda_0(w)$ is obtained from $\lambda_0(w')$ by replacing the last occurrence of u by $u\tau v$.*
- *If $w = w'b$, the tree-sequence $\lambda_0(w)$ is obtained from $\lambda_0(w')$ by replacing the first occurrence of v by $u\tau v$.*
- *If $w = w'\overline{a}$, we consider the first occurrence of v in $\lambda_0(w')$ and the trees T_1 and T_2 directly preceding and following it. The tree-sequence $\lambda_0(w)$ is obtained from $\lambda_0(w')$ by*

replacing the subword T_1vT_2 by the tree $\sigma(T_1,T_2)$.

- *If $w = w'\overline{b}$, we consider the last occurrence of u in $\lambda_0(w')$ and the trees T_1 and T_2 directly preceding and following it. The tree-sequence $\lambda_0(w)$ is obtained from $\lambda_0(w')$ by replacing the subword T_1uT_2 by the tree $\sigma(T_1,T_2)$.*

We applied the mapping λ_0 to the word $w = ba\overline{a}a\overline{b}\overline{a}$. The different steps are represented in Figure 89.

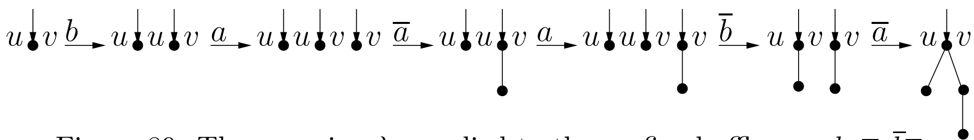


Figure 89: The mapping λ_0 applied to the prefix-shuffle $w = ba\overline{a}a\overline{b}\overline{a}$.

It is easily seen by induction that the number of v (resp. u) in $\lambda_0(w)$ is $|w|_a - |w|_{\overline{a}} + 1$ (resp. $|w|_b - |w|_{\overline{b}} + 1$). Hence, the mapping λ_0 is well defined on prefix-shuffles. Moreover, the first letter u and last letter v are never replaced by anything. Observe also (by induction) that the letters u always precede the letters v in $\lambda_0(w)$. Thus, $\lambda_0(w)$ is indeed a tree-sequence. If w is a parenthesis-shuffle, there is exactly one letter u and one letter v in $\lambda_0(w)$, hence $\lambda_0(w)$ is a three letter word uTv.

Definition 3.16 *The mapping λ'_0 associates with a parenthesis-shuffle w the unique tree T in the tree-sequence $\lambda_0(w) = uTv$.*

Observe that, for any prefix-shuffle w, the total number of edges in the trees $t_1,\dots,t_k$ of the tree-sequence $\lambda_0(w) = ut_1u\dots t_{i-1}ut_ivt_{i+1}\dots t_kv$ is $|w|_{\overline{a}} + |w|_{\overline{b}}$. Hence, if w is parenthesis-shuffle of size n, the tree $\lambda'_0(w)$ has size n.

We now define the mapping λ_1 which associates *binary trees* with prefix-shuffles. A binary tree is a (planted plane) tree for which each vertex is either of degree 3, a *node*, or of degree 1, a *leaf*. The size of a binary tree is defined as the number of its nodes. It is well-known that binary trees of size n (i.e. with n nodes) are in one-to-one correspondence with trees of size n (i.e. with n edges).

In a binary tree, the two sons of a node are called *left son* and *right son*. In counter-clockwise order around a node we find the father (or the root), the left son and the right son (see Figure 90). A *left leaf* (resp. *right leaf*) is a leaf which is a left son (resp. right son). As before, we compare vertices according to their order of appearance around the tree and we shall talk about the *first* and *last* leaf. Moreover, a leaf will be either *active* or *inactive*. Graphically, active leaves will be represented by circles and inactive ones by squares.

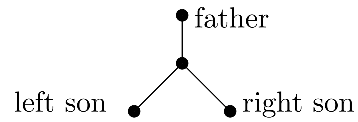
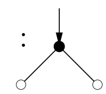


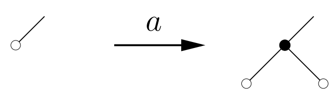
Figure 90: Left and right son of a node

Definition 3.17 *The mapping λ_1 is recursively defined on prefix-shuffles by the following rules:*

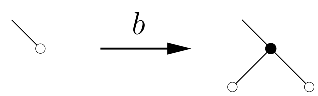
- *If $w = \epsilon$ is the empty word, $\lambda_1(w)$ is the binary tree B_1 consisting of a root, a node and two active leaves.*

B_1 :


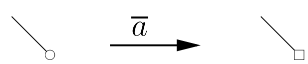
- *If $w = w'a$, the tree $\lambda_1(w)$ is obtained from $\lambda_1(w')$ by replacing the last active left leaf by B_1.*



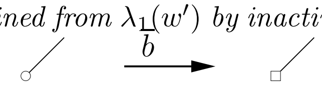
- *If $w = w'b$, the tree $\lambda_1(w)$ is obtained from $\lambda_1(w')$ by replacing the first active right leaf by B_1.*



- *If $w = w'\overline{a}$, the tree $\lambda_1(w)$ is obtained from $\lambda_1(w')$ by inactivating the first active right leaf.*



- *If $w = w'\overline{b}$, the tree $\lambda_1(w)$ is obtained from $\lambda_1(w')$ by inactivating the last active left leaf.*



We applied the mapping λ_1 to the word $w = ba\overline{a}a\overline{b}\overline{a}$. The different steps are represented in Figure 91.

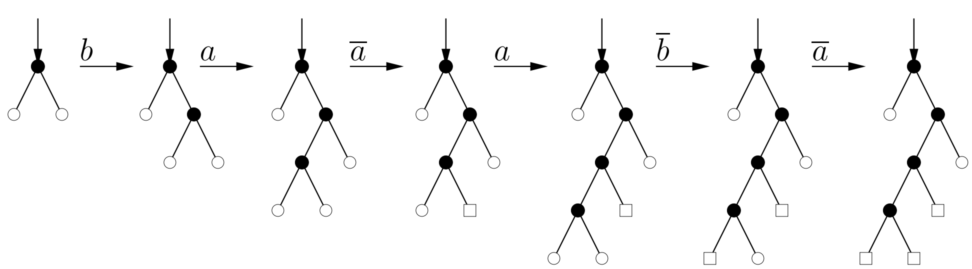


Figure 91: The mapping λ_1 on the word $w = ba\overline{a}a\overline{b}\overline{a}$.

It is easily seen by induction that the number of active right leaves (resp. left leaves) in $\lambda_1(w)$ is $|w|_a - |w|_{\overline{a}} + 1$ (resp. $|w|_b - |w|_{\overline{b}} + 1$). Hence, the mapping λ_1 is well defined on prefix-shuffles. Observe that the binary tree $\lambda_1(w)$ has $|w|_a + |w|_b + 1$ nodes. Observe also (by induction) that active left leaves always precede active right leaves in $\lambda_1(w)$. Moreover, if w is a parenthesis-shuffle, only the first left leaf and the last right leaf are active (since they can never be inactivated).

Definition 3.18 *The mapping λ'_1 associates with a parenthesis-shuffle w of size n the binary tree of size $n+1$ obtained from $\lambda_1(w)$ by inactivating the two active leaves.*

We now make some informal remarks explaining why the mapping $w \mapsto (\lambda_0(w), \lambda_1(w))$ is injective. It is, of course, possible to decide from $(\lambda_0(w), \lambda_1(w))$ if w is the empty word. Indeed, w is the empty word iff $\lambda_1(w) = B_1$ (equivalently iff $\lambda_0(w) = \tau$). Otherwise, the remarks below show that the last letter α of $w = w'\alpha$ can be determined as well as $\lambda_0(w')$ and $\lambda_1(w')$. So any prefix-shuffle w can be entirely recovered from $(\lambda_0(w), \lambda_1(w))$.

Remarks:

• For any prefix-shuffle w, the number of letters u (resp. v) in the tree-sequence $\lambda_0(w)$ is equal to the number of active left leaves (resp. right leaves) in the binary tree $\lambda_1(w)$. Furthermore, it can be shown by induction that the size of the tree t_i lying between the i^{th} and $i+1^{th}$ letters u, v in $\lambda_0(w)$ is the number of inactive leaves lying between the i^{th} and $i+1^{th}$ active leaves in $\lambda_1(w)$.

• The three following statements are equivalent:

- the word w is not empty and the last letter α of $w = w'\alpha$ is in $\{a, b\}$,
- there is a sequence $u\tau v$ in $\lambda_0(w)$,
- there is an active left leaf and an active right leaf which are siblings.

In this case, $\lambda_1(w')$ is obtained from $\lambda_1(w)$ by deleting the two actives leaves and making the father an active leaf ℓ. Moreover, $\alpha = a$ (resp. $\alpha = b$) if ℓ is a left leaf (resp. right leaf) in $\lambda_1(w')$ in which case $\lambda_0(w')$ is obtained from $\lambda_0(w)$ by replacing the subword $u\tau v$ by u (resp. v).

• If the last letter α of $w = w'\alpha$ is in $\{\overline{a}, \overline{b}\}$, we know from the above remark that the tree T lying between the last letter u and the first letter v in the tree-sequence $\lambda_0(w)$ has size $k > 0$. Since $k > 0$, the tree T admits a (unique) preimage (T_1, T_2) by the mapping σ. Let k' be the size of the tree T_1. Then $k' < k$. We know that there are k inactive leaves lying between the last active left leaf and the first active right leaf in $\lambda_1(w)$. The binary tree $\lambda_1(w')$ is obtained from $\lambda_1(w)$ by activating the $k'+1^{th}$ leaf ℓ encountered when following the border of the tree starting from the last active left leaf. Moreover, $\alpha = \overline{a}$ (resp. $\alpha = \overline{b}$) if ℓ is a right leaf (resp. left leaf), in which case the tree-sequence $\lambda_0(w')$ is obtained from $\lambda_0(w)$ by replacing T by $T_1 v T_2$ (resp. $T_1 u T_2$).

From these remarks, we see that the mapping $w \mapsto (\lambda_0(w), \lambda_1(w))$ is injective. It can be shown, with the same ideas, that it is bijective on the set of pairs consisting of a tree-sequence S and a binary tree B with active and inactive leaves satisfying the following conditions:
- the active left leaves precede the active right leaves in B,
- the number of active left leaves (resp. right leaves) in B is the same as the number of u (resp. v) in S,
- the number of inactive leaves lying between the i^{th} and $i+1^{th}$ active leaves in B is the size of the tree lying between the i^{th} and $i+1^{th}$ letters u, v in S.

We now define the mapping Λ of Cori *et al.* on parenthesis-shuffles.

Definition 3.19 *The mapping $w \mapsto (\lambda'_0(w), \lambda'_1(w))$ defined on parenthesis-shuffles is denoted Λ.*

We know that Λ associates with a parenthesis-shuffle of size n a pair consisting of a tree of size n and a binary tree of size $n+1$. The remarks above should convince the reader that the mapping Λ is a bijection between these two sets of objects.

3.4.2 The bijections Φ and Λ are isomorphic

We now return to our business and prove that the bijection Λ of Cori *et al.* and our bijection Φ are isomorphic. Before stating precisely this result, we define a (non-classical) bijection θ between binary trees and trees. By composition, this allows us to define a bijection Θ between binary trees and non-crossing partitions.

Let e be an edge of a binary tree. The edge e is said to be branching if one of its vertices is a right son and the other is a left son or the root-vertex. Intuitively, this means that the edge e is non-parallel to its parent-edge. For instance, the branching edges of the binary tree in Figure 92 are indicated by thick lines.

Definition 3.20 *Let B be a binary tree. The tree $\theta(B)$ is obtained by contracting every non-branching edge. The non-crossing partition $\Theta(B)$ is the image of $\theta(B)$ by the mapping Υ^{-1} (see Figure 75).*

We applied the mapping Θ to the binary tree of Figure 92.

The mapping Θ is a bijection between binary trees of size n (n nodes) and trees of size n (n edges). The proof is omitted here since we will not use this property.
We now state the main result of this section.

Theorem 3.21 *Let M_T be a tree-rooted map and $w = \Xi(M_T)$ its associated parenthesis-shuffle. Let $\varphi_0(\vec{M}^T)$ and $\varphi_1(\vec{M}^T)$ be the tree and the non-crossing partition obtained from*

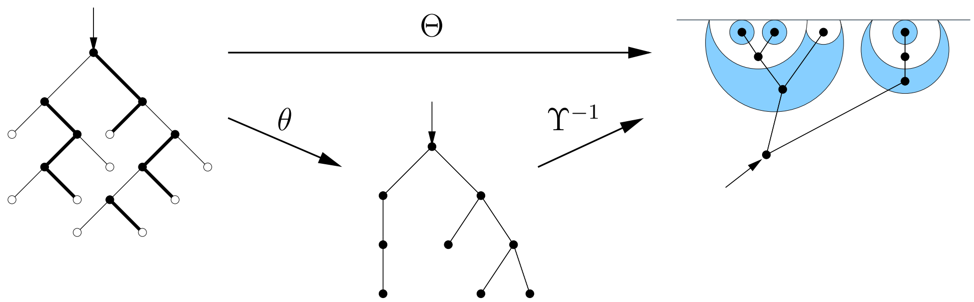


Figure 92: The mappings θ and Θ.

M_T by the mapping Φ. Let $\lambda'_0(w)$ and $\lambda'_1(w)$ be the tree and binary tree obtained from w by the mapping Λ. Then $\varphi_0(\vec{M}^T) = \lambda'_0(w)$ and $\varphi_1(\vec{M}^T) = \Theta(\lambda'_1(w))$.

This relation between the mappings Λ and Φ is represented by Figure 87. As an illustration, we applied the mapping Φ to the tree-rooted map M_T of Figure 93 and we applied the mapping Λ to $w = \Xi(M_T) = ba\bar{a}a\bar{b}\bar{a}$. The rest of this section is devoted to the proof of Theorem 3.21.

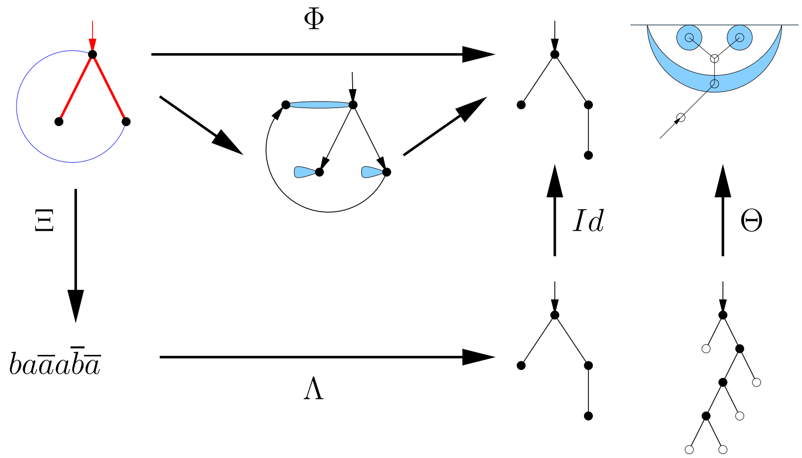


Figure 93: The isomorphism between Λ and Φ.

3.4.3 Prefix-maps

The mappings λ'_0 and λ'_1 are defined on parenthesis-shuffles from the more general mappings λ_0 and λ_1 defined on prefix-shuffles. In order to relate $\varphi_0(\vec{M}^T)$ and $\lambda'_0(w)$ (resp. $\varphi_1(\vec{M}^T)$ and $\lambda'_1(w)$) we need to define the *prefix-maps* which are in one-to-one correspondence with prefix-shuffles. As we will see, prefix-maps are tree-oriented maps together with some *dangling* heads in the root-face. In Subsections 3.4.4 and 3.4.5 we shall extend the mappings

φ_0 and φ_1 defined in Section 3.3 to prefix-maps.

For any prefix-shuffle w we denote by w_a (resp. w_b) the subword of w consisting of the letters $a, \overline{a}$ (resp. $b, \overline{b}$). The words w_a and w_b are prefixes of parenthesis systems. We say that an occurrence of a letter $c = a, b$ is *paired* with an occurrence of $\overline{c}$ if the subword of w_c lying between these two letters is a parenthesis system. There are $|w|_a - |w|_{\overline{a}}$ non-paired letters a and $|w|_b - |w|_{\overline{b}}$ non-paired letters b in w. We denote by w_a^+ the parenthesis system obtained from w_a by adding $|w|_a - |w|_{\overline{a}}$ letters $\overline{a}$ at the end of this word.

Let w be a prefix-shuffle. We define T_w as the tree associated to the parenthesis system w_a^+, that is, T_w is such that, making the tour of T_w and writing a the first time we follow an edge and $\overline{a}$ the second time, we obtain w_a^+. We orient the edges of T_w from the root to the leaves. Then, we add half-edges to T_w by looking at the position of the letters b and $\overline{b}$ in w. More precisely, we read the word w and while making the tour of T according to the letters $a, \overline{a}$, we insert heads for the letters b and tails for the letters $\overline{b}$. If an occurrence of b and an occurrence of $\overline{b}$ are paired in w we connect the corresponding head and tail. We obtain an oriented map together with some heads called *dangling heads* corresponding to non-paired letters b of w. In the tree T_w, the edges corresponding to non-paired letters a are called *active* while the others are called *inactive*. We denote by M_w, and call *prefix-map associated with* w, the oriented map (with dangling heads and active edges) obtained. For instance, the prefix-map associated with $bab\overline{a}a\overline{b}a\overline{b}\overline{a}ab$ has been represented in Figure 94 (the active edges are dashed).
Observe that T_w is a spanning tree of the prefix-map M_w. The orientation of M_w is the tree-orientation associated to the spanning tree T_w by the mapping δ defined in Section 3.3. In particular, when w is a parenthesis-shuffle, the prefix-map M_w is a map (i.e. it has no active edge and no dangling head except for the root) which is tree-oriented. More precisely, if $w = \Xi(M_T)$, the tree-oriented map M_w is $\vec{M}^T \equiv \delta(M_T)$.

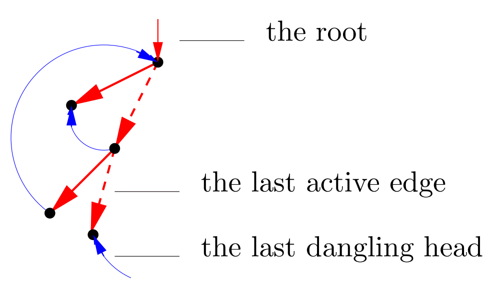


Figure 94: The prefix-map associated to $bab\overline{a}a\overline{b}a\overline{b}\overline{a}ab$.

Let w be a prefix-shuffle. The heads of active edges in the prefix map M_w are called

rooting heads, and their ends are called *rooting vertices*. By convention, the root is considered as a rooting head. As before, we compare active edges (resp. rooting vertices, dangling heads) of M_w according to their order of appearance around T_w. By convention, the root is considered as the first rooting head.
Let w^+ be the word w followed by $|w|_a - |w|_{\overline{a}}$ letters $\overline{a}$. We obtain w^+ by making the tour of the tree T_w and writing a the first time we follow an edge of the tree, $\overline{a}$ the second time, b when we cross a head not in the tree and $\overline{b}$ when we cross a tail not in the tree. Each prefix of w^+ corresponds to a given time in this journey. In particular, w corresponds to a given corner c of a vertex v. The $|w|_a - |w|_{\overline{a}}$ letters $\overline{a}$ at the end of w^+ correspond to the left border of active edges followed from c to the root. Thus, the active edges are the edges on the directed path of T_w from the root to v. Note that an active edge precedes another one if it appears before on the path from the root to v. Therefore, v is the last rooting vertex and c is the corner at the left of the last rooting head. Moreover, active edges are directed from a rooting vertex to the next one (for the appearance order). In particular, the next-to-last rooting vertex (if it exists) is the origin of the last active edge.

We now explore the relation between M_w and $M_{w\alpha}$ when α is a letter in $\{a, \overline{a}, b, \overline{b}\}$.

Lemma 3.22 *Let c be the corner at the left of the last rooting head of M_w.*

- *M_{wa} is obtained from M_w by adding an edge e in the corner c. It is oriented from this corner to a vertex not present in M_w. The edge e is the last active edge of M_{wa}.*
- *M_{wb} is obtained from M_w by adding a dangling head h in the corner c. The head h is the last dangling head of M_{wb}.*
- *$M_{w\overline{a}}$ is obtained from M_w by inactivating the last active edge e. The origin of e becomes the last rooting vertex.*
- *$M_{w\overline{b}}$ is obtained from M_w by adding a tail in the corner c and connecting it to the last dangling head.*

In any case, the appearance order on the edges, half-edges and vertices present in M_w is the same in $M_{w\alpha}$.

Proof: As mentioned above, the corner c is the corner reached when the word w is written during the tour of T_w in M_w.
• Case $\alpha = a$. The letter a added to w is not paired. Therefore, it corresponds to a new active edge e added to T_w. This new edge is added in the corner c. The edge e is oriented from c to a new vertex (since it is leaf of T_{wa}). All active edges of M_w are encountered before c around the spanning tree T_w. Therefore, e is the last active edge of M_{wa}.
• Case $\alpha = b$. The letter b added to w is not paired. Therefore, it corresponds to a new dangling head h. This new head is added in the corner c. All dangling heads of M_w are

encountered before c around the spanning tree T_w. Therefore, h is the last dangling head of M_{wb}.

• Case $\alpha = \overline{a}$. The last letter a of w is paired with the letter $\overline{a}$ added to w. This last letter a corresponds to the last active edge. Therefore, the last active edge e of M_w is inactivated. We know that the next-to-last rooting vertex of M_w is the origin v of the last active edge e. Therefore, v becomes the last rooting vertex.

• Case $\alpha = \overline{b}$. The last letter b of w is paired with the letter $\overline{b}$ added to w. This last letter b corresponds to the last dangling head h'. Hence, $M_{w\overline{b}}$ is obtained from M_w by adding a tail h in the corner c and connecting it to h'. $\square$

This completes our study of prefix-maps. We are now ready to extend the mappings φ_0 and φ_1 to prefix maps and to prove Theorem 3.21.

3.4.4 The trees $\varphi_0(\vec{M}^T)$ and $\lambda'_0(w)$ are the same

In this subsection, we prove that, when $w = \Xi(M_T)$, the trees $\varphi_0(\vec{M}^T)$ and $\lambda'_0(w)$ are the same.

Let w be a prefix-shuffle and M_w the corresponding prefix-map. Note that any vertex of M_w is incident to at least one head. The *prefix-forest* of w, denoted by F_w, is obtained by deleting the tails of active edges and then applying the vertex explosion process of Figure 78 (we forget about the cells corresponding to the parts of the non-crossing partition). We will prove that the prefix-forest is indeed a forest (i.e. a collection of trees) in Proposition 3.23. For instance, we represented the prefix-forest of $w = bab\overline{a}a\overline{b}a\overline{b}\overline{a}ab$ in Figure 95.

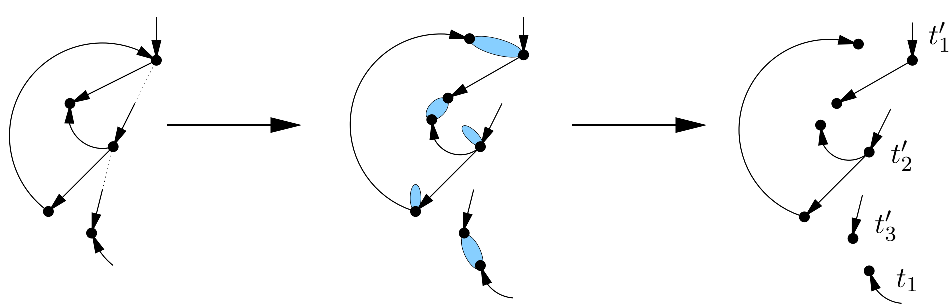


Figure 95: The prefix-forest F_w (on the right).

Note that, if $w = \Xi(M_T)$ is a parenthesis-shuffle, the prefix-map M_w is $\vec{M}^T$ and no edge is active. Thus, in this case, the prefix-forest F_w is the tree $\varphi_0(\vec{M}^T)$. We now prove a relation between the prefix-forest F_w and the tree-sequence $\lambda_0(w)$.

Proposition 3.23 *Let w be a prefix-shuffle. Let $h_1 < \cdots < h_k$ be the dangling heads and $h'_1 < \cdots < h'_l$ be the rooting heads of the prefix-map M_w (linearly ordered by the appearance* *order). The prefix-forest F_w is a collection of $k+l$ trees $t_1,\ldots,t_k,t'_1,\ldots,t'_l$. The root of the tree $t_i, i=1,\ldots,k$ is h_i and the root of the tree $t'_i, i=1,\ldots,l$ is h'_i. Moreover, the tree-sequence $\lambda_0(w)$ is $ut_1u\ldots ut_kut'_lv\ldots vt'_1v$.*

Proof: We use Lemma 3.22 and prove the property by induction on the length of w.
If w is the empty word, the prefix-map M_w is the tree τ reduced to a vertex and a root. Hence, the prefix-forest F_w is reduced to a single tree $\tau = t'_1$. The tree-sequence $\lambda_0(w')$ is equal to $u\tau v$ thus the property is satisfied. If $w' = w\alpha$, we suppose the lemma true for w, we write $\lambda_0(w) = ut_1u\ldots ut_kut'_lv\ldots vt'_1v$ and study separately the four possible cases.

• Case $\alpha = a$. The prefix-map M_{wa} is obtained from M_w by adding an edge e incident to the last rooting vertex. The edge e is the last active edge of M_{wa}. It is oriented toward a new vertex v not present in M_w. The tail of e is deleted in the construction of F_{wa} and its head $h = h'_{l+1}$ is only incident to v. Therefore, F_{wa} is obtained from F_w by adding the tree $\tau = t'_{l+1}$ (the tree reduced to a root and a vertex) rooted on the last rooting head h.
By definition, $\lambda_0(wa) = ut_1u\ldots ut_ku\tau vt'_lv\ldots vt'_1v$, so we observe that the property is satisfied by wa.

• Case $\alpha = b$. The prefix-map M_{wb} is obtained from M_w by adding a dangling head $h = h_{k+1}$ in the corner at the left of the last rooting head h'_l. Therefore, during the vertex explosion process h "steals" the tree t'_l rooted on h'_l in F_w (see Figure 96). That is, in F_{wb} the tree rooted on h'_l is reduced to a vertex and the tree rooted on h is t'_l. The head h is the last dangling head of M_{wb}.

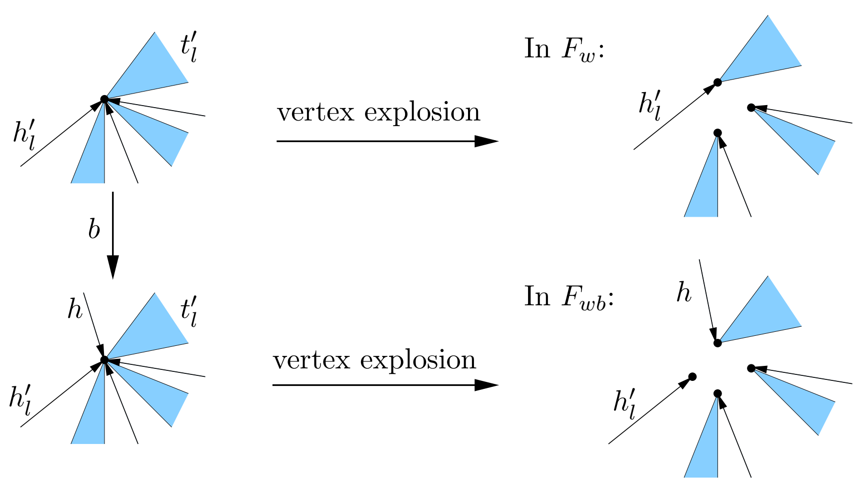


Figure 96: The case $\alpha = b$.

By definition, $\lambda_0(wb) = ut_1u\ldots ut_kut'_lu\tau vt'_{l-1}\ldots vt'_1v$, so we observe that the property is satisfied by wb.

• Case $\alpha = \overline{a}$. The prefix-map $M_{w\overline{a}}$ is obtained from M_w by inactivating the last active edge e. The origin v of e is the next-to-last rooting vertex of M_w. Moreover, e is the first edge encountered in clockwise order around v starting from h'_{l-1}. In $F_{w\overline{a}}$, the head h'_l is part of the edge e which links the tree t'_l to the tree t'_{l-1} rooted on h'_{l-1} (see Figure 97). Therefore, the tree rooted on h'_{l-1} in $F_{w\overline{a}}$ is $t = \sigma(t'_l, t'_{l-1})$.

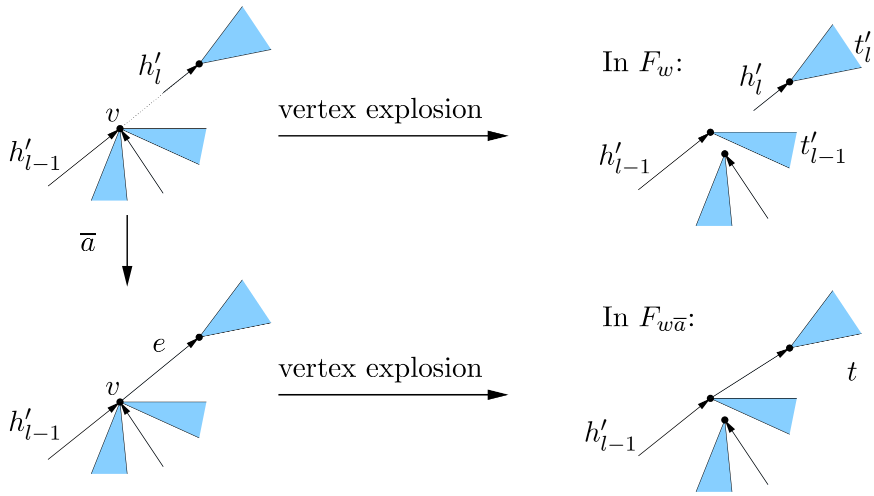


Figure 97: The case $\alpha = \overline{a}$.

By definition, $\lambda_0(w\overline{a}) = ut_1u \dots ut_kutvt'_{l-2} \dots vt'_1v$, so we observe that the property is satisfied by $w\overline{a}$.

• Case $\alpha = \overline{b}$. The prefix-map $M_{w\overline{b}}$ is obtained from M_w by adding a tail in the corner at the left of the last rooting head h'_l and connecting it to the last dangling head h_k. In $F_{w\overline{b}}$, the head h_k is part of an edge e which links the tree t_k to the tree t'_l rooted on h'_l. Therefore, the tree rooted on h'_l in $F_{w\overline{b}}$ is $t = \sigma(t_k, t'_l)$. The illustration would be the same as Figure 97 except $h'_{l-1}, h'_l, t'_{l-1}, t'_l$ would be replaced by h'_l, h_k, t'_l, t_k respectively.

By definition, $\lambda_0(w\overline{b}) = ut_1u \dots t_{k-1}utvt'_{l-1}v \dots vt'_1v$, so we observe that the property is satisfied by $w\overline{b}$.

$\square$

As mentioned above, when w is a parenthesis-shuffle $w = \Xi(M_T)$, the prefix-map M_w is the tree-oriented map $\vec{M}^T$ and the prefix-forest F_w is the tree $\varphi_0(\vec{M}^T)$. Therefore, Proposition 3.23 implies that the tree-sequence $\lambda_0(w)$ is equal to $u\varphi_0(\vec{M}^T)v$. Thus, the trees $\lambda'_0(w)$ and $\varphi_0(\vec{M}^T)$ are the same.

$\square$

3.4.5 The partitions $\varphi_1(\vec{M}^T)$ and $\Theta \circ \lambda'_1(w)$ are the same

In this subsection, we prove that, when $w = \Xi(M_T)$, the non-crossing partition $\varphi_1(\vec{M}^T)$ is the image of the binary tree $\lambda'_1(w)$ by the mapping Θ defined in Definition 3.20.

Let M_T be a tree-rooted map. We call *partition-tree* of M_T the tree $P = \Upsilon \circ \varphi_1(\vec{M}^T)$ (the mapping Υ is represented in Figure 75). Observe that the tree P can be drawn directly on the map obtained after the vertex explosion process of Figure 78. To do so, one keeps the cells corresponding to the vertices of $\vec{M}^T/$ (These cells are glued to the first corner of the vertices of the tree $\varphi_0(\vec{M}^T)$). Then, one draws a vertex in each face of M_T and in each cell corresponding to a vertex of M_T: this gives the vertices of P. The edges of P join vertices in adjacent cells and faces. The tree is rooted canonically. In particular, the root-vertex of P lies in the root-face of M_T. This construction is illustrated in Figure 98.

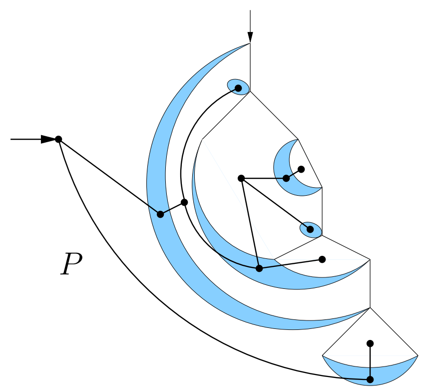


Figure 98: The partition-tree of a tree-rooted map.

We want to extend this construction to prefix-maps. We need some extra vocabulary. Consider a prefix-shuffle w and the corresponding prefix map M_w. We denote by $M_w^\star$ the map obtained after the vertex explosion process when one keeps the cells corresponding to the vertices of M_w. A face of $M_w^\star$ is said *white* if it corresponds to a face of M_w and *black* if it corresponds to a vertex of M_w. For instance, the map $M_w^\star$ in Figure 99 has 2 white faces and 4 black faces. We call *regular* the edges of M_w, and *permeable* the edges that separate black and white faces. The map $M_w^\star$ inherits the root of M_w. In particular, it has the same root-face. The map $M_w^\star$ has $k = |w|_b - |w|_{\bar{b}}$ dangling heads which are all in the root-face. We can compare these heads according to their order of appearance *around* the root-face, that is, when following its border in counterclockwise direction starting from the root. We denote by $h_1, \ldots, h_k$ the heads of $M_w^\star$ encountered in this order around the root-face.

We define the *partition-tree* P_w of the prefix-map M_w as follows. (We shall prove later that the partition-tree is indeed a tree.) We draw a vertex in each face of $M_w^\star$. The vertex v_0 drawn in the root-face is called the *exterior vertex*. We draw k additional vertices $v_1, \ldots, v_k$ in the root-face, each associated to a dangling head (v_i is associated to h_i). These are the vertices of P_w. The edges of P_w are the *duals* of permeable edges. We need to be more precise. If e is a permeable edge that is not incident to the root-face, its dual joins the vertices drawn in the incident black and white faces. If e is a permeable edge incident to the root-face

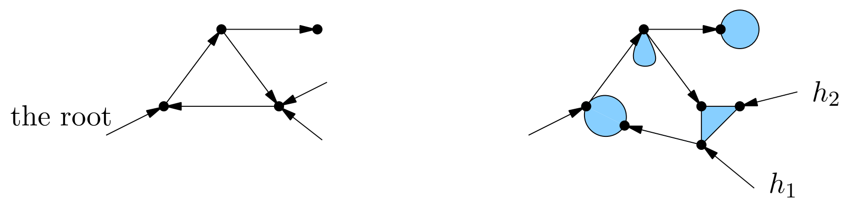


Figure 99: The prefix-map associated to $w = baa\bar{b}bb\bar{a}a$ and the map $M_w^\star$.

and a black face f, its dual joins the vertex drawn in f to v_i if h_i is the last dangling head encountered before e around the root-face, or to v_0 if no dangling head precedes e. Note that the partition-tree P_w can be drawn in such a way that no edge of P_w intersects another. For instance, the partition-tree associated to $w = baa\bar{b}bb\bar{a}a$ is shown in Figure 100.

Moreover the vertices of the partition-tree have an *activity*. We call *white* and *black* the vertices of P_w corresponding to *white* and *black* faces of $M_w^\star$. The *active* white vertices are $v_0, \ldots, v_k$. The *active* black vertices are the vertices corresponding to rooting vertices of M_w (see Subsection 3.4.3 where the notion of *rooting vertex* is introduced). The other vertices are said to be *inactive*.

It remains to define the root of the partition-tree. Consider the first edge e followed around the root-face of $M_w^\star$. It is a permeable edge. Its dual e^* in P_w joins the exterior vertex v_0 to the vertex drawn in the black face corresponding to the root-vertex of M_w. The root of P_w is incident to v_0 and follows e^* in counterclockwise direction around v_0. This root is indicated in Figure 100.

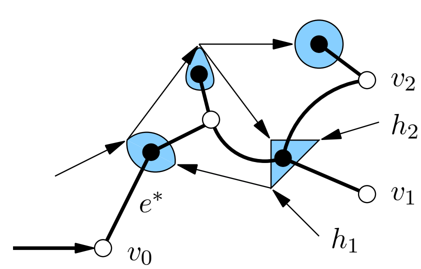


Figure 100: The partition-tree P_w (thick lines) drawn on $M_w^\star$ ($w = baa\bar{b}bb\bar{a}a$).

Observe that, when $w = \Xi(M_T)$ is a parenthesis-shuffle, the map $M_w = \vec{M}^T$ has no dangling heads and the partition-tree P_w is $\Upsilon \circ \varphi_1(\vec{M}^T)$.

We now relate the partition-tree P_w to the binary tree $\lambda_1(w)$.

Proposition 3.24 *For all prefix-shuffle w, the partition-tree P_w is equal to $\theta \circ \lambda_1(w)$ where $\lambda_1(w)$ is the binary tree defined in Definition 3.17 and θ is the mapping defined in Definition 3.20.*

Proposition 3.24 implies that for any parenthesis-shuffle $w = \Xi(M_T)$ we have $P_w = \theta \circ \lambda'_1(w)$. Given that $P_w = \Upsilon \circ \varphi_1(\vec{M}^T)$, we obtain $\varphi_1(\vec{M}^T) = \Theta \circ \lambda'_1(w)$.

The rest of this subsection is devoted to the proof of Proposition 3.24. We first describe a recursive construction of the partition-tree P_w. That is, we describe how to obtain $P_{w\alpha}$ from P_w when α is a letter in $\{a, \overline{a}, b, \overline{b}\}$ (Lemma 3.25). Then we describe a recursive construction of $\theta \circ \lambda_1(w)$ (Lemma 3.26). We conclude the proof by induction on the length of w.

Recursive construction of the partition-tree P_w

The recursive description of the partition-tree requires to define an order on active vertices. Let w be a prefix-shuffle and M_w be the associated prefix-map. The rooting vertices of M_w can be compared by their order of appearance around the spanning tree T_w of M_w. The active black vertices inherit their order from the rooting vertices. The black vertex of P_w corresponding to the root-vertex of M_w is the first element for this order. We can also compare the dangling heads $h_1, \ldots, h_k$ of M_w according to their order of appearance around T_w. This order *is the same* as the order of appearance around the root-face of $M_w^\bigstar$. Indeed, the order of appearance around the root-face of $M_w^\bigstar$ is also the order of appearance around the root-face of M_w. Furthermore, the deletion of an edge of M_w not in T_w does not modify this order. By deleting all the edges not in T_w we obtain the appearance order around T_w. The active white vertices inherit their order from the dangling heads. The exterior vertex v_0 is considered the first element. That is, v_i *precedes* v_j for $0 \leq i \leq j \leq k$.

Let v be a vertex of a tree which is not a leaf. We call *leftmost son* (resp. *rightmost son*) of v the son following (resp. preceding) the father of v (or the root) in counterclockwise direction around v (see Figure 101).

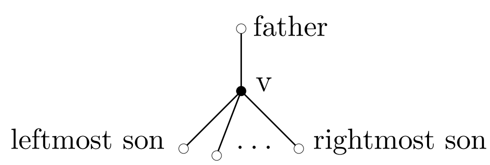


Figure 101: A vertex and its leftmost and rightmost sons.

We are now ready to describe the relation between the partition-tree P_w and the partition-tree $P_{w\alpha}$ when α is a letter in $\{a, \overline{a}, b, \overline{b}\}$.

Lemma 3.25 *The partition-tree P_w is a tree. Moreover,*

- *the partition-tree P_{wa} is obtained from P_w by adding a new leaf which becomes the last* *active black vertex. This leaf is the leftmost son of the last active white vertex,*

- *the partition-tree P_{wb} is obtained from P_w by adding a new leaf which becomes the last active white vertex. This leaf is the rightmost son of the last active black vertex,*

- *the partition-tree $P_{w\overline{a}}$ is obtained from P_w by inactivating the last active black vertex,*

- *the partition-tree $P_{w\overline{b}}$ is obtained from P_w by inactivating the last active white vertex.*

To illustrate this lemma we have represented the evolution of a partition-tree in Figure 102. Active vertices are represented by circles and inactive ones by squares. The white (resp. black) active vertices are denoted $v_0, v_1, \ldots$ (resp. $r_1, r_2, \ldots$).

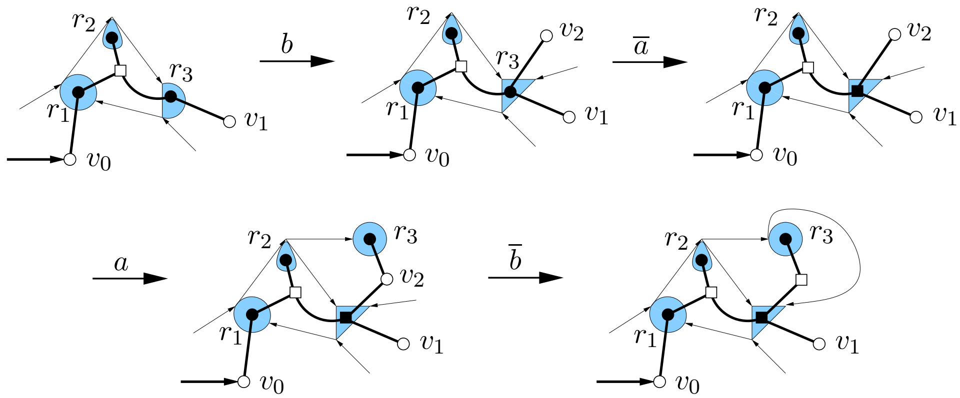


Figure 102: Evolution of the partition-tree from $w = baa\overline{b}b$ to $w = baa\overline{b}bb\overline{a}a\overline{b}$.

Before we embark on the proof, we need to define a correspondence E (resp. V) between the heads of M_w and the edges (resp. vertices distinct from v_0) of P_w. The correspondences E and V are represented in Figure 103.

Consider a head h of M_w and its end v in $M_w^\star$. The edge following h in counterclockwise direction around v is a permeable edge. The dual of this edge in the partition-tree P_w is denoted $E(h)$. The correspondence E between heads of M_w and edges of P_w is one-to-one. The edge $E(h)$ is incident to a white and to a black vertex. If h is in the tree T_w (in particular, if h is the root), we define $V(h)$ as the black vertex incident to $E(h)$. Else $V(h)$ is the white vertex incident to $E(h)$. The correspondence V is a bijection between heads of M_w and vertices of P_w distinct from v_0. Indeed, black vertices of P_w correspond to vertices of M_w which are in one-to-one correspondence with heads in T_w, white vertices distinct from $v_0, \ldots, v_k$ correspond to faces of M_w which are in one-to-one correspondence with heads not in T_w (a face f is associated with the head we cross when we first enter f during the tour of T_w), and the vertices $v_1, \ldots, v_k$ are in one-to-one correspondence with the dangling heads $h_1, \ldots, h_k$.

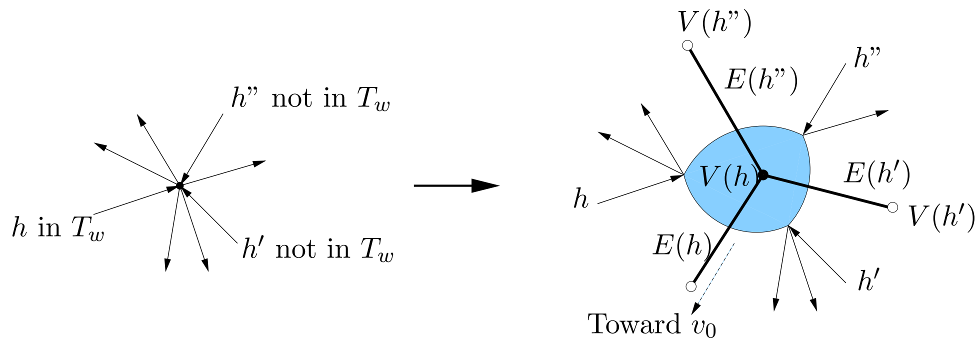


Figure 103: Left: a typical vertex of the prefix map M_w incident with three heads: h in T_w and h' , h'' not in T_w. Right: the correspondence E (resp. V) between heads of M_w and edges (resp. vertices) of P_w.

Proof: We prove the lemma by induction on the length of w. If w is the empty word, P_w is a tree. Suppose now, by induction hypothesis, that P_w is a tree. We first show the following property: *for any head h of M_w, the edge $E(h)$ links $V(h)$ to its father in P_w.* The mapping $V \circ E^{-1}$ is a bijection from the edges of P_w to the vertices of P_w distinct from its root-vertex v_0. Moreover an edge e of P_w is always incident to the vertex $V \circ E^{-1}(e)$ in P_w. Since P_w is a tree, the only possibility is that any edge e of P_w links the vertex $V \circ E^{-1}(e)$ to its father in P_w.

We are now ready to study separately the different cases $\alpha = a, \overline{a}, b, \overline{b}$. We use Lemma 3.22 and denote by c the corner of M_w at the left of the last rooting head of M_w.

• Case $\alpha = a$.

- The prefix-map M_{wa} is obtained from M_w by adding a new edge e in the corner c oriented away from c. Let h be the head of e and s its end. The vertex s is the last rooting vertex in M_{wa}. The partition-tree P_{wa} is obtained from P_w by adding the edge $E(h)$ and the black vertex $V(h)$ to P_w (see Figure 104). By definition, the vertex $V(h)$ is the last active black vertex in P_{wa}.

- By definition, the corner c is situated after any dangling head around T_w. Hence, it is situated after any dangling head around the root-face of $M_w^{\bigstar}$. Therefore, the edge $E(h)$ joins $V(h)$ to the last active white vertex v_k. Moreover, since $V(h)$ is only incident to $E(h)$ and P_w is a tree, we check that P_{wa} is a tree and $V(h)$ a leaf.

- It remains to show that $V(h)$ is the leftmost son of v_k. By definition, the permeable edges that have their dual incident to v_k are situated between h_k (or the root h_0 of $M_w^{\bigstar}$ if $k = 0$) and c around the root-face of $M_w^{\bigstar}$. The dual of the first of these permeable edge is $E(h_k)$ and the dual of the last of them is $E(h)$. If $k \neq 0$, we know that $E(h_k)$ links $v_k = V(h_k)$ to its father in P_w. Therefore, $V(h)$ is the leftmost son of v_k. If $k = 0$, we know (by definition) that the root of P_w follows $E(h_0)$ in counterclockwise direction around v_0. Therefore, $V(h)$ is the leftmost son of v_0.

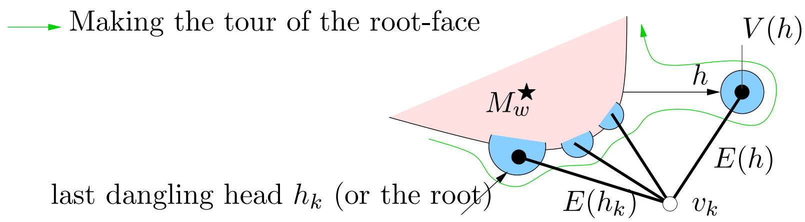


Figure 104: The new vertex $V(h)$ is the leftmost son of v_k.

• Case $\alpha = b$.
We denote by h and v the last rooting head and vertex.
- The prefix-map M_{wb} is obtained from M_w by adding a dangling head h_{k+1} in the corner c. It is the last dangling head of M_{wb}. The partition-tree P_{wb} is obtained by adding the vertex $v_{k+1} = V(h_{k+1})$ and the edge $E(h_{k+1})$ to P_w (see Figure 105). By definition, v_{k+1} is the last active white vertex of P_{wb}.
- The dangling head h_{k+1} is incident to v in M_{wb}. Hence, the edge $E(h_{k+1})$ joins v_{k+1} to the last active black vertex $V(h)$ of P_w. Moreover, since v_{k+1} is only incident to $E(h_{k+1})$ and P_w is a tree, P_{wb} is a tree and v_{k+1} a leaf.
- It remains to prove that v_{k+1} is the rightmost son of $V(h)$. By definition, $E(h_{k+1})$ and $E(h)$ are respectively the dual of the permeable edges preceding and following the head h in counterclockwise direction around its end. Therefore, $E(h)$ follows $E(h_{k+1})$ in counterclockwise direction around $V(h)$. Given that $E(h)$ links $V(h)$ to its father, v_{k+1} is the rightmost son of $V(h)$.

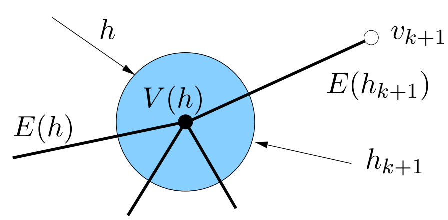


Figure 105: The new vertex v_{k+1} is the rightmost son of $V(h)$.

• Case $\alpha = \overline{a}$.
The prefix-map $M_{w\overline{a}}$ is obtained from M_w by inactivating the last active edge e. Thus, $P_{w\overline{a}}$ is obtained from P_w by inactivating the last active black vertex.
• Case $\alpha = \overline{b}$.
The prefix-map $M_{w\overline{b}}$ is obtained from M_w by adding a tail in the corner c and connecting it to the last dangling head h_k. This creates a new face of M_w (hence of $M_w^\star$) and lowers by one the number of dangling heads. The last active white vertex v_k is trapped in the new face of $M_{w\overline{b}}$. Hence, $P_{w\overline{b}}$ is obtained from P_w by inactivating the last active black vertex v_k.

□

Recursive construction of the tree $\theta \circ \lambda_1(w)$.

We continue the proof of Proposition 3.24. We now describe the relation between the trees $\theta \circ \lambda_1(w)$ and $\theta \circ \lambda_1(w\alpha)$ when α is a letter in $\{a, \overline{a}, b, \overline{b}\}$ (the mapping λ_1 is defined in Definition 3.17).

We first need to define a correspondence between the leaves of a binary tree B and the vertices of the tree $\theta(B)$. An edge of B is said *left* (resp. *right*) if it links a node to its left son (resp. right son). We consider a leaf l of B. If l is a left (resp. right) leaf, the path from l to the root begins with a non-empty sequence of left (resp. right) edges. By definition, only the last edge $e(l)$ of this sequence is branching except if l is the first left leaf in which case no edge is branching. We associate the first left leaf of B with the root-vertex of $\theta(B)$ and we associate any other leaf l with the son of the branching edge $e(l)$ in $\theta(B)$. This correspondence is one-to-one. For instance, the leaves $l_1, \ldots, l_6$ of the binary tree B in Figure 106 are associated with the vertices $v_1, \ldots, v_6$ of the tree $\theta(B)$.

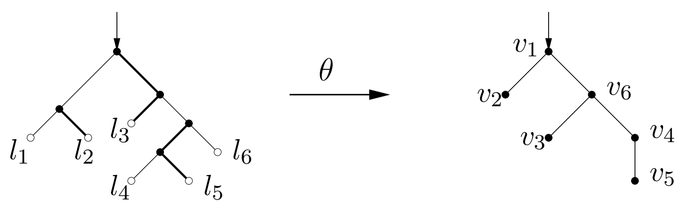


Figure 106: Correspondence between leaves of B and vertices of $\theta(B)$.

Consider a prefix-shuffle w. In the binary tree $\lambda_1(w)$, leaves are either active or inactive. We say that a vertex of $\theta \circ \lambda_1(w)$ is left, right, active, inactive if the associated leaf of $\lambda_1(w)$ is so. Moreover, the leaves of the binary tree $\lambda_1(w)$ can be compared by their order of appearance around this tree. The vertices of $\theta \circ \lambda_1(w)$ inherit this order. For instance, the root-vertex of $\theta \circ \lambda_1(w)$ is the first active left vertex (recall that the first left leaf of $\lambda_1(w)$ is always active).

We are now ready to state the last lemma which is the counterpart of Lemma 3.25.

Lemma 3.26 *Let $\mathcal{T}$ be the tree $\theta \circ \lambda_1(w)$ and $\mathcal{T}_\alpha = \theta \circ \lambda_1(w\alpha)$ for α in $\{a, b, \overline{a}, \overline{b}\}$.*

- *The tree $\mathcal{T}_a$ is obtained from $\mathcal{T}$ by adding a new leaf which becomes the first active right vertex. This leaf is the leftmost son of the last active left vertex.*

- *The tree $\mathcal{T}_b$ is obtained from $\mathcal{T}$ by adding a new leaf which becomes the last active left vertex. This leaf is the rightmost son of the first right vertex.*

- *The tree $\mathcal{T}_{\overline{a}}$ is obtained from $\mathcal{T}$ by inactivating the first active right vertex.*
- *The tree $\mathcal{T}_{\overline{b}}$ is obtained from $\mathcal{T}$ by inactivating the last active left vertex.*

Proof: We study separately the four cases $\alpha = a, b, \overline{a}, \overline{b}$.
• Case $\alpha = a$. By definition of the mapping λ_1 (Definition 3.17), the binary tree $\lambda_1(wa)$ is obtained from $\lambda_1(w)$ by replacing the last active left leaf l by a node with two leaves l_l and l_r. The left leaf l_l replaces l as the last left leaf. The right leaf l_r becomes the first right leaf. The edge from l to l_r is branching. The other branching edges are unchanged. Therefore, $\mathcal{T}_a$ is obtained from $\mathcal{T}$ by adding a new leaf. This leaf is associated with l_r hence becomes the first active right vertex. The father of this leaf was associated with l in $\mathcal{T}$ and is associated with l_l in $\mathcal{T}_a$. Therefore, it was and remains the last active left vertex. It is easily seen that the new leaf becomes its leftmost son.
• The case $\alpha = b$ is symmetric to the case $\alpha = a$. We do not detail it.
• Case $\alpha = \overline{a}$. The binary tree $\lambda_1(w\overline{a})$ is obtained from $\lambda_1(w)$ by inactivating the first active right leaf. Therefore, $\mathcal{T}_{\overline{a}}$ is obtained from $\mathcal{T}$ by inactivating the first active right vertex.
• The case $\alpha = \overline{b}$ is symmetric to the case $\alpha = \overline{a}$. □

Recursive proof of Proposition 3.24.

We want to show that, for any prefix-shuffle w, the partition-tree P_w is the tree $\theta \circ \lambda_1(w)$. We show by induction the following more precise property: for any prefix-shuffle w,
- the partition-tree P_w is equal to $\theta \circ \lambda_1(w)$,
- the active and inactive vertices of P_w and $\theta \circ \lambda_1(w)$ are the same,
- the white (resp. black) vertices of P_w correspond to left (resp. right) vertices of $\theta \circ \lambda_1(w)$,
- the order on white (resp. black) vertices of P_w is equal (resp. inverse) to the order on left (resp. right) vertices of $\theta \circ \lambda_1(w)$.

Suppose that w is the empty word. The partition-tree P_w has one edge, an active white vertex which is its root-vertex and an active black vertex. Similarly, $\theta \circ \lambda_1(w)$ has one edge, an active left vertex which is its root-vertex and an active right vertex. Hence, we check that the property is true. In view of Lemma 3.25 and Lemma 3.26, it is clear that the property is true by induction on the set of prefix-shuffles. □

This concludes the proof of Proposition 3.24 and Theorem 3.21. □

Part III

Combinatorial maps and the Tutte polynomial

In the following chapters, we exhibit and exploit a characterization of the Tutte polynomial based on combinatorial embeddings. Our characterization is valid for general graphs (as opposed to planar graphs). In Chapter 5, we define a notion of activity, the *embedding-activity*, for spanning trees. We prove that the Tutte polynomial is the generating function of spanning trees counted by internal and external embedding-activities. We compare this characterization of the Tutte polynomial to earlier ones. We also take a glimpse at the applications of our characterization to be developed in the following chapters. In Chapter 6, we define a partition of the set of subgraphs based on the embedding activities. Each part of this partition is associated with a spanning tree. The partition of the set of subgraphs is used in order to define a bijection Φ between subgraphs and orientations. This bijection extends the correspondence between spanning trees and tree-orientations that we exhibited in Chapter 3. In Chapter 7, we study the restriction of the bijection Φ to several classes of subgraphs. Among other results, we obtain an interpretation for all the evaluations $T_G(i,j), 0 \leq i,j \leq 2$ of the Tutte polynomial in terms of orientations. For instance the strongly connected orientations are counted by $T_G(0,2)$ while the acyclic orientations are counted by $T_G(2,0)$. The strength of our approach is to derive all our results from a unique bijection Φ specialized in various ways. Some of the results are expressed in terms of outdegree sequences. For instance, we obtain a bijection between forests and outdegree sequences (this answers a question by Stanley [Stan 80a]). We also obtain a bijection between spanning trees and root-connected outdegree sequences. Lastly, in Chapter 8 we define a bijection between spanning trees and the recurrent configurations of the sandpile model. Combining our results, we obtain a bijection between recurrent configurations and root-connected outdegree sequences which leaves the configurations at level 0 unchanged (this answers a question by Gioan [Gioa 06]).

Before we get started, we summarize the definitions and notations needed for the four following chapters.

4.1 Definitions and notations

We denote by $\mathbb{N}$ the set of non-negative integers. For any set S, we denote by $|S|$ its cardinality. For any sets S_1, S_2, we denote by $S_1 \vartriangle S_2$ the symmetric difference of S_1 and S_2. If $S \subseteq S'$ and S' is clear from the context, we denote by $\overline{S}$ the complement of S, that is, $S' \setminus S$. If $S \subseteq S'$ and $s \in S'$, we write $S+s$ and $S-s$ for $S \cup \{s\}$ and $S \setminus \{s\}$ respectively (whether s belongs to S or not).

4.1.1 Graphs

In the following chapters we consider finite, undirected graphs. Loops and multiple edges are allowed. Formally, a *graph* $G = (V, E)$ is a finite set of *vertices* V, a finite set of *edges* E and a relation of *incidence* in $V \times E$ such that each edge e is incident to either one or two vertices. The *endpoints* of an edge e are the vertices incident to e. A *cycle* is a set of edges that form a simple closed path. A *cut* is a set of edges C whose deletion increases the number of connected components and such that the endpoints of every edge in C are in distinct components of the resulting graph. A cut is shown in Figure 107. Given a subset of vertices U, the *cut defined by* U is the set of edges with one endpoint in U and one endpoint in $\overline{U}$. A *cocycle* is a cut which is minimal for inclusion (equivalently it is a cut whose deletion increases the number of connected components by *one*). For instance, the set of edges $\{f, g, h\}$ in Figure 107 is a cocycle.

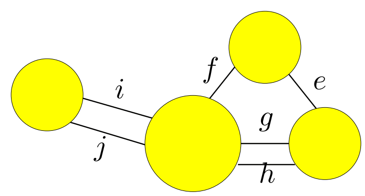


Figure 107: The cut $\{e, f, g, h, i, j\}$ and the connected components after deletion of this cut (shaded regions).

Let $G = (V, E)$ be a graph. A *spanning subgraph* of G is a graph $G' = (V, E')$ where $E' \subseteq E$. *All the subgraphs considered in the following chapters are spanning* and we shall not further precise it. A subgraph is entirely determined by its edge set and, by convenience, we shall identify the subgraph with its edge set. A forest is an acyclic graph. A *tree* is a connected forest. A *spanning tree* is a (spanning) subgraph which is a tree. Given a tree T and a vertex distinguished as the *root-vertex* we shall use the usual *family vocabulary* and talk about the *father*, *son*, *ancestors* and *descendants* of vertices in T. By convention, a vertex is considered to be an ancestor and a descendant of itself. If a vertex of the graph G is distinguished as the *root-vertex* we implicitly consider it to be the root-vertex of every spanning tree.

Let T be a spanning tree of the graph G. An edge of G is said to be *internal* if it is in T and *external* otherwise. The *fundamental cycle* (resp. *cocycle*) of an external (resp. internal) edge e is the set of edges e' such that the subgraph $T - e' + e$ (resp. $T - e + e'$) is a spanning tree. Observe that the fundamental cycle C of an external edge e is a cycle contained in $T + e$ (C is made of e and the path of T between the endpoints of e). Similarly, the fundamental cocycle D of an internal edge e is a cocycle contained in $\overline{T} + e$ (D is made of the edges linking the two subtrees obtained from T by removing e). Observe also that, if e is internal and e' is external, then e is in the fundamental cycle of e' if and only if e' is in the fundamental cocycle of e.

4.1.2 Embeddings

We recall the notion of *combinatorial map* introduced by Cori and Machì [Cori 75, Cori 92]. A *combinatorial map* (or *map* for short) $\mathcal{G} = (H, \sigma, \alpha)$ is a set of *half-edges* H, a permutation σ and an involution without fixed point α on H such that the group generated by σ and α acts transitively on H. A map is *rooted* if one of the half-edges is distinguished as the *root*. For $h_0 \in H$, we denote by $\mathcal{G} = (H, \sigma, \alpha, h_0)$ the map (H, σ, α) rooted on h_0. From now on *all our maps are rooted.*

Given a map $\mathcal{G} = (H, \sigma, \alpha, h_0)$, we consider the *underlying* graph $G = (V, E)$, where V is the set of cycles of σ, E is the set of cycles of α and the incidence relation is to have at least one common half-edge. We represented the underlying graph of the map $\mathcal{G} = (H, \sigma, \alpha)$ on the left of Figure 108, where the set of half-edges is $H = \{a, a', b, b', c, c', d, d', e, e', f, f'\}$, the involution α is $(a, a')(b, b')(c, c')(d, d')(e, e')(f, f')$ in cyclic notation and the permutation σ is $(a, f', b, d)(d')(a', e, f, c)(e', b', c')$. Graphically, we keep track of the cycles of σ by drawing the half-edges of each cycle in counterclockwise order around the corresponding vertex. Hence, our drawing characterizes the map $\mathcal{G}$ since the order around vertices give the cycles of the permutation σ and the edges give the cycles of the involution α. On the right of Figure 108, we represented the map $\mathcal{G}' = (H, \sigma', \alpha)$, where $\sigma' = (a, f', b, d)(d')(a', e, c, f)(e', b', c')$. The maps $\mathcal{G}$ and $\mathcal{G}'$ have isomorphic underlying graphs.

Note that the underlying graph of a map $\mathcal{G} = (H, \sigma, \alpha)$ is always connected since σ and α act transitively on H. A *combinatorial embedding* (or *embedding* for short) of a connected graph G is a map $\mathcal{G} = (H, \sigma, \alpha)$ whose underlying graph is isomorphic to G (together with an explicit bijection between the set H and the set of half-edges of G). When an embedding $\mathcal{G}$ of G is given we shall write the edges of G as pairs of half-edges (writing for instance $e = \{h, h'\}$). Moreover, we call *root-vertex* the vertex incident to the root and *root-edge* the edge containing the root. In the following, we use the terms *combinatorial map* and *embedded graph* interchangeably. *We do not require our graphs to be planar.*

Intuitively, a combinatorial embedding corresponds to the choice of a cyclic order on the edges around each vertex. This order can also be seen as a local planar embedding. As explained in the introduction of this thesis (Subsection 0.1.3), there is a one-to-one correspondence between the *combinatorial embeddings of graphs* and the *cellular embeddings of graphs in surfaces* (defined up to homeomorphism).

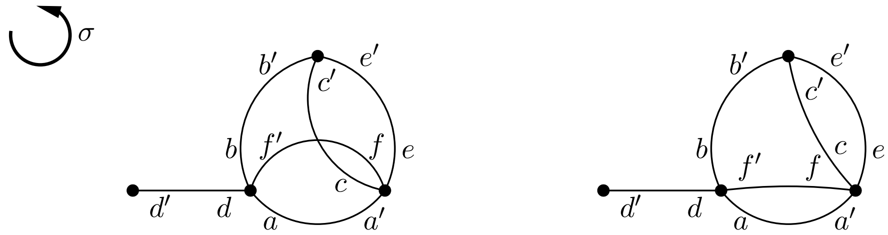


Figure 108: Two embeddings of the same graph.

4.1.3 Orientations and outdegree sequences

Let G be a connected graph and let $\mathcal{G}$ be an embedding of G. An orientation is a choice of a direction for each edge of G, that is to say, a function $\mathcal{O}$ which associates to any edge $e = \{h_1, h_2\}$ one of the ordered pairs (h_1, h_2) or (h_1, h_2). Note that loops have two possible directions. We call $\mathcal{O}(e)$ an *arc*, or *oriented edge*. If $\mathcal{O}(e) = (h_1, h_2)$ we call h_1 the *tail* and h_2 the *head*. We call *origin* and *end* of $\mathcal{O}(e)$ the endpoint of the tail and head respectively. Graphically, we represent an arc by an arrow going from the origin to the end (see Figure 109).

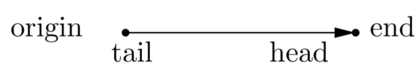


Figure 109: Half-edges and endpoints of arcs.

A *directed path* is a sequence of arcs $(a_1, a_2, \ldots, a_k)$ such that the end of a_i is the origin of a_{i+1} for $1 \le i \le k-1$. A *directed cycle* is a simple directed closed path. A *directed cocycle* is a set of arcs $a_1, \ldots, a_k$ whose deletion disconnects the graph into two components and such that all arcs are directed toward the same component. If the orientation $\mathcal{O}$ is not clear from the context, we shall say that a path, cycle, or cocycle is *$\mathcal{O}$-directed*. An orientation is said to be *acyclic* if there is no directed cycle (resp. cocycle). An orientation is said to be *totally cyclic* or *strongly connected* if there is no directed cocycle.

We say that a vertex v is *reachable* from a vertex u if there is a directed path $(a_1, a_2, \ldots, a_k)$ such that the origin of a_1 is u and the end of a_k is v. If v is *reachable* from u in the orientation $\mathcal{O}$ denote it by $u \overset{\mathcal{O}}{\to} v$. An orientation is said to be *u-connected* if every vertex is reachable from u. Observe that an orientation $\mathcal{O}$ is totally cyclic if and only if the origin of every arc is reachable from its end. Equivalently, $\mathcal{O}$ is totally cyclic if every pair of vertices are reachable from one another.

The *outdegree sequence* of an orientation $\mathcal{O}$ of the graph $G = (V, E)$ is the function $\delta : V \mapsto \mathbb{N}$ that associates to every vertex the number of incident tails. We say that $\mathcal{O}$ is a *δ-orientation*. The outdegree sequences are strongly related to the *cycle flips*, that is, the reversing of every edge in a directed cycle. Indeed, it is known that the outdegree sequences are in one-to-one correspondence with the equivalence classes of orientations up to *cycle flips* (see Lemma 7.12).

There are nice characterizations of the functions $\delta : V \mapsto \mathbb{N}$ that are outdegree sequences. Given a function $\delta : V \mapsto \mathbb{N}$ we define the *excess* of any subset of vertices $U \subseteq V$ by

$$exc_\delta(U) = \left(\sum_{u\in U} \delta(u)\right) - |G_U|,$$

where $|G_U|$ is the number of edges of G having both endpoints in U. By definition, if δ is the outdegree sequence of an orientation $\mathcal{O}$, the sum $\sum_{u\in U} \delta(u)$ is the number of tails incident with vertices in U. From this number, exactly $|G_U|$ are part of edges with both endpoints in U. Hence, the excess $exc_\delta(U)$ corresponds to the number of tails incident with vertices in U in the cut defined by U. This property is illustrated in Figure 110. It is clear that if $\delta : V \mapsto \mathbb{N}$ is an outdegree sequence, then the excess of V is 0 and the excess of any subset $U \subseteq V$ is non-negative. In fact, the converse is also true: every function $\delta : V \mapsto \mathbb{N}$ satisfying these two conditions is an outdegree sequence [Fels 04].

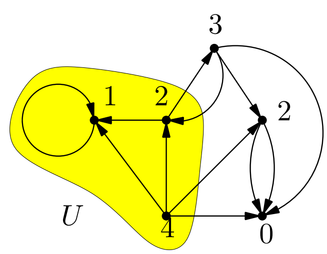


Figure 110: The excess of the subset U is $exc_\delta(U) = (4 + 2 + 1) - 4 = 3$.

We now prove that the reachability properties between vertices in a directed graph only depend on the outdegree sequence of the orientation.

Lemma 4.1 *Let $G = (V, E)$ be a graph and let u, v be two vertices. Let $\mathcal{O}$ be an orientation of G and let δ be its outdegree sequence. Then v is reachable from u if and only if there is no subset of vertices $U \subseteq V$ containing u and not v and such that $exc_\delta(U) = 0$.*

Proof: Lemma 4.1 is illustrated in Figure 111. Observe that the excess of a subset $U \subseteq V$ is 0 if and only if the cut defined by U is directed toward U.

• Suppose there is a subset of vertices $U \subseteq V$ containing u and not v such that $exc_\delta(U) = 0$. Then, the cut defined by U is directed toward U. Thus, there is no directed path from U to $\overline{U}$. Hence v is not reachable from v.

• Conversely, suppose v is not reachable from u. Consider the set of vertices U reachable from u. The subset U contains u but not v. Moreover, the cut defined by U is directed toward U, hence $exc_\delta(U) = 0$.

□

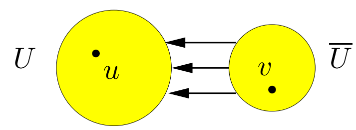


Figure 111: Reachability is a property of the outdegree sequence.

Since the reachability properties only depend on the outdegree sequence of the orientation, we can define an outdegree sequence δ to be *u-connected* or *strongly connected* if the δ-orientations are. The u-connected outdegree sequences were considered in [Gioa 06] in connection with the *cycle/cocycle reversing system* (see Section 7.7.1).

Remark: From the characterization of outdegree sequences given above and Lemma 4.1 it is possible to characterize *u-connected* and *strongly connected* outdegree sequences. Let $G = (V, E)$ be a graph and $\delta : V \mapsto \mathbb{N}$ be a mapping such that $\sum_{v \in V} \delta(v) = |E|$. The mapping δ is a strongly connected outdegree sequence if and only if the excess of any subset $U \subsetneq V$ is positive. The mapping δ is a u-connected outdegree sequence if and only if the excess of any subset $U \subsetneq V$ is non-negative and is positive whenever $u \in U$.

4.1.4 The sandpile model

The sandpile model is a dynamical system introduced in statistical physics in order to study self-organized criticality [Bak 87, Dhar 90]. It appeared independently in combinatorics as the *chip firing game* [Björ 91]. Roughly speaking, the model consists of grains of sand toppling through edges when there are too many on the same vertex. *Recurrent configurations* play an important role in the model: they correspond to configurations that can be observed after a long period of time. Despite its simplicity, the sandpile model displays interesting enumerative [Cori 03, Dhar 92, Meri 97] and algebraic properties [Cori 00, Dhar 95].

Let $G = (V, E)$ be a graph with a vertex v_0 distinguished as the *root-vertex*. A *configuration* of the sandpile model is a function $\mathcal{S} : V \mapsto \mathbb{N}$, where $\mathcal{S}(v)$ represents the number of grains of sand on v. A vertex v is *unstable* if $\mathcal{S}(v)$ is greater than or equal to its degree $\deg(v)$. An unstable vertex v can *topple* by sending a grain of sand through each of the incident edges. This leads to the new configuration $\mathcal{S}'$ defined by $\mathcal{S}'(u) = \mathcal{S}(u) + \deg(u, v)$ for all $u \neq v$ and $\mathcal{S}'(v) = \mathcal{S}(v) - \deg(v, *)$, where $\deg(u, v)$ is the number of edges with endpoints

u, v and $\deg(v, *)$ is the number of non-loop edges incident to v. We denote this transition by $\mathcal{S} \overset{v}{\dashrightarrow} \mathcal{S}'$. An evolution of the system is represented in Figure 112.

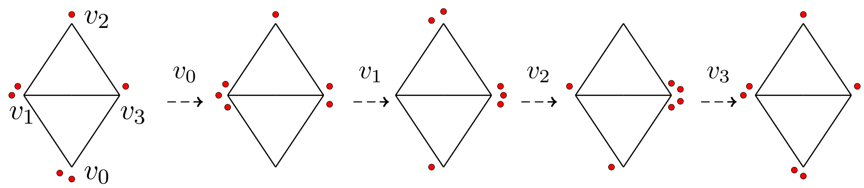


Figure 112: A recurrent configuration and the evolution rule.

A configuration is *stable* if every vertex $v \neq v_0$ is stable. A stable configuration $\mathcal{S}$ is *recurrent* if $\mathcal{S}(v_0) = \deg(v_0)$ and if there is a labeling of the vertices in V by $v_0, v_1, \ldots, v_{|V|-1}$ such that $\mathcal{S} \overset{v_0}{\dashrightarrow} \mathcal{S}_1 \overset{v_1}{\dashrightarrow} \ldots \overset{v_{|V|-1}}{\dashrightarrow} \mathcal{S}_{|V|} = \mathcal{S}$. This means that after toppling the root-vertex v_0, there is a valid sequence of toppling involving each vertex once that gets back to the initial configuration. For instance, the configuration at the left of Figure 112 is recurrent. The *level* of a recurrent configuration $\mathcal{S}$ is

$$\mathrm{level}(\mathcal{S}) = \left(\sum_{v \in V} \mathcal{S}(v) \right) - |E|.$$

4.1.5 The Tutte polynomial

We recall the *subgraph expansion* of the Tutte polynomial. We postpone the presentation of the other characterizations of this polynomial to Chapter 5 (Section 5.3).

Definition 4.2 *The* Tutte polynomial *of a graph $G = (V, E)$ is*

$$T_G(x, y) = \sum_{S \subseteq E} (x-1)^{c(S)-c(G)} (y-1)^{c(S)+|S|-|V|}, \tag{79}$$

where the sum is over all subgraphs S and $c(S)$ (resp. $c(G)$) denotes the number of connected components of S (resp. G).

For example, if G is the triangle K_3 there are 8 subgraphs. The subgraph with no edge has contribution $(x-1)^2$, each subgraph with one edge has contribution $(x-1)$, each subgraph with two edges has contribution 1 and the subgraph with three edges has contribution $(y-1)$. Summing up these contributions, we get $T_{K_3}(x, y) = (x-1)^2 + 3(x-1) + 3 + (y-1) = x^2 + x + y$.

The *subgraph expansion* (79) of the Tutte polynomial is the generating function of subgraphs according to two parameters: the (normalized) number of connected components $c(S) - c(G)$ and the *cyclomatic number* $c(S) + |S| - |V|$. The *cyclomatic number* is the maximum number of edges that can be removed from S without increasing the number of connected components. In particular, $c(S) + |S| - |V| = 0$ if and only if S is a forest. From the subgraph expansion (79), it is easy to check that whenever G is the disjoint union of two graphs $G = G_1 \cup G_2$, then $T_G(x,y) = T_{G_1}(x,y) \times T_{G_2}(x,y)$. This relation allows us to restrict our attention to connected graphs. *In the following chapters, all the graphs we consider are connected.*

Before we close this section, we recall the relations of induction satisfied by the Tutte polynomial [Tutt 54]. (These relations reminiscent of the relations of induction of the chromatic polynomial [Whit 32a] are easy to prove from (79)).

Proposition 4.3 ***(Tutte)*** *Let G be a graph and e be any edge of G. The Tutte polynomial of G satisfies:*

$$T_G(x,y) = \left| \begin{array}{ll} x \cdot T_{G/e}(x,y) & \textit{if } e \textit{ is an isthmus,} \\ y \cdot T_{G\backslash e}(x,y) & \textit{if } e \textit{ is a loop,} \\ T_{G/e}(x,y) + T_{G\backslash e}(x,y) & \textit{if } e \textit{ is neither a loop nor an isthmus.} \end{array} \right. \quad (80)$$

This concludes our presentation of the notions needed for the four following chapters. We will now present a new characterization of the Tutte polynomial and exploit its numerous consequences.

Chapter 5

Characterization of the Tutte polynomial via combinatorial embeddings

Abstract: We give a new characterization of the Tutte polynomial of graphs. Our characterization is formally close (but inequivalent) to the original definition given by Tutte as the generating function of spanning trees counted according to *activities*. Tutte's notion of activity requires to choose a *linear order* on the edge set. We define a new notion of activity, the *embedding-activity*, which requires to choose a *combinatorial embedding* of the graph, that is, a cyclic order of the edges around each vertex. We prove that the Tutte polynomial equals the generating function of spanning trees counted according to *embedding-activities*.

Résumé : Nous présentons une nouvelle caractérisation du polynôme de Tutte des graphes. Notre caractérisation est proche dans sa formulation (mais non équivalente) à la première définition donnée par Tutte comme la série génératrice des arbres couvrants comptés selon leurs *activités*. La caractérisation de Tutte demande d'ordonner linéairement l'ensemble des arêtes. Nous définissons une nouvelle notion d'activité, *l'activité de plongement*, qui demande de choisir un plongement combinatoire du graphe, soit un ordre cyclique des arêtes autour de chaque sommet. Nous montrons que le polynôme de Tutte est égal à la série génératrice des arbres couvrants comptés selon leurs *activités de plongement*.

5.1 Introduction

In 1954, Tutte defined a graph invariant that he named *dichromate* because he thought of it as a bivariate generalization of the *chromatic polynomial* [Tutt 54]. The first definition by Tutte was a generating function of spanning trees counted according to their *activities*. Since then, this polynomial, which is now known as the *Tutte polynomial*, has been widely studied (see for instance, [Bryl 91] and references therein). We refer the reader to [Boll 98, Chapter X] for an easy-to-read but comprehensive survey of the properties and applications of the Tutte polynomial.

In this chapter, we give a new characterization of the Tutte polynomial of graphs. Our characterization is formally close (but not equivalent) to the original definition by Tutte in terms of the *ordering-activities* of spanning trees (compare (85) and (84)). Tutte's notion of activity requires to choose a *linear order* on the edge set. The Tutte polynomial is then the generating function of spanning trees counted according to their (internal and external) ordering-activities (this generating function being, in fact, independent of the linear order). Our characterization of the Tutte polynomial requires instead to choose an *embedding* of the graph, that is, a cyclic order for the incidences of edges around each vertex. Once the embedding is chosen, one can define the (internal and external) *embedding-activities* of spanning trees. We prove that the Tutte polynomial is equal to the generating function of spanning trees counted according to their (internal and external) embedding-activities (this generating function being, in fact, independent of the embedding).

This chapter is organized as follows. In Section 5.2, we study the tour of spanning trees. In Section 5.3, we define the embedding activities and characterize the Tutte polynomial as the generating function of spanning trees counted by embedding-activities. We also compare this characterization to earlier definitions of the Tutte polynomial. In Section 5.4, we give the proof of the characterization of the Tutte polynomial by embedding activities. Lastly, in Section 5.5, we take a glimpse at the results to be developed in the following chapters.

5.2 The tour of spanning trees

We first define the *tour* of spanning trees. Informally, the *tour* is a walk around the tree that follows internal edges and crosses external edges. A graphical representation of the tour is given in Figure 113. We already encountered this notion in chapter 3 in the case of planar embeddings. We will now define it below for general embeddings.

Let $\mathcal{G} = (H, \sigma, \alpha)$ be an embedding of the graph $G = (V, E)$. Given a spanning tree T, we

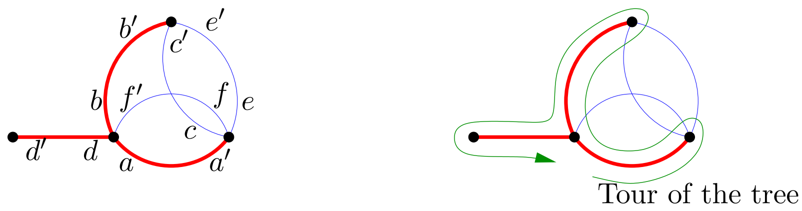


Figure 113: Intuitive representation of the tour of a spanning tree (indicated by thick lines).

define the *motion function* t on the set H of half-edges by:

$$t(h) = \left| \begin{array}{ll} \sigma(h) & \text{if } h \text{ is external,} \\ \sigma\alpha(h) & \text{if } h \text{ is internal.} \end{array} \right. \tag{81}$$

Clearly, the motion function t is a bijection on H (since the inverse mapping is easily seen to be $t^{-1}(h) = \sigma^{-1}(h)$ if $\sigma^{-1}(h)$ is external and $t^{-1}(h) = \alpha\sigma^{-1}(h)$ if $\sigma^{-1}(h)$ is internal). In fact, we will prove shortly that the motion function t is a cyclic permutation. For instance, the motion function of the embedded graph in Figure 113 is the cyclic permutation $(a, e, f, c, a', f', b, c', e', b', d, d')$. The cyclic order defined by the motion function t on the set of half-edges is what we call the *tour of the tree* T.

Our proof that the motion function t is a cyclic permutation is by induction on the number of edges of the graph. This proof requires to define the deletion and contraction of edges in embedded graphs. Our definitions preserve the *cyclic order* of the half-edges around each vertex. We represented the result of deleting and contracting the edge $e = \{b, b'\}$ in Figure 114.

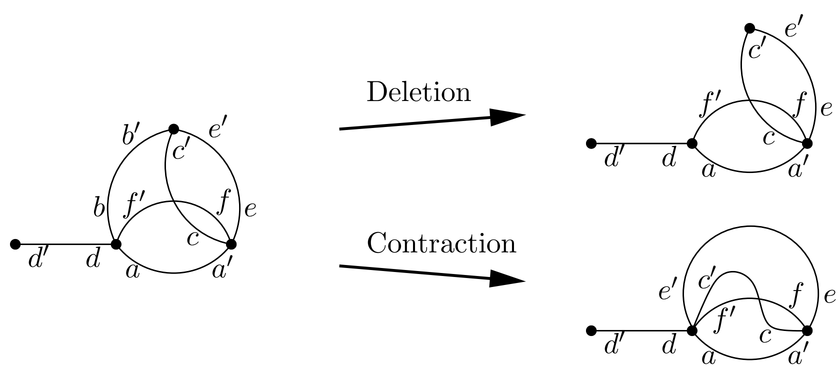


Figure 114: Deletion and contraction of the edge $e = \{b, b'\}$.

Let G be a graph and let e be an edge. If e is not an isthmus (resp. loop), we denote by $G_{\backslash e}$ (resp. $G_{/e}$) the graph obtained by deleting e (resp. contracting e). Let $\mathcal{G} = (H, \sigma, \alpha)$ be an embedding of the graph G and let $e = \{h_1, h_2\}$ be an edge. If e is not an isthmus, we define the embeddings $\mathcal{G}_{\backslash e} = (H', \sigma', \alpha_{\backslash e})$ of $G_{\backslash e}$ by $H' = H \setminus \{h_1, h_2\}$, the involution α' equals the involution α restricted to H' and

$$\sigma_{\backslash e}(h) = \left| \begin{array}{ll} \sigma\sigma\sigma(h) & \text{if } (\sigma(h) = h_1 \text{ and } \sigma(h_1) = h_2) \text{ or } (\sigma(h) = h_2 \text{ and } \sigma(h_2) = h_1), \\ \sigma\sigma(h) & \text{if } (\sigma(h) = h_1 \text{ and } \sigma(h_1) \neq h_2) \text{ or } (\sigma(h) = h_2 \text{ and } \sigma(h_2) \neq h_1), \\ \sigma(h) & \text{otherwise.} \end{array} \right. \tag{82}$$

Similarly, if e is not a loop, we define the embeddings $\mathcal{G}_{/e} = (H', \sigma', \alpha_{/e})$ of $G_{/e}$ by $H' = H \setminus \{h_1, h_2\}$, the involution α' equals the involution α restricted to H' and

$$\sigma_{/e}(h) = \left| \begin{array}{ll} \sigma\sigma(h) & \text{if } (\sigma(h) = h_1 \text{ and } \sigma(h_2) = h_2) \text{ or } (\sigma(h) = h_2 \text{ and } \sigma(h_1) = h_1), \\ \sigma\alpha\sigma(h) & \text{if } (\sigma(h) = h_1 \text{ and } \sigma(h_2) \neq h_2) \text{ or } (\sigma(h) = h_2 \text{ and } \sigma(h_1) \neq h_1), \\ \sigma(h) & \text{otherwise.} \end{array} \right. \tag{83}$$

We now describe the effect of a contraction or deletion on the motion function.

Lemma 5.1 *Let $\mathcal{G} = (H, \sigma, \alpha)$ be an embedded graph, let T be a spanning tree and let t be the corresponding motion function. For all external (resp. internal) edge $e = \{h_1, h_2\}$, the spanning tree T (resp. $T - e$) of $\mathcal{G}_{\backslash e}$ (resp. $\mathcal{G}_{/e}$) defines a motion function t' on $H \setminus \{h_1, h_2\}$ such that*

$$t'(h) = \left| \begin{array}{ll} t \circ t \circ t(h) & \textit{if } (t(h) = h_1 \textit{ and } t(h_1) = h_2) \textit{ or } (t(h) = h_2 \textit{ and } t(h_2) = h_1), \\ t \circ t(h) & \textit{if } (t(h) = h_1 \textit{ and } t(h_1) \neq h_2) \textit{ or } (t(h) = h_2 \textit{ and } t(h_2) \neq h_1), \\ t(h) & \textit{otherwise.} \end{array} \right.$$

Proof: Lemma 5.1 follows immediately from the definitions and Equations (82) and (83). □

Remark: Another way of stating Lemma 5.1 is to say that the cycles of the permutation t' are obtained from the cycles of t by erasing h_1 and h_2. Consider, for instance, the embedded graph and the spanning tree represented in Figure 113. The motion function is the cycle $t = (a, e, f, c, a', f', b, c', e', b', d, d')$. If we delete the edge the external edge $\{e, e'\}$ (resp. internal edge $\{b, b'\}$), the motion function becomes $t' = (a, f, c, a', f', b, c', b', d, d')$ (resp. $t' = (a, e, f, c, a', f', c', e', d, d')$).

We are now ready to prove the main result this section:

Proposition 5.2 *For any embedded graph and any spanning tree, the motion function is a cyclic permutation.*

Proof: We prove the lemma by induction on the number of edges of the graph. The property is obviously true for the graph reduced to a loop and the graph reduced to an isthmus. We assume the property holds for all graphs with at most $n \geq 1$ edges and consider an embedded graph $\mathcal{G}$ with $n + 1$ edges. Let T be a spanning tree and t the corresponding motion function. We know that t is a permutation, that is, a product of cycles. We consider an edge $e = \{h_1, h_2\}$.

• *In any cycle of the motion function t, there is a half-edge $h \neq h_1, h_2$.*

First note that $t(h_i) \neq h_i$ for $i = 1, 2$. Indeed, if e is external, this would mean $\sigma(h_i) = h_i$ which is excluded or e would be an isthmus not in the spanning tree. Similarly, if e is internal, we would have $\sigma\alpha(h_i) = h_i$ which is excluded or e would be a loop in the spanning tree. Moreover, we cannot have ($t(h_1) = h_2$ and $t(h_2) = h_1$). Indeed, this would mean that e is either an isolated loop (if e is external) or an isolated isthmus (if e is internal) contradicting our hypothesis that $\mathcal{G}$ is connected and has more than one edge.

• *The motion function is cyclic.*

If e is external (resp. internal), we consider the motion function t' defined by the spanning tree T (resp. $T - e$) on $\mathcal{G}_{\backslash e}$ (resp. $\mathcal{G}_{/e}$). By Lemma 5.1, the cycles of t' are the cycles of t where the half-edges h_1, h_2 are erased. Suppose now that t is not cyclic. Then t has at least two cycles each containing a half-edge $h \neq h_1, h_2$. Therefore, t' has at least two non-empty cycles, which contradicts our induction hypothesis.

□

5.3 The Tutte polynomial of embedded graphs

We now define the *embedding-activities* of spanning trees. We consider an embedded graph $\mathcal{G}$ and a spanning tree T. By Proposition 5.2, the motion function is a cyclic permutation on the set H of half-edges, hence defines a cyclic order on H. If the embedding $\mathcal{G}$ is rooted, that is, a half-edge $h \in H$ is distinguished as the *root*, we can consider the linear order for which h is the smallest element.

Definition 5.3 *Let $\mathcal{G} = (H, \sigma, \alpha, h)$ be an embedded graph and let T be a spanning tree. We define the* $(\mathcal{G}, T)$-order *on the set H of half-edges by $h < t(h) < t^2(h) < \ldots < t^{|H|-1}(h)$, where t is the motion function. (Note that the $(\mathcal{G}, T)$-order is a linear order on H since t is a cyclic permutation.) We define the $(\mathcal{G}, T)$-order on the edge set by setting $e = \{h_1, h_2\} < e' = \{h'_1, h'_2\}$ if $\min(h_1, h_2) < \min(h'_1, h'_2)$. (Note that this is also a linear order.)*

Example: Consider the embedded graph $\mathcal{G}$ rooted on a and the spanning tree T represented in Figure 113. The $(\mathcal{G}, T)$-order on the half-edges is $a < e < f < c < a' < f' < b < c' < e' < b' < d < d'$. Therefore, the $(\mathcal{G}, T)$-order on the edges is $\{a, a'\} < \{e, e'\} < \{f, f'\} < \{c, c'\} < \{b, b'\} < \{d, d'\}$.

We are now ready to define the *embedding-activity*.

Definition 5.4 *Let $\mathcal{G}$ be a rooted embedded graph and T be a spanning tree. We say that an external (resp. internal) edge is* $(\mathcal{G}, T)$-active *(or* embedding-active *if $\mathcal{G}$ and T are clear from the context) if it is minimal for the $(\mathcal{G}, T)$-order in its fundamental cycle (resp. cocycle).*

Example: In Figure 113, the $(\mathcal{G}, T)$-order on the edges is $\{a, a'\} < \{e, e'\} < \{f, f'\} < \{c, c'\} < \{b, b'\} < \{d, d'\}$. Hence, the internal active edges are $\{a, a'\}$ and $\{d, d'\}$ and there is no external active edge. For instance, $\{e, e'\}$ is not active since $\{a, a'\}$ is in its fundamental cycle.

We are now ready to give a characterization of the Tutte polynomial based on embedding-activities. This characterization of the Tutte polynomial by embedding-activities is reminiscent of Tutte's original characterization [Tutt 54]. We urge to say that these two characterizations are not equivalent.

Theorem 5.5 *Let $\mathcal{G}$ be any rooted embedding of the connected graph G (with at least one edge). The Tutte polynomial of G is equal to*

$$T_G(x, y) = \sum_{T \text{ spanning tree}} x^{\mathcal{I}(T)} y^{\mathcal{E}(T)}, \tag{84}$$

where the sum is over all spanning trees and $\mathcal{I}(T)$ (resp. $\mathcal{E}(T)$) is the number of $(\mathcal{G}, T)$-active internal (resp. external) edges.

Example: We represented the spanning trees of K_3 in Figure 115. If the embedding is rooted on the half-edge a, then the embedding-active edges are the one indicated by a $\star$. Hence, the spanning trees (taken from left to right) have respective contributions x, x^2 and y and the Tutte polynomial is $T_{K_3}(x, y) = x^2 + x + y$.

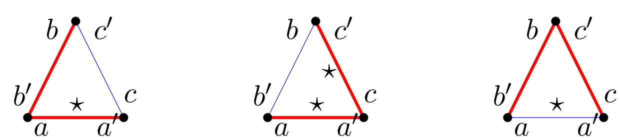


Figure 115: The embedding-activities of the spanning trees of K_3.

Note that Theorem 5.5 implies that the sum in (84) does not depend on the embedding, whereas the summands clearly depends on it. We postpone the proof of Theorem 5.5 to the next section. In the rest of this section we present some other characterizations of the Tutte polynomial which will serve as an element of comparison with the present work.

The characterization of the Tutte polynomial given in Theorem 5.5 is reminiscent of the first definition given by Tutte in 1954 [Tutt 54]. Tutte's characterization is also a generating function of spanning trees counted according to some activities, the *ordering-activities*. In order to define the ordering-activities we need to choose a linear ordering of the edge set (instead of an embedding). Let G be a graph whose edge set is linearly ordered. Then, given a spanning tree T, an external (resp. internal) edge is said to be *ordering-active* if it is minimal in its fundamental cycle (resp. cocycle). Tutte proved in [Tutt 54] that

$$T_G(x,y) = \sum_{T \text{ spanning tree}} x^{i(T)} y^{e(T)}, \qquad (85)$$

where the sum is over all spanning trees and $i(T)$ (resp. $e(T)$) is the number of internal (resp. external) ordering-active edges. We indicated the ordering-activities of the spanning trees of K_3 in Figure 116.

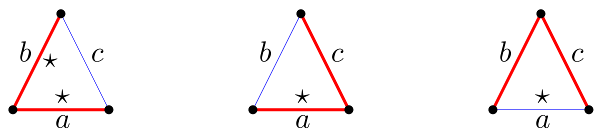


Figure 116: The ordering-activities of the spanning trees of K_3 (indicated by a $\star$) for the linear order $a < b < c$. The spanning trees (taken from left to right) have respective contribution x^2, x and y.

Tutte's characterization implies that the sum in (85) does not depend on the ordering of the edge set (whereas the summands clearly depends on that order). This characterization is easily proved by induction. Indeed, it is simple to prove that the induction relation of Proposition 4.3 holds *for the edge having the largest label.*

We emphasize that Theorem 5.5 is *not* a special case of Tutte's result since the $(\mathcal{G}, T)$-order is a linear order on the edge set that *depends* on the tree T.

The characterization of the Tutte polynomial in terms of the ordering-activities of spanning trees is sometimes thought of as slightly unnatural. It is true that the dependence of this characterization on a particular linear ordering of the edge set is a bit puzzling. We want to argue that an embedding may be a less arbitrary structure than a linear order on the edge set. As a matter of fact, there are a number of mathematical conjectures dealing with the Tutte polynomial, or sometimes the chromatic polynomial, of *planar graphs.* A graph is *planar* if and only if can be embedded in the sphere. Equivalently, it has an embedding (H, σ, α) with Euler characteristic equal to 2. For instance, the four color theorem can be stated as: $T_{\mathcal{G}}(-3, 0) \neq 0$ *for any loopless planar embedding* $\mathcal{G}$.

We now present a characterization of the Tutte polynomial given by Las Vergnas as the generating function of orientations counted according to their *cyclic-activities* [Las 84b]. Let G be a graph whose edge set is linearly ordered. Given an orientation $\mathcal{O}$, a cyclic (resp. acyclic) edge is said to be *cyclic-active* (resp. *acyclic-active*) if it is minimal in an $\mathcal{O}$-directed cycle (resp. cocycle). It was proved in [Las 84b] that

$$T_G(x,y) = \sum_{\mathcal{O} \text{ orientation}} \left(\frac{x}{2}\right)^{ic(\mathcal{O})} \left(\frac{y}{2}\right)^{ec(\mathcal{O})}, \tag{86}$$

where the sum is over all orientations and $ic(\mathcal{O})$ (resp. $ec(\mathcal{O})$) is the number of cyclic-active (resp. acyclic-active) edges. We indicated the cyclic-activities of the orientations of K_3 in Figure 117.

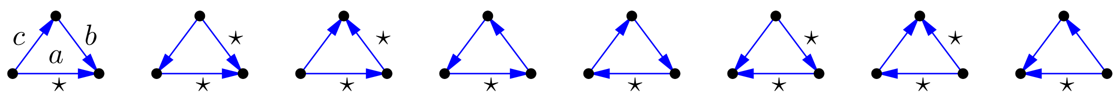


Figure 117: The cyclic-activities of the spanning trees of K_3 (indicated by a $\star$) for the linear order $a < b < c$. The orientations have respective contribution $\frac{x}{2}, \frac{x^2}{4}, \frac{x^2}{4}, \frac{y}{2}, \frac{y}{2}, \frac{x^2}{4}, \frac{x^2}{4}$ and $\frac{x}{2}$.

Comparing the Characterizations (85) and (86) makes it appealing to look for a correspondence between spanning trees and orientations in which each spanning tree T having ordering-activities $(i(T), e(T))$ is associated with $2^{i(T)+e(T)}$ orientations $\mathcal{O}$ having cyclic-activities $(ic(\mathcal{O}), ec(\mathcal{O})) = (i(T), e(T)$. This was first done in [Las 84a]. Another correspondence was defined in [Gioa 05] which has the advantage of being extendable to the context of oriented matroids. In some senses, the correspondence we establish in Chapter 6 between subgraphs and orientations can be seen as the counterpart of [Gioa 05] for embedding activities.

Lastly, Gessel and Wang [Gess 79] introduced a notion of external activity, the *DFS-activity* (based on the depth-first search algorithm) which was further investigated in [Gess 95, Gess 96]. Consider a connected graph G, a linear order on the vertex set and a linear order on the edge set (the latter can be derived from the former if G is simple). Consider a forest F of G. We define the root-vertex of any tree T of F as the smallest vertex in T. An edge $e \notin F$ having both endpoints in the same tree T of F is called *external*. The *fundamental cycle* C of e is the union of e and the path in T between its two endpoints. The external edge e is *DFS-active* if one of its endpoints, say u, is the ancestor of the other and $e < e_u$, where e_u is the internal edge in C incident to u. It was proved in [Gess 96] that

$$T_G(x,y) = \sum_{F \text{ forest}} (x-1)^{c(F)-1} y^{ed(F)}, \tag{87}$$

where the sum is over all forests and $ed(F)$ is the number of external DFS-active edges. We indicated the external DFS-active edges for the forests of K_3 in Figure 118.

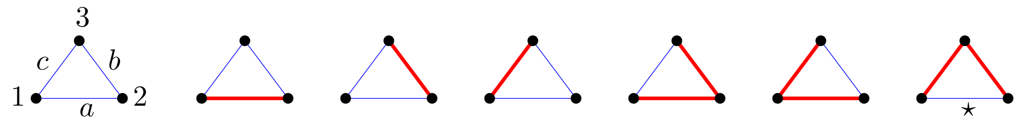


Figure 118: The external DFS-active edges of the 7 forests the of K_3 for the linear order $a < b < c$. There only one external DFS-active edge which is indicated by a $\star$. The forests have respective contribution $(x-1)^2, (x-1), (x-1), (x-1), 1, 1$ and y.

We will prove shortly (Lemma 5.8) a characterization for external embedding-active edges which is very close (in its formulation) to the definition of external DFS-active edges. In fact it is the same except that in the case of embedding-activities the comparison between e and e_u is made according to the $(\mathcal{G}, T)$-order. In other words, the condition $e < e_u$ for DFS-activity is replaced by the condition that e_0, e, e_u *are in cyclic order around* u for embedding-activity, where e_0 is the edge linking u to its father in T (or the root-edge if u is the root-vertex). It does not seem to exist a nice way of defining a DFS-activity for internal edges. The intuitive reason for this is that *linear* local orders (as opposed to *cyclic* local orders) do not behave well when an edge is contracted.

5.4 Proofs of the characterization of the Tutte polynomial by embedding activities

In this section we prove the characterization of the Tutte polynomial given by Theorem 5.5. We also establish several lemmas that will be useful in the following chapters.

Lemma 5.6 *Let $\mathcal{G}$ be an embedded graph. Let T be a spanning tree and let $e = \{h_1, h_2\}$ be an internal edge. Assume that $h_1 < h_2$ (for the $(\mathcal{G}, T)$-order) and denote by v_1 and v_2 the endpoints of h_1 and h_2 respectively. Then, v_1 is the father of v_2 in T. Moreover, the half-edges h such that $h_1 < h \leq h_2$ are the half-edges incident to a descendant of v_2.*

Proof: Let t be the motion function associated to the tree T (t is defined by (81)). We consider the subtrees T_1 and T_2 obtained from T by deleting e with the convention that h_1 is incident to T_1 and h_2 is incident to T_2. Let h be any half-edge distinct from h_1 and h_2. By definition of t, the half-edges h and $t(h)$ are incident to the same subtree T_i. Therefore, the $(\mathcal{G}, T)$-order is such that $h_0 < l_1 < \cdots < l_i < h_1 < l'_1 < \cdots < l'_j < h_2 < l''_1 < \cdots < l''_k$ where $l'_1, \ldots, l'_j, h_2$ are the half-edges incident with the subtree T_2 not containing the root-vertex v_0. Since the subtree T_2 does not contain v_0 its vertices are the descendants of v_2 in T.

□

Lemma 5.7 *With the same assumption as in Lemma 5.6, let $e = \{h_1, h_2\}$ with $h_1 < h_2$ be an internal edge and let $e' = \{h'_1, h'_2\}$ with $h'_1 < h'_2$ be an external edge.*
• *Then, e is in the fundamental cycle of e' (equivalently, e' is in the fundamental cocycle of e) if and only if $h_1 < h'_1 < h_2 < h'_2$ or $h'_1 < h_1 < h'_2 < h_2$.*
• *Suppose that e is in the fundamental cycle of e' and denote by v_1, v_2, v'_1, v'_2 the endpoints of h_1, h_2, h'_1, h'_2 respectively. Recall that v_1 is the father of v_2 in T (Lemma 5.6) and that exactly one of the vertices v'_1, v'_2 is a descendant of v_2. If $e < e'$, then v'_1 is the descendant of v_2, else it is v'_2.*

Proof:
• Let V_2 be the set of descendants of v_2. Recall that the edge e' is in the fundamental cocycle of e if and only if it has one endpoint in V_2 and the other in $\overline{V_2}$. By Lemma 5.6, this is equivalent to the fact that exactly one of the half-edges h'_1, h'_2 is in $\{h' : h_1 < h' \leq h_2\}$. Thus, e' is in the fundamental cocycle of e if and only if $h_1 < h'_1 < h_2 < h'_2$ or $h'_1 < h_1 < h'_2 < h_2$.
• Suppose that e is in the fundamental cycle of e'. By the preceding point, $e < e'$ implies $h_1 < h'_1 < h_2 < h'_2$. In this case, h'_1 is incident to a descendant of v_2 by Lemma 5.6. Similarly, $e' < e$ implies $h'_1 < h_1 < h'_2 < h_2$, hence h'_2 is incident to a descendant of v_2. □

Lemma 5.8 *An external edge $e' = \{h'_1, h'_2\}$ with $h'_1 < h'_2$ is $(\mathcal{G}, T)$-active if and only if the endpoint of h'_1 is an ancestor of the endpoint of h'_2.*

Proof: Denote by v'_1 and v'_2 the endpoints of h'_1 and h'_2 respectively.
• Suppose v'_1 is an ancestor of v'_2. We want to prove that e' is active. Let $e = \{h_1, h_2\}$ with $h_1 < h_2$ be an internal edge in the fundamental cycle of e'. The edge e is in the path of T between v'_1 and v'_2. Denote by v_1 and v_2 the endpoints of h_1 and h_2 respectively. Recall that v_1 is the father of v_2 (Lemma 5.6). Since v'_2 is a descendant of v_2, we have $e' < e$ by Lemma 5.7. The edge e' is less than any edge in its fundamental cycle hence it is $(\mathcal{G}, T)$-active.
• Suppose that v'_1 is not an ancestor of v'_2. Then the edge $e = \{h_1, h_2\}$ with $h_1 < h_2$ linking v'_1 to its father in T is in the fundamental cycle of e'. If we denote by v_1 and v_2 the endpoints of h_1 and h_2 respectively, we get $v_2 = v'_1$ by Lemma 5.6. Since the endpoint v'_1 of h'_1 is a descendant of the endpoint v_2 of h_2, we get $e < e'$ by Lemma 5.7. Thus, e' is not $(\mathcal{G}, T)$-active.

□

Lemma 5.9 *Let $\mathcal{G}$ be a rooted embedded graph with edge set E and half-edge set H. Let T be a spanning tree and $e = \{h_1, h_2\}$ be an edge not containing the root. If e is external (resp. internal), the $(\mathcal{G}_{\backslash e}, T)$-order (resp. $(\mathcal{G}_{/e}, T - e)$-order) on $H \setminus \{h_1, h_2\}$ and $E - e$ is simply the restriction of the $(\mathcal{G}, T)$-order to these sets.*

Proof: By Lemma 5.1, we see that if e is external (resp. internal), the $(\mathcal{G}_{\backslash e}, T)$-order (resp. $(\mathcal{G}_{/e}, T-e)$-order) on the half-edge set $H\setminus\{h_1, h_2\}$ is simply the restriction of the $(\mathcal{G}, T)$-order to this set. The same property follows immediately for the edge set.

$\square$

Proof of Theorem 5.5: We associate to the rooted embedded graph $\mathcal{G}$ the polynomial

$$\mathcal{T}_{\mathcal{G}}(x,y) = \sum_{T \text{ spanning tree}} x^{\mathcal{I}(T)} y^{\mathcal{E}(T)},$$

where $\mathcal{I}(T)$ (resp. $\mathcal{E}(T)$) is the number of embedding-active internal (resp. external) edges. We want to show that the polynomial $\mathcal{T}_{\mathcal{G}}(x,y)$ is equal to the Tutte polynomial $T_G(x,y)$ of G. We proceed by induction on the number of edges, using the induction relations (80) of the Tutte polynomial.

• The graphs with one edge are the graph L reduced to a loop and the graph I reduced to an isthmus. The graph L (resp. I) has a unique rooted embedding $\mathcal{L}$ (resp. $\mathcal{I}$). We check that $T_L(x,y) = y = \mathcal{T}_{\mathcal{L}}(x,y)$ and $T_I(x,y) = x = \mathcal{T}_{\mathcal{I}}(x,y)$.

• We assume the property holds for all (connected) graphs with at most $n \geq 1$ edges and consider a rooted embedding $\mathcal{G} = (H, \sigma, \alpha, h_0)$ of a graph G with $n+1$ edges. We denote by v_0 the vertex incident to the root h_0 and e_0 the edge containing h_0. We denote by $h_* = \sigma^{-1}(h_0)$ the half-edge preceding h_0 around v_0 and by $e_* = \{h_*, h'_*\}$ the edge containing h_*.

We study separately the 3 different cases of the induction relation (80).

Case 1: *The edge e_* is neither an isthmus nor a loop.*
The set $\mathbb{T}$ of spanning trees of G can be partitioned into $\mathbb{T} = \mathbb{T}_1 \cup \mathbb{T}_2$, where $\mathbb{T}_1$ (resp. $\mathbb{T}_2$) is the set of spanning trees containing (resp. not containing) the edge e_*. The set $\mathbb{T}_1$ (resp. $\mathbb{T}_2$) is in bijection by the mapping $\Phi_1 : T \mapsto T - e_*$ (resp. $\Phi_2 : T \mapsto T$) with the spanning trees of $G_{/e_*}$ (resp. $G_{\backslash e_*}$). We want to show e_* is never embedding-active and that the mappings Φ_i preserve the embedding-activities: for any tree T in $\mathbb{T}_1$ (resp. $\mathbb{T}_2$), an edge is $(\mathcal{G}, T)$-active if and only if it is $(\mathcal{G}_{/e_*}, T - e_*)$-active (resp. $(\mathcal{G}_{\backslash e_*}, T)$-active). We are going to prove successively the following four points:

• *The edges e_* and e_0 are distinct.*
First note that $h_0 \neq h_*$ or we would have $\sigma(h_*) = h_*$ implying that v_0 has degree one hence that e_* is an isthmus. Also, $h_0 \neq h'_*$ or we would have $\sigma(h_*) = \alpha(h_*)$ implying that e_* is a loop. Thus, $e_* = \{h_*, h'_*\}$ does not contain h_0.

• *Given any spanning tree, the edge e_* is maximal in its fundamental cycle or cocycle.*
Let T be a spanning tree of G. Suppose first that the edge e_* is internal. In this case, the motion function t satisfies, $t(h'_*) = \sigma\alpha(h'_*) = h_0$. Hence, h'_* is the greatest half-edge for the $(\mathcal{G}, T)$-order. Let $e = \{h, h'\}$ with $h < h'$ be an edge in the fundamental cocycle of e_*. By Lemma 5.7, the half-edges h, h', h_*, h'_* satisfy $h < h_* < h' < h'_*$. Hence, the edge e is less

than e_*. Thus, the edge e_* is maximal in its fundamental cocycle. Suppose now that the edge e_* is external. In this case, $t(h_*) = \sigma(h_*) = h_0$ hence, h_* is the greatest half-edge for the $(\mathcal{G}, T)$-order. Let $e = \{h, h'\}$ with $h < h'$ be an edge in the fundamental cycle of e_*. By Lemma 5.7, the half-edges h, h', h_*, h'_* satisfy $h < h'_* < h' < h_*$, hence, the edge e is less than e_*. Thus, the edge e_* is maximal in its fundamental cycle.

• *For any tree T in $\mathbb{T}_1$ (resp. $\mathbb{T}_2$), the $(\mathcal{G}, T)$-active and $(\mathcal{G}_{/e_*}, T - e_*)$-active (resp. $(\mathcal{G}_{\setminus e_*}, T)$-active) edges are the same.*

First note that e_* is never alone in its fundamental cycle or cocycle (or e_* would be a loop or isthmus). Hence, by the preceding point, e_* is never embedding-active. We now look at the embedding-activities of the other edges. Let T be a tree in $\mathbb{T}_1$ (i.e. containing e_*). Let e be an external (resp. internal) edge distinct from e_* and let C be its fundamental cycle (resp. cocycle). The fundamental cycle (resp. cocycle) of e in $(\mathcal{G}_{/e_*}, T - e_*)$ is $C - e_*$. Note that the $(\mathcal{G}, T)$-minimal element of C is in $C - e_*$ (since, if e_* is in C then e is in the fundamental cycle of e_* hence $e < e_*$ for the $(\mathcal{G}, T)$-order). Moreover, by Lemma 5.9, the $(\mathcal{G}, T)$-order and $(\mathcal{G}_{/e_*}, T - e_*)$-order coincide on $C - e_*$. Hence, the $(\mathcal{G}, T)$-minimal element of C is the $(\mathcal{G}_{/e_*}, T - e_*)$-minimal element in $C - e_*$. Therefore, the edge e is $(\mathcal{G}, T)$-active if and only if it is $(\mathcal{G}_{/e_*}, T - e_*)$-active.

The case where T is a tree in $\mathbb{T}_2$ (i.e. not containing e_*) is identical.

• *The polynomial $\mathcal{T}_{\mathcal{G}}(x, y)$ is equal to the Tutte polynomial $T_G(x, y)$.*

From the properties above, we have

$$\begin{aligned}
\mathcal{T}_{\mathcal{G}}(x, y) &\equiv \sum_{T \text{ spanning tree of } G} x^{\mathcal{I}(T)} y^{\mathcal{E}(T)} \\
&= \sum_{T \in \mathbb{T}_1} x^{\mathcal{I}(T)} y^{\mathcal{E}(T)} + \sum_{T \in \mathbb{T}_2} x^{\mathcal{I}(T)} y^{\mathcal{E}(T)} \\
&= \sum_{T \in \mathbb{T}_1} x^{\mathcal{I}'(T-e_*)} y^{\mathcal{E}'(T-e_*)} + \sum_{T \in \mathbb{T}_2} x^{\mathcal{I}''(T)} y^{\mathcal{E}''(T)} \qquad (88)
\end{aligned}$$

where $\mathcal{I}'(T - e_*)$, $\mathcal{E}'(T - e_*)$, $\mathcal{I}''(T)$, $\mathcal{I}''(T)$ are respectively the number of internal $(\mathcal{G}_{/e_*}, T - e_*)$-active, external $(\mathcal{G}_{/e_*}, T - e_*)$-active, $(\mathcal{G}_{\setminus e_*}, T)$-active and external $(\mathcal{G}_{\setminus e_*}, T)$-active edges.

In the right-hand side of (88) we recognize the polynomials $\mathcal{T}_{\mathcal{G}_{/e_*}}(x, y)$ and $\mathcal{T}_{\mathcal{G}_{\setminus e_*}}(x, y)$. By the induction hypothesis, these polynomials are the Tutte polynomials $T_{G_{/e_*}}(x, y)$ and $T_{G_{\setminus e_*}}(x, y)$. Thus,

$$\mathcal{T}_{\mathcal{G}}(x, y) = \mathcal{T}_{\mathcal{G}_{/e_*}}(x, y) + \mathcal{T}_{\mathcal{G}_{\setminus e_*}}(x, y) = T_{G_{/e_*}}(x, y) + T_{G_{\setminus e_*}}(x, y). \qquad (89)$$

In view of the induction relation of Proposition 4.3, this is the Tutte polynomial $T_G(x, y)$.

Case 2: *The edge e_* is an isthmus.*

Since e_* is an isthmus, it is in every spanning tree. Moreover, being alone in its fundamental

cocycle, it is always active. We want to show that for any spanning tree T, the embedding-activity of any edge other than e_* is the same in $(\mathcal{G}, T)$ and in $(\mathcal{G}_{/e_*}, T - e_*)$. Before we do that, we must cope with a (little) technical difficulty: the edge e_* might be equal to e_0 in which case we should specify how to root the graph $\mathcal{G}_{/e_*}$.
First note that $h_0 \neq h'_*$ or we would have $\sigma(h_*) = \alpha(h_*)$ implying that e_* is a loop. Suppose now that $h_0 = h_*$ (equivalently, $\sigma(h_*) = h_*$). In this case, we define the root of $\mathcal{G}_{/e_*}$ to be $h_1 = \sigma(h'_*)$ (h_1 is not an half-edge of e_* or e_* would be an isolated isthmus).

• *For any spanning tree T of G, the $(\mathcal{G}, T)$-order and the $(\mathcal{G}_{/e_*}, T - e_*)$-order coincide on $E - e_*$.*
If $e_* \neq e_0$ the property is given by Lemma 5.9. Now suppose that $e_* = e_0$ (that is $h_* = h_0$). Since e_* is internal, the motion function t satisfies $t(h'_*) = h_0$ and $t(h_0) = h_1$. Therefore, the $(\mathcal{G}, T)$-order on half-edges is $h_0 = h_* < h_1 < t(h_1) < \ldots < h'_*$. Let us denote by $\mathcal{G}_1$ the embedded graph $\mathcal{G}$ rooted on h_1. The $(\mathcal{G}_1, T)$-order on half-edges is $h_1 < t(h_1) < \ldots < h'_* < h_0 = h_*$. Thus, the $(\mathcal{G}_1, T)$-order and $(\mathcal{G}, T)$-order coincide on $E - e_*$. Moreover, by Lemma 5.9, the $(\mathcal{G}_1, T)$-order and $(\mathcal{G}_{/e_*}, T)$-order coincide on $E - e_*$.

• *For any spanning tree T, the set of $(\mathcal{G}, T)$-active edges distinct from e_* is the set of $(\mathcal{G}_{/e_*}, T - e_*)$-active edges.*
For any tree T and any external (resp. internal) edge $e \neq e_*$, the fundamental cycle (resp. cocycle) of e does not contain e_* and is the same in $(\mathcal{G}, T)$ and in $(\mathcal{G}_{/e_*}, T - e_*)$. Since the $(\mathcal{G}, T)$-order and the $(\mathcal{G}_{/e_*}, T - e_*)$-order coincide on $E - e_*$, the edge e is $(\mathcal{G}, T)$-active if and only if it is $(\mathcal{G}_{/e_*}, T - e_*)$-active.

• *The polynomial $\mathcal{T}_{\mathcal{G}}(x, y)$ is equal to the Tutte polynomial $T_G(x, y)$.*
From the properties above, we have

$$
\begin{aligned}
\mathcal{T}_{\mathcal{G}}(x,y) &\equiv \sum_{T \text{ spanning tree of } G} x^{\mathcal{I}(T)} y^{\mathcal{E}(T)} \\
&= \sum_{T \text{ spanning tree of } G} x^{1+\mathcal{I}'(T-e_*)} y^{\mathcal{E}'(T-e_*)} \\
&= x \cdot \sum_{T \text{ spanning tree of } G} x^{\mathcal{I}'(T-e_*)} y^{\mathcal{E}'(T-e_*)} \qquad (90)
\end{aligned}
$$

where $\mathcal{I}'(T - e_*)$ and $\mathcal{E}'(T - e_*)$ are respectively the number of internal $(\mathcal{G}_{/e_*}, T - e_*)$-active and external $(\mathcal{G}_{/e_*}, T - e_*)$-active edges.
In the right-hand side of (90) we recognize the sum as being $\mathcal{T}_{\mathcal{G}_{/e_*}}(x, y)$. By the induction hypothesis, we know this polynomial to be equal to the Tutte polynomial $T_{G_{/e_*}}(x, y)$. Thus,

$$\mathcal{T}_{\mathcal{G}}(x,y) = x \cdot \mathcal{T}_{\mathcal{G}_{/e_*}}(x,y) = x \cdot T_{G_{/e_*}}(x,y). \qquad (91)$$

In view of the induction relation of Proposition 4.3, this is the Tutte polynomial $T_G(x, y)$.

Case 3: *The edge e_* is a loop.*
This case is dual to Case 2.
Since e_* is a loop, it is always external and always active. We want to show that for any spanning tree T, the embedding-activity of any edge other than e_* is the same in $(\mathcal{G}, T)$ and in $(\mathcal{G}_{\backslash e_*}, T)$. Before we do that, we must choose a root for $\mathcal{G}_{\backslash e_*}$ when $e_* = e_0$. We see that $h_0 \neq h_*$ or we would have $\sigma(h_*) = h_*$ implying that e_* is an isthmus. Suppose now that $h_0 = h'_*$ (equivalently, $\alpha(h_*) = \sigma(h_*)$). In this case, we define the root of $\mathcal{G}_{\backslash e_*}$ to be $h_1 = \sigma(h_0)$ (h_1 is not an half-edge of e_* or e_* would be an isolated loop).

• *For any spanning tree T of G, the $(\mathcal{G}, T)$-order and the $(\mathcal{G}_{\backslash e_*}, T)$-order coincide on $E - e_*$.*
The proof of Case 2 can be copied verbatim except "e_* is internal" is replaced by "e_* is external".

• *For any spanning tree T, the set of $(\mathcal{G}, T)$-active edges distinct from e_* is the set of $(\mathcal{G}_{\backslash e_*}, T)$-active edges.*
The proof of Case 2 can be copied verbatim.

• *The polynomial $\mathcal{T}_{\mathcal{G}}(x, y)$ is equal to the Tutte polynomial $T_G(x, y)$.*
From the properties above, we have

$$\begin{aligned} \mathcal{T}_{\mathcal{G}}(x,y) &\equiv \sum_{T \text{ spanning tree of } G} x^{\mathcal{I}(T)} y^{\mathcal{E}(T)} \\ &= y \cdot \sum_{T \text{ spanning tree of } G} x^{\mathcal{I}''(T)} y^{\mathcal{E}''(T)} \end{aligned} \tag{92}$$

where $\mathcal{I}''(T)$ and $\mathcal{E}''(T)$ are respectively the number of internal $(\mathcal{G}_{\backslash e_*}, T)$-active and external $(\mathcal{G}_{\backslash e_*}, T)$-active edges.
In the right-hand side of (92) we recognize the sum as being $\mathcal{T}_{\mathcal{G}_{\backslash e_*}}(x, y)$. By the induction hypothesis, we know this polynomial to be equal to the Tutte polynomial $T_{G_{\backslash e_*}}(x, y)$. Thus,

$$\mathcal{T}_{\mathcal{G}}(x,y) = y \cdot \mathcal{T}_{\mathcal{G}_{\backslash e_*}}(x,y) = y \cdot T_{G_{\backslash e_*}}(x,y). \tag{93}$$

In view of the induction relation of Proposition 4.3, this is the Tutte polynomial $T_G(x, y)$. □

5.5 A glimpse at the results contained in the next chapters

We now take a glimpse at the results to be developed in the following chapters. In order to present these results, we define two mappings Γ and Λ on the set of spanning trees of a graph. Consider a graph G with a distinguished vertex v_0. The mapping Γ is a bijection between the spanning trees of G and the v_0-connected outdegree sequences. The mapping Λ is a bijection between the spanning trees of G and the recurrent configurations of the sandpile model. These two bijections are very close in their formulations (see Definitions

5.10 and 5.11) and are both related to a mapping Φ from spanning trees to orientations. The mapping Φ will be extended into a bijection between subgraphs and orientations in Chapter 6. The bijection Γ between spanning trees and v_0-connected outdegree sequences will be extended into a bijection between forests and outdegree sequences in Chapter 7. The bijection Λ between spanning trees and recurrent sandpile configurations will be studied in Chapter 8.

We first define a mapping Φ between from spanning trees to orientations. The mapping Φ is reminiscent of the mapping δ studied in Chapter 3 between spanning trees and tree-orientations. We only give here a reformulation based on combinatorial embeddings (this presentation is more convenient for the extensions to be presented in the following chapters). Consider an embedded graph $\mathcal{G} = (H, \sigma, \alpha, h_0)$. Recall that the *tour of a spanning tree* T has been defined in Subsection 4.1.2 as a way of visiting every half-edge of $\mathcal{G}$ in cyclic order. This tour is based on the *motion function* (giving the next half-edge in the cyclic order) defined on H by (81). The tour of T defines a linear order, the $(\mathcal{G}, T)$*-order*, on H for which the root h_0 is the least element.

We now define an orientation $\mathcal{O}_T$ of $\mathcal{G}$ associated to the spanning tree T by:

$$\text{For any edge } e = \{h_1, h_2\} \text{ with } h_1 < h_2,\ \mathcal{O}_T(e) = \left| \begin{array}{ll} (h_1, h_2) & \text{if } e \text{ is internal,} \\ (h_2, h_1) & \text{if } e \text{ is external.} \end{array} \right. \tag{94}$$

We illustrate this definition in Figure 119 (left).

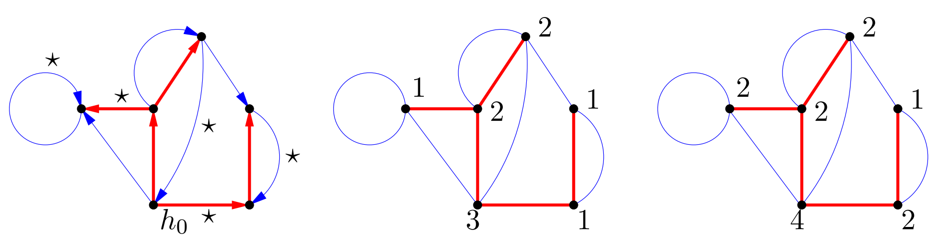


Figure 119: Left: Orientation $\mathcal{O}_T$ associated the spanning tree T (indicated by thick lines) and active edges (indicated by a star). Middle: outdegree sequence $\Gamma(T)$. Right: recurrent configuration of the sandpile model $\Lambda(T)$.

Let v_0 be the root-vertex of G. Observe that the spanning tree T is oriented from its root-vertex v_0 to its leaves in $\mathcal{O}_T$. Indeed, it is clear from the definitions and Lemma 5.6 that every internal edge is oriented from father to son. This property implies that for every spanning tree T the orientation $\mathcal{O}_T$ is v_0-connected.

The mapping $\Phi : T \mapsto \mathcal{O}_T$ from spanning trees to v_0-connected orientations is not bijective. However, it is injective and in Chapter 6 we will extend it into a bijection between subgraphs and orientations. For the time being, let us observe (the proof will be given in Chapter 6) that the tree T can be recovered from the orientation $\mathcal{O}_T$ by the following procedure:

Procedure Construct-tree:
Initialization: Initialize the *current half-edge* h to be the root h_0. Initialize the tree T and the set of visited arcs F to be empty.
Core: Do:
C1: If the edge e containing h is not in F and h is a tail then add e to T.
Add e to F.
C2: Move to the next half-edge around T:
If e is in T, then set the current half-edge h to be $\sigma\alpha(h)$, else set it to be $\sigma(h)$.
Repeat until the current half-edge h is h_0.
End: Return the tree T.

In the procedure **Construct-tree** we keep track of the set F of edges already visited. The decision of adding an edge e to the tree T or not is taken when e is visited for the *first* time. The principle of procedure **Construct-tree**, which consists in constructing a tree T while making its tour, will appear again in the next chapters.

Building on the mapping $\Phi : T \mapsto \mathcal{O}_T$ we define two mappings Γ and Λ.

Definition 5.10 *Let $\mathcal{G}$ be an embedded graph. The mapping Γ associates with any spanning tree T the outdegree sequence of the orientation $\mathcal{O}_T$.*

Definition 5.11 *Let $\mathcal{G}$ be an embedded graph and let V be the vertex set. The mapping Λ associates with any spanning tree T the configuration $\mathcal{S}_T : V \mapsto \mathbb{N}$, where $\mathcal{S}_T(v)$ is the number of tails plus the number of external $(\mathcal{G}, T)$-active heads incident to v in the orientation $\mathcal{O}_T$.*

The mappings Γ and Λ are illustrated in Figure 119. As observed above, the orientation $\mathcal{O}_T$ is always v_0-connected. We shall prove in Chapter 7 that Γ is a bijection between spanning tree and v_0-connected outdegree sequences. As for the mapping Λ, we shall prove in Chapter 8 that it is a bijection between spanning trees and recurrent configurations of the sandpile model. Moreover, the number of external $(\mathcal{G}, T)$-active edges is easily seen to be the level of the configuration $\Lambda(T)$. This gives a new bijective proof of a result by Merino linking external activities to the level of recurrent sandpile configurations [Cori 03, Meri 97] (see Chapter 8). The two mappings Γ and Λ are very similar and coincide on *internal* trees, that is, trees that have external activity 0. Thus, the mapping $\Gamma \circ \Lambda^{-1}$ is a bijection between recurrent configurations of the sandpile model and v_0-connected outdegree sequences that leaves the configurations at level 0 unchanged. This answers a problem raised by Gioan [Gioa 06].

We now highlight a relation (to be exploited in Chapter 7) between the embedding-activities of the spanning tree T and the acyclicity or strong connectivity of the associated orientation $\mathcal{O}_T$.

Lemma 5.12 *Let $\mathcal{G}$ be an embedded graph ant let T be a spanning tree. The fundamental cycle (resp. cocycle) of an external (resp. internal) edge e is $\mathcal{O}_T$-directed if and only if e is $(\mathcal{G}, T)$-active.*

Lemma 5.12 is illustrated by Figures 120 and 121. From this lemma we deduce that if $\mathcal{O}_T$ is acyclic (resp. strongly connected) then T is *internal* (resp. *external*), that is, has no external (resp. internal) active edge. In fact, we shall prove in Chapter 7 that the converse is true: if the tree T is internal (resp. external), then the orientation $\mathcal{O}_T$ is acyclic (resp. strongly connected).

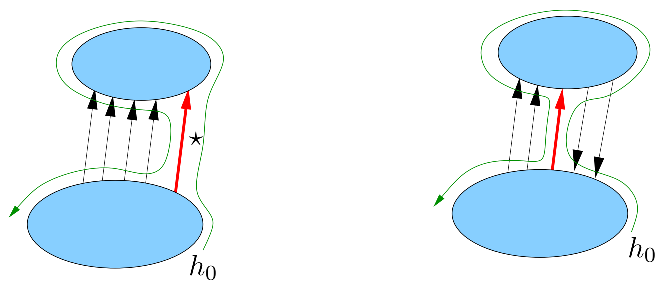


Figure 120: Fundamental cocycles of an active internal edge (left) and of a non-active internal edge (right).

Figure 121: Fundamental cycles of an active external edge (left) and of a non-active external edge (right).

Proof: Consider an edge $e = \{h_1, h_2\}$ with $h_1 < h_2$ and denote by v_1 and v_2 the endpoints of h_1 and h_2 respectively.

• Suppose that e is internal. We want to prove that the fundamental cocycle D of e is directed if and only if e is $(\mathcal{G}, T)$-active. Recall that v_1 is the father of v_2 by Lemma 5.6. Let V_2 be the set of descendants of v_2. Recall that D is the cocycle defined by V_2. By definition, the

arc $\mathcal{O}_T(e)$ is directed toward $v_2 \in V_2$. By Lemma 5.7, for all edge $e' = \{h'_1, h'_2\}$ with $h'_1 < h'_2$ in $D - e$, the arc $\mathcal{O}_T(e') = (h'_2, h'_1)$ is directed toward V_2 if and only if $e < e'$. Therefore, the fundamental cocycle D is directed if and only if e is minimal in D, that is, if e is $(\mathcal{G}, T)$-active.
• Suppose that e is external. We want to prove that the fundamental cycle C of e is directed if and only if e is $(\mathcal{G}, T)$-active. Recall that $C - e$ is the path in T between v_1 and v_2. Since $\mathcal{O}_T(e)$ is directed toward v_1, the cycle C is directed if and only if the path $C - e$ is directed from v_1 to v_2. Since every edge in $C - e \subseteq T$ is directed from father to son (Lemma 5.6), the cycle C is directed if and only if v_1 is an ancestor of v_2. This is precisely the characterization of external $(\mathcal{G}, T)$-active edges given by Lemma 5.8. $\square$

Until now we have looked at mappings defined on the set of spanning trees. In order to extend these mappings to general subgraphs we will now associate a spanning tree to every subgraph. This will be our first task in the next chapter.

Chapter 6

Partition of the set of subgraphs and a bijection between subgraphs and orientations

Abstract: In the previous chapter, we defined the *embedding activities* of spanning trees. In the present chapter, we define a partition of the set of subgraphs based on embedding activities. Each part of the partition is associated to a spanning-tree. We use this partition in order to define a bijection Φ between subgraphs and orientations that displays nice properties. The bijection Φ will be further investigated in the next chapter.

Résumé: Dans le chapitre précèdent nous avons défini les *activités de plongement* des arbres couvrants. Dans le présent chapitre, nous définissons une partition de l'ensemble des sous-graphes basée sur les activités de plongement. Chaque part de la partition est associée à un arbre couvrant. Nous utilisons cette partition des sous-graphes pour définir une bijection Φ entre les sous-graphes et les orientations qui a des propriétés intéressantes. Nous étudierons la bijection Φ plus en détail dans le prochain chapitre.

6.1 Introduction

In the previous chapter we defined the embedding-activities of edges. This definition was based on the tour of spanning trees. We then characterized the Tutte polynomial as the generating function of spanning trees counted by internal and external embedding-activities. In the present chapter we use the embedding-activities in order to define a partition of the set of subgraphs. Our partition is the counterpart for embedding-activities of some partitions based on other characterizations of the Tutte polynomial [Tutt 54, Gess 79]. Indeed, a partition of the set of subgraphs was defined for the notion of ordering-activity introduced by Tutte in [Tutt 54] as well as for the notion of external DFS-activities introduced by Gessel and Wang in [Gess 79] (these two notions were recalled in Section 5.3 of Chapter 5). These partitions have been used extensively to extract informations about the Tutte polynomial [Bari 79, Crap 69, Gess 95, Gess 96, Gess 79, Gord 90].

In Chapter 5 (Section 5.5) we defined a mapping Φ between spanning trees and orientations. This mapping was just a reformulation of the bijection between spanning trees and *tree-orientations* defined in Chapter 3. Building on our partition of the set of subgraphs we will extend the mapping Φ into a general bijection between subgraphs and orientations. We shall see in the next chapter that the mapping Φ has a lot of interesting specializations.

The outline of this chapter is as follows. In Section 6.2, we define a partition of the set of subgraphs indexed by spanning trees. In Section 6.3, we exploit our partition in order to define a general bijection between subgraphs and orientations. Lastly, in Section 6.4 we comment on the case of planar graphs and on the computational aspects of our bijection.

6.2 A partition of the set of subgraphs indexed by spanning trees

In this section we define a partition of the set of subgraphs for any embedded graph. Each part of this partition is associated with a spanning tree.

Let $\mathcal{G}$ be an embedded graph. Given a spanning tree T, we consider the set of subgraphs that can be obtained from T by removing some internal $(\mathcal{G}, T)$-active edges and adding some external $(\mathcal{G}, T)$-active edges. Observe that this set is an interval in the boolean lattice of the subgraphs of $\mathcal{G}$ (i.e. subsets of edges). We call *tree-interval* and denote by $[T^-, T^+]$ the set of subgraphs obtained from a spanning tree T. We represented the tree-intervals corresponding to each of the 5 spanning trees of the embedded graph in Figure 122.

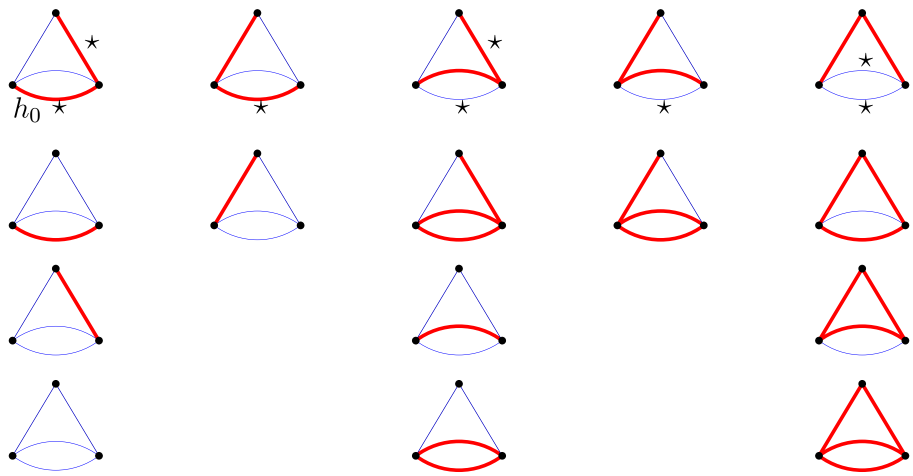


Figure 122: The tree-intervals corresponding to each of the 5 spanning trees (first line). Active edges are indicated by a $\star$.

We first give some properties of the subgraphs in the tree-interval $[T^-, T^+]$.

Lemma 6.1 *Let $\mathcal{G}$ be an embedded graph and let T be a spanning tree. Let e be an internal (resp. external) $(\mathcal{G}, T)$-active edge. The fundamental cocycle (resp. cycle) of e is contained in $\overline{S} + e$ (resp. $S + e$) for any subgraph S in $[T^-, T^+]$.*

Proof: If e is internal and $(\mathcal{G}, T)$-active, no edge in its fundamental cocycle D is $(\mathcal{G}, T)$-active (since their fundamental cycle contains e). Since no edge of $D - e$ is in T nor is $(\mathcal{G}, T)$-active, none is in S. Hence, $D \subseteq \overline{S} + e$. Similarly, if e is external $(\mathcal{G}, T)$-active, its fundamental cycle is contained in $S + e$.

□

Lemma 6.2 *Let $\mathcal{G}$ be an embedded graph. Let T be a spanning tree and let S be a subgraph in $[T^-, T^+]$ having $c(S)$ connected components. Then $c(S) - 1$, (resp. $e(S) + c(S) - |V|$) is the number of edges in $\overline{S} \cap T$ (resp. $S \cap \overline{T}$).*

Proof: Consider any subgraph S in $[T^-, T^+]$. By Lemma 6.1, removing an internal $(\mathcal{G}, T)$-active edge from S increases $c(S)$ by one and leaves $e(S) + c(S)$ unchanged. Similarly, adding an external $(\mathcal{G}, T)$-active edge to S leaves $c(S)$ unchanged and increases $e(S) + c(S)$ by one. Moreover, $c(T) - 1 = 0$ and $e(T) + c(T) - |V| = 0$. Therefore, Lemma 6.2 holds for every subgraph S in $[T^-, T^+]$ by induction on the number of edges in $S \bigtriangleup T$.

□

By Lemma 6.2, the connected subgraphs in $[T^-, T^+]$ are the subgraphs in $[T, T^+]$ (the subgraphs obtained from T by adding some external $(\mathcal{G}, T)$-active edges). Similarly, the forests in $[T^-, T^+]$ are the subgraphs in $[T^-, T]$ (the subgraphs obtained from T

by removing some internal $(\mathcal{G}, T)$-active edges). These properties are illustrated in Figure 123.

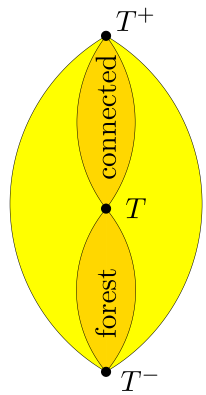


Figure 123: The tree-interval $[T^-, T^+]$, the sub-interval $[T, T^+]$ of connected subgraphs and the sub-interval $[T^-, T]$ of forests.

We are now ready to state and comment on the main result of this section.

Theorem 6.3 *Let $G = (V, E)$ be a graph and let $\mathcal{G}$ be an embedding of G. The tree-intervals form a partition of the set of subgraphs of G:*

$$2^E = \biguplus_{T \textit{ spanning tree}} [T^-, T^+],$$

where the disjoint union is over all spanning trees of G.

The counterpart of this theorem is known for the notion of ordering-activity introduced by Tutte in [Tutt 54] as well as for the notion of external DFS-activities introduced by Gessel and Wang in [Gess 79] (these two notions were recalled in Section 5.3 of Chapter 5). This property has been used extensively to extract informations about the Tutte polynomial [Bari 79, Crap 69, Gess 95, Gess 96, Gess 79, Gord 90]. Theorem 6.3 constitutes the key link between the subgraph expansions (79) and spanning tree expansions (84) of the Tutte polynomial. Indeed, given Lemma 6.2, we get

$$\sum_{S \in [T^-, T^+]} (x-1)^{c(S)-1}(y-1)^{e(S)+c(S)-|V|} = (x-1+1)^{\mathcal{I}(T)}(y-1+1)^{\mathcal{E}(T)} = x^{\mathcal{I}(T)}y^{\mathcal{E}(T)},$$

where $\mathcal{I}(T)$ (resp. $\mathcal{E}(T)$) is the number of internal (resp. external) $(\mathcal{G}, T)$-active edges. Summing over all spanning trees gives the identity:

$$\sum_{S \text{ subgraph}} (x-1)^{c(S)-1}(y-1)^{e(S)+c(S)-|V|} = \sum_{T \text{ spanning tree}} x^{\mathcal{I}(T)}y^{\mathcal{E}(T)}.$$

Remark. As observed in [Gord 90], the partition of the set of subgraphs gives several other expansions of the Tutte polynomial. For instance, the tree-intervals can be partitioned into *forest-intervals*. The *forest-interval* of a forest F in $[T^-, T^+]$ is the set $[F, F^+]$ of subgraphs obtained from F by adding some external $(\mathcal{G}, T)$-active edges. Since

$$[T^-, T^+] = \biguplus_{F \text{ forest in } [T^-,T^+]} [F, F^+],$$

the partition into tree-intervals given by Theorem 6.3 leads to a partition into forest-intervals:

$$2^E = \biguplus_{F \text{ forest}} [F, F^+].$$

Given Lemma 6.2, we get

$$\sum_{S\in[F,F^+]} (x-1)^{c(S)-1}(y-1)^{e(S)+c(S)-|V|} = (x-1)^{c(F)-1}(y-1+1)^{\mathcal{E}(T)} = (x-1)^{c(F)-1}y^{\mathcal{E}(T)},$$

for any forest in $[T^-, T^+]$. Summing up over forests, gives the *forest expansion*

$$T_G(x,y) = \sum_{F \text{ forest}} (x-1)^{c(F)-1}y^{\mathcal{E}(F)},$$

where $\mathcal{E}(F)$ is the number of $(\mathcal{G}, T)$-active edges for the spanning tree T such that $F \in [T^-, T^+]$. Observe the similarity with the characterization (87) of the Tutte polynomial based on DFS-activities.

In order to prove Theorem 6.3 we define a mapping Δ from subgraphs to spanning trees.

Definition 6.4 *Let $\mathcal{G}$ be an embedded graph rooted on h_0 and let S be a subgraph. The spanning tree $T = \Delta(S)$ is defined by the following procedure:*

Initialization: *Initialize the* current half-edge *h to be the root h_0. Initialize the tree T and the set of visited edges F to be empty.*

Core: *Do:*

C1: *If the edge e containing h is not in F, then decide whether to add e to T according to the following rule:*

If (e is in S and is in no cycle $C \subseteq S \cap \overline{F}$) or

(e is not in S and is in a cocycle $D \subseteq \overline{S} \cap \overline{F}$),

Then add e to T.

Endif.

Endif.

Add e to F.

C2: *Move to the next half-edge around T:*

If e is in T, then set the current half-edge h to be $\sigma\alpha(h)$, else set it to be $\sigma(h)$.

Repeat until the current half-edge h is h_0.

End: *Return the tree T.*

An execution of the procedure Δ is illustrated in Figure 124.

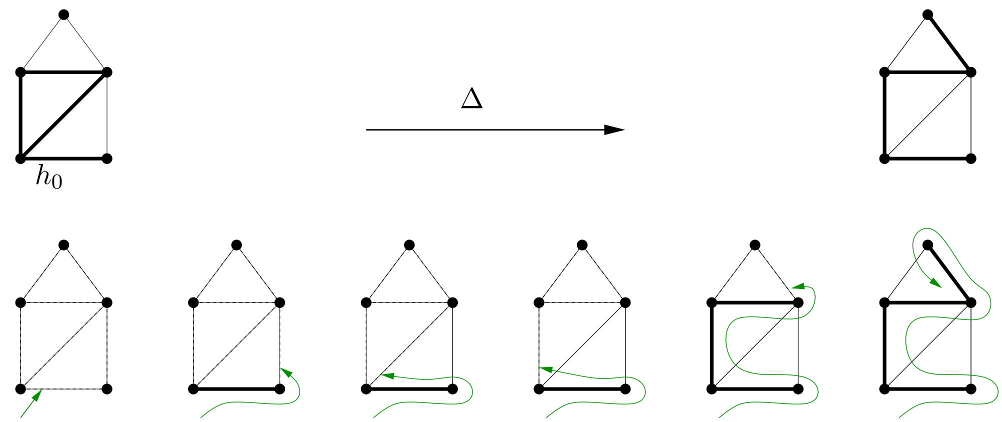


Figure 124: The mapping Δ and some intermediate steps. The dashed lines correspond to the set $\overline{F}$ of unvisited edges.

There is a *direct* proof that the mapping Δ is well defined on every subgraph (that is, the procedure terminates and returns a spanning tree). But we shall only prove an (a priory weaker) result: the mapping Δ is well defined on every tree-interval and $\Delta(S) = T$ for any subgraph S in $[T^-, T^+]$ (Proposition 6.5). This will prove that the tree-intervals are disjoint. Moreover, the cardinality of the tree-interval $[T^-, T^+]$ is $2^{\mathcal{I}(T)+\mathcal{E}(T)}$, where $\mathcal{I}(T)$ and $\mathcal{E}(T)$ are the number of internal and external $(\mathcal{G}, T)$-active edges. Therefore, the number of subgraphs contained in some tree-intervals is

$$\left| \bigcup_{T \text{ spanning tree}} [T^-, T^+] \right| = \sum_{T \text{ spanning tree}} \left|[T^-, T^+]\right| = \sum_{T \text{ spanning tree}} 2^{\mathcal{I}(T)+\mathcal{E}(T)}.$$

By Theorem 5.5, this sum is the specialization $T_G(2, 2)$ of the Tutte polynomial counting the subgraphs of G (as is clear from (79)). This counting argument proves that every subgraph belongs to a tree-interval. Thus, we only need to prove the following proposition.

Proposition 6.5 *Let $\mathcal{G}$ be an embedded graph. Let T be a spanning tree and let S be a subgraph in the tree-interval $[T^-, T^+]$. The procedure Δ is well defined on S and returns the tree T.*

Before proving Proposition 6.5, we need to recall a classical result of graph theory.

Lemma 6.6 (Elimination) *The symmetric difference of two cycles (resp. cocycles) C and C' is a union of cycles (resp. cocycles).*

Lemma 6.6 is illustrated by Figure 125.

We now characterize the edges in the symmetric difference $S \vartriangle T$.

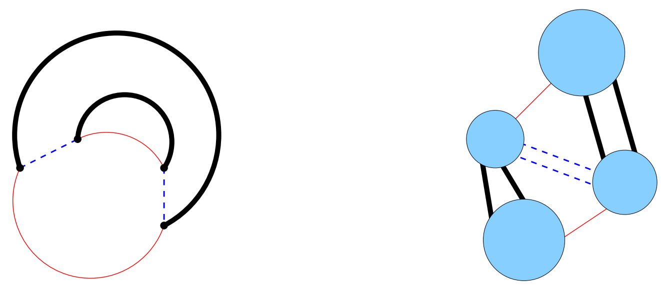


Figure 125: Two cycles (resp. cocycles) C and C' (thin and thick lines) and their intersection (dashed lines).

Lemma 6.7 *Let $\mathcal{G}$ be an embedded graph. Let T be a spanning tree and let S be a subgraph in the tree-interval $[T^-, T^+]$.*
(i) An edge e is in $S \cap \overline{T}$ if and only if e is minimal (for the $(\mathcal{G}, T)$-order) in a cycle $C \subseteq S$.
(ii) An edge e is in $\overline{S} \cap T$ if and only if e is minimal (for the $(\mathcal{G}, T)$-order) in a cocycle $D \subseteq \overline{S}$.

Proof: We give the proof of (i); the proof of (ii) is similar.
• Suppose e is in $S \cap \overline{T}$. Then e is $(\mathcal{G}, T)$-active, that is, e is minimal in its fundamental cycle C. Moreover, by Lemma 6.1, C is contained in S.
• Suppose e is minimal in a cycle $C \subseteq S$. We want to prove that e is in $\overline{T}$. Suppose the contrary. Then, there is an edge $e' \neq e$ in $C \cap \overline{T}$ (since T has no cycle). Take the least edge e' in $C \cap \overline{T}$ and consider its fundamental cycle C'. The edge e' is $(\mathcal{G}, T)$-active, that is, e' is minimal in C'. In particular, e is not in C'. This situation is represented in Figure 126. Since e is in $C \vartriangle C'$ and e' is not, there is a cycle $C_1 \subseteq C \vartriangle C'$ containing e and not e' (Lemma 6.6). By Lemma 6.1, the fundamental cycle C' of e' is contained is $S + e'$, thus $C_1 \subseteq C \vartriangle C' \subseteq S$. Note that e is minimal in the cycle $C_1 \subseteq S$ (since e is minimal in C and $e' > e$ is minimal in C'). Moreover, the least edge in $C_1 \cap \overline{T}$ (this edge exists since T has no cycle) is in $C \cap \overline{T} - e'$ (since $C' \subseteq T + e'$), hence is greater than e'. We can repeat this operation again in order to produce an infinite sequence $C_0 = C, C_1, C_2, \dots$ of cycles with e minimal in C_i and $C_i \subseteq S$ for all $i \geq 0$. But the minimal element of $C_i \cap \overline{T}$ is strictly increasing with i. This is impossible.
$\square$

Proof of Proposition 6.5. We consider a subgraph S in the tree-interval $[T_0^-, T_0^+]$. We denote by H the set of half-edges. We denote by t the motion function associated with spanning tree T_0 and we denote by $h_i = t^i(h_0)$ the i^{th} half-edge for the $(\mathcal{G}, T_0)$-order. For any half-edge h, we denote $F_h = \{e = \{h_1, h_2\} / \min(h_1, h_2) < h\}$ and $T_h = T_0 \cap F_h$.
We adopt the notations h, e, F and T of the procedure Δ (for instance, h denotes the current half-edge) and we compare half-edges according to the $(\mathcal{G}, T_0)$-order. We want to prove that, for all $i \leq |H|$, *at the beginning of the i^{th} core step, $h = h_i$, $F = F_h$ and $T = T_h$.* We proceed by induction on i. The property holds for the first core step ($i = 0$) since $h = h_0$ and

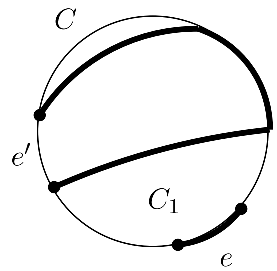


Figure 126: The cycle C (circle), some edges in the tree T (indicated by thick lines) and the edges e and e'.

$F_{h_0} = T_{h_0} = \emptyset$. Consider now the i^{th} core step. Suppose first that the edge e containing the current half-edge h is not in F. By the induction hypothesis, $F = F_h$ thus e is greater than any edge in F and less than any edge in $\overline{F} - e$. By Lemma 6.7, if e is in S, then it is in $\overline{T_0}$ if and only if it is in a cycle $C \subseteq S \cap \overline{F}$. Also, if e is in $\overline{S}$, then it is in T_0 if and only if it is in a cocycle $D \subseteq \overline{S} \cap \overline{F}$. Therefore, the edge e is added to T at the step **C1** if and only if it is in T_0. Suppose now that the edge e is already in F at the beginning of the i^{th} core step. Then, by the induction hypothesis, e is in $T = T_h = T_0 \cap F_h = T_0 \cap F$ if and only if it is in T_0. Whether the edge e is in F or not at the beginning of the step **C1**, the edge e is in T at the beginning of the step **C2** if and only if it is in T_0. Therefore, the current half-edge at the beginning of the $(i+1)^{th}$ core step, is $t(h) = h_{i+1}$. Thus, the property holds for all $i \leq |H|$ by induction. In particular, the procedure Δ stops after $|H|$ core steps and returns the spanning tree $T = T_{h_{|H|-1}} = T_0$.

□

This concludes the proof of Theorem 6.3.

□

6.3 A bijection between subgraphs and orientations

In this section we define a bijection Φ between subgraphs and orientations. This task might not seem very challenging but we will prove in the next section that Φ has numerous interesting specializations. The bijection Φ is an extension of the correspondence $T \mapsto \mathcal{O}_T$ between spanning trees and orientations defined in Chapter 5 (Section 5.5). For instance, the image by Φ of the spanning tree T and the image of a subgraph S in $[T^-, T^+]$ are shown in Figure 127.

Definition 6.8 *Let $\mathcal{G}$ be an embedded graph. Let T be a spanning tree and let S be a subgraph in the tree-interval $[T^-, T^+]$. The orientation $\mathcal{O}_S = \Phi(S)$ is defined as follows. For any edge $e = \{h_1, h_2\}$ with $h_1 < h_2$ (for the $(\mathcal{G}, T)$-order), the arc $\mathcal{O}_S(e)$ is (h_1, h_2) if and only if -*

either e is in T and its fundamental cocycle contains no edge in the symmetric difference $S \vartriangle T$ - or if e is not in T and its fundamental cycle contains some edges in $S \vartriangle T$; the arc $\mathcal{O}_S(e)$ is (h_2, h_1) otherwise.

Recall that a subgraph S is in the tree-interval $[T^-, T^+]$ if and only if every edge in the symmetric difference $S \vartriangle T$ is $(\mathcal{G}, T)$-active. Let S be a subgraph in $[T^-, T^+]$ and let e be any edge of $\mathcal{G}$. We say that the arc $\mathcal{O}_S(e)$ is *reverse* if $\mathcal{O}_S(e) \neq \mathcal{O}_T(e)$. Observe that the arc $\mathcal{O}_S(e)$ is reverse if and only if the fundamental cycle or cocycle of e (with respect to the spanning tree T) contains an edge of $S \vartriangle T$ (compare for instance the orientations $\mathcal{O}_S$ and $\mathcal{O}_T$ in Figure 127). In particular, Definition 6.8 of the mapping Φ extends the definition (94) given for spanning trees in in Chapter 5.

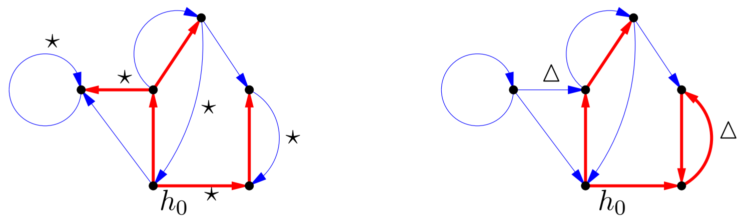


Figure 127: The orientations $\mathcal{O}_T$ and $\mathcal{O}_S$ associated with a spanning tree T and a subgraph S in $[T^-, T^+]$. The edges in the symmetric difference $S \vartriangle T$ are indicated by a $\vartriangle$.

The main result of this section is that the mapping Φ is a bijection between subgraphs and orientations. For instance, we have represented in Figure 128 the image by Φ of the subgraphs represented in Figure 122.

Theorem 6.9 *Let $\mathcal{G}$ be an embedded graph. The mapping Φ establishes a bijection between the subgraphs and the orientations of G.*

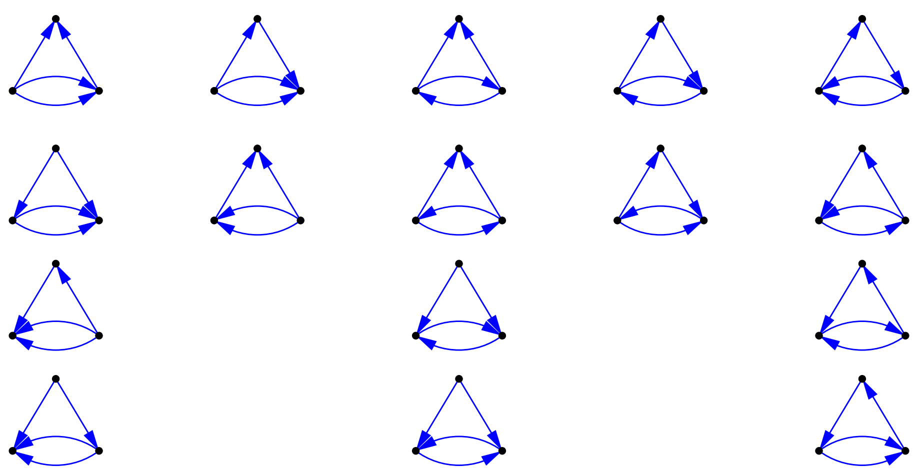


Figure 128: The image by Φ of the subgraphs in Figure 122.

In order to prove Theorem 6.9, we define a mapping Ψ from orientations to subgraphs. We shall prove that Ψ is the inverse of Φ.

Definition 6.10 *Let $\mathcal{G}$ be an embedded graph and let $\mathcal{O}$ be an orientation. We define the subgraph $S = \Psi(\mathcal{O})$ by the procedure described below. The procedure Ψ visits the half-edges in sequential order. The set of visited edges is denoted by F. If C is a set of edges that intersects the set F of visited edges, we denote by $e_{first}(C)$ and $h_{first}(C)$ the first visited edge and half-edge of C respectively ($e_{first}(C)$ contains $h_{first}(C)$). In this case, C is said to be* tail-first *if $h_{first}(C)$ is a tail and* head-first *otherwise.*

Initialization: *Initialize the* current half-edge *h to be the root h_0. Initialize the subgraph S, the tree T and the set of visited edges F to be empty.*

Core: *Do:*

C1: *If the edge e containing h is not in F, then decide whether to add e to S and T:*

- *If h is a tail, then*
  (a) If e is in a directed cycle $C \subseteq \overline{F}$, then add e to S but not to T.
  (b) If e is in a head-first directed cocycle $D \nsubseteq \overline{F}$ such that for all directed cocycle D' with $e_{first}(D') = e_{first}(D)$ either $e \in D'$ or ($D \vartriangle D' \nsubseteq \overline{F}$ and $e_{first}(D \vartriangle D') \in D'$), then do not add e to S nor to T.
  (c) Else, add e to S and to T.

- *If h is a head, then*
  (a′) If e is in a directed cocycle $D \subseteq \overline{F}$, then add e to T but not to S.
  (b′) If e is in a tail-first directed cycle $C \nsubseteq \overline{F}$ such that for all directed cycle C' with $e_{first}(C') = e_{first}(C)$ either $e \in C'$ or ($C \vartriangle C' \nsubseteq \overline{F}$ and $e_{first}(C \vartriangle C') \in C'$), then add e to S and to T.
  (c′) Else, do not add e to S nor to T.

Add e to F.

C2: *Move to the next half-edge around T:*

If e is in T, then set the current half-edge h to be $\sigma\alpha(h)$, else set it to be $\sigma(h)$.

Repeat until the current half-edge h is h_0.

End: *Return the subgraph S.*

In the procedure Ψ the conditions (a) and (b) (resp. (a') and (b')) are incompatible. Indeed the following lemma is a classical result of graph theory [Mint 66].

Lemma 6.11 *[Mint 66] Every arc (of an oriented graph) is either in a directed cycle or a directed cocycle but not both.*

Proof: (Hint) is the origin of the arc reachable from its end? □

We are now going to prove that Φ and Ψ are inverse mappings.

Proposition 6.12 *Let $\mathcal{G}$ be an embedded graph and let S be a subgraph. The mapping Ψ is well defined on the orientation $\Phi(S)$ (the procedure terminates) and $\Psi \circ \Phi(S) = S$.*

Proposition 6.12 implies that the mapping Φ is injective. Since there are as many subgraphs and orientations ($2^{|E|}$), it implies that Φ is bijective and that Ψ and Φ are reverse mappings. The rest of this section is devoted to the proof of proposition 6.12. Observe that Ψ is a variation on the procedure **Construct-tree** presented in Chapter 5 (Section 5.5). The differences lie in the extra Conditions (a), (b), (a'), (b') which are now needed in order to cope with reverse edges. In Lemmas 6.13 to 6.17 we express some properties characterizing reverse edges.

We first need some definitions. Let $\mathcal{G}$ be an embedded graph and $\mathcal{O}$ be an orientation. Suppose that the edges and half-edges of $\mathcal{G}$ are linearly ordered. For any set of edges C, we denote by $e_{\min}(C)$ and $h_{\min}(C)$ the minimal edge and half-edge of C respectively. We say that C is *tail-min* if $h_{\min}(C)$ is a tail and *head-min* otherwise. A directed cycle (resp. cocycle) is *tight* if any directed cycle (resp. cocycle) $C' \neq C$ with $e_{\min}(C') = e_{\min}(C)$ satisfies $e_{\min}(C \bigtriangleup C') \in C'$. For instance, if the edges of the graph in Figure 129 are ordered by $a < b < c < d < e < f < g$, the directed cycles (a,h,g,f,e,c) and (b,g,f,e,c) are tight whereas (a,h,g,d,c) is not.

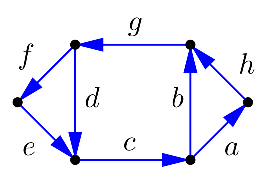


Figure 129: The directed cycles (a,h,g,f,e,c) and (b,g,f,e,c) are tight whereas (a,h,g,d,c) is not.

In Lemmas 6.13 to 6.17 we consider an embedded graph $\mathcal{G}$, a spanning tree T and a subgraph S in the tree-interval $[T^-, T^+]$. We consider the orientation $\mathcal{O}_S = \Phi(S)$ and compare edges and half-edges according to the $(\mathcal{G}, T)$-order.

Lemma 6.13 *The fundamental cycle (resp. cocycle) of any edge in $S \cap \overline{T}$ (resp. $\overline{S} \cap T$) is $\mathcal{O}_S$-directed and tail-min (resp. head-min).*

Proof: If e is in $S \cap \overline{T}$ (resp. $\overline{S} \cap T$), then every edge e' in its fundamental cycle (resp. cocycle) C is reverse ($\mathcal{O}_S(e') \neq \mathcal{O}_T(e')$). By Lemma 5.12, the cycle (resp. cocycle) C is $\mathcal{O}_T$-directed, hence it is $\mathcal{O}_S$-directed. Since e is $(\mathcal{G}, T)$-active, the minimal edge $e_{\min}(C)$ is e.

Hence, $h_{\min}(C)$ is the least half-edge of e. By definition of $\mathcal{O}_S$, the least half-edge of $\mathcal{O}_S(e)$ is a tail (resp. head). Hence, C is tail-min (resp. head-min).

□

Lemma 6.14 *Let e be a reverse edge ($\mathcal{O}_S(e) \neq \mathcal{O}_T(e)$). Then, e is in S if an only if it is in a directed cycle (otherwise it is in a directed cocycle by Lemma 6.11).*

Proof:
• Suppose that e is in S. We want to prove that e is in a directed cycle. If e is in $S \cap \overline{T}$, its fundamental cycle is directed by Lemma 6.13. If e is in $S \cap T$ there is an edge $e' \in S \cap \overline{T}$ in its fundamental cocycle (since e is reverse). Therefore, e is in the fundamental cycle of e' which is directed by Lemma 6.13.
• A similar argument proves that if e is in $\overline{S}$, then it is in a directed cocycle. In this case, e is not in a directed cycle by Lemma 6.11.

□

We now need to recall a classical result of graph theory (which is closely related to the axioms of oriented matroid theory [Björ 93]).

Lemma 6.15 (Orthogonality) *Let D be a cocycle and let V_1 and V_2 be the connected components after deletion of D. If a directed cycle C contains an arc oriented from V_1 to V_2 then it also contains an arc oriented from V_2 to V_1.*

Lemma 6.15 is illustrated by Figure 130.

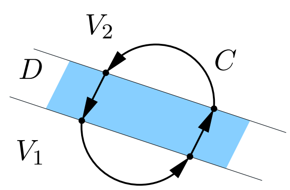


Figure 130: A directed cycle crossing a cocycle.

Lemma 6.16 *An edge e is in $S \cap \overline{T}$ (resp. $\overline{S} \cap T$) if and only if it is minimal in a tail-min (resp. head-min) directed cycle (resp. cocycle).*

Proof: We only prove that if an edge is minimal in a tail-min directed cycle then it is in $\in S \cap \overline{T}$. The reverse implication is given by Lemma 6.13. The proof of the dual equivalence (e is minimal in a tail-min directed cycle if and only if e is in $\overline{S} \cap T$) is similar.
Let $e = \{h_1, h_2\}$ with $h_1 < h_2$ be a minimal edge in a tail-min directed cycle C. We want to prove that e is in $S \cap \overline{T}$. Observe first that $\mathcal{O}_S(e) = (h_1, h_2)$ (since $h_{\min}(C) = h_1$ and C is tail-min). We now prove successively the following points.
- *The edge e is not in $\overline{S} \cap T$.* Otherwise, the edge e would be both in a directed cycle C and

in a directed cocycle by Lemma 6.13.

- *The edge e is not in $S \cap T$.* Suppose the contrary. Since e is in T, the arc $\mathcal{O}_S(e) = (h_1, h_2) = \mathcal{O}_T(e)$ is not reverse. Let D be the fundamental cocycle of e. Let v_1 and v_2 be the endpoints of h_1 and h_2 respectively and let V_2 be set of descendants of v_2. Recall that v_1 is the father of v_2 in T (Lemma 5.6) and that D is the cocycle defined by V_2. Since the cycle C is directed and the arc $\mathcal{O}_S(e)$ in $C \cap D$ is directed toward V_2, there is an edge e' in $C \cap D$ with $\mathcal{O}_S(e')$ directed away from V_2 by Lemma 6.15. This situation is represented in Figure 131. Since e is minimal in the cycle C, we have $e < e'$. Therefore, the arc $\mathcal{O}_T(e')$ is directed toward V_2 by Lemma 5.7. Thus, e' is reverse. The edge e' is reverse and contained in a directed cycle, therefore it is in S by Lemma 6.14. We have shown that e' is in $S \cap \overline{T}$. But this is impossible since $e < e'$ is in the fundamental cycle of e'.

- *The edge e is in $S \cap \overline{T}$.* We know from the preceding points that e is in $\overline{T}$. Hence, $\mathcal{O}_T(e) = (h_2, h_1) \neq \mathcal{O}_S(e)$. Thus, e is reverse in a directed cycle. Therefore, e is in S by Lemma 6.14.

$\square$

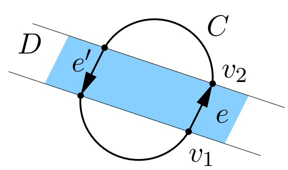


Figure 131: The directed cycle C, the fundamental cocycle D and the edges e and e'.

Lemma 6.17 *The fundamental cycle (resp. cocycle) of any edge in $S \cap \overline{T}$ (resp. $\overline{S} \cap T$) is tight.*

Proof: We prove that the fundamental cycle of an edge in $S \cap \overline{T}$ is tight. The proof of the dual property (concerning edges in $\overline{S} \cap T$) is similar. Let e^* be in $S \cap \overline{T}$. Recall that $e^* = e_{\min}(C)$. By Lemma 6.13, the fundamental cycle C of e^* is directed. We want to prove that C is tight. Suppose not and consider a directed cycle C' with $e_{\min}(C') = e_{\min}(C) = e^*$ and $e = e_{\min}(C \vartriangle C') \in C$. The edge e is in the fundamental cycle C of e^*, hence e^* is in fundamental cocycle D of e. This situation is represented in Figure 132. Let v_1 and v_2 be the endpoints of e with v_1 father of v_2 in T. Let V_2 be the set of descendants of v_2. Recall that D is the cocycle defined by V_2. The edge e is in the fundamental cycle of e^* which is $(\mathcal{G}, T)$ active, hence $e^* < e$. Therefore, the arc $\mathcal{O}_T(e^*)$ is directed away from V_2 by Lemma 5.7. Since e^* is in $S \cap \overline{T}$, the arc $\mathcal{O}_S(e^*)$ is reverse, hence is directed toward V_2. Since the cycle C' is directed and the arc $\mathcal{O}(e^*)$ in $C' \cap D$ is directed toward V_2, there is an arc $\mathcal{O}_S(e')$ in $C' \cap D$ oriented away from V_2 by Lemma 6.15. Observe that e' is not in the fundamental cycle C since $C \subseteq T + e^*$ and $D \subseteq \overline{T} + e$. Thus, e' is in $C \vartriangle C'$ and $e' > e$. Hence, by Lemma 5.7, the arc $\mathcal{O}_T(e')$ in the fundamental cocycle D of e is directed toward V_2. Thus, the arc

$\mathcal{O}_S(e') \neq \mathcal{O}_T(e')$ is reverse. Since e' is reverse and contained in a directed cycle, it is in S by Lemma 6.14. We have shown that e' is in $S \cap \overline{T}$. But this is impossible. Indeed e' is not $(\mathcal{G}, T)$-active since its fundamental cycle contains e which is less than e'.

□

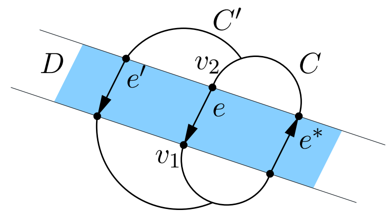


Figure 132: The directed cycles C and C' and the cocycle D.

Proof of Proposition 6.12. We consider a subgraph S_0 in the tree-interval $[T_0^-, T_0^+]$ and the orientation $\mathcal{O}_{S_0} = \Phi(S_0)$. We want to prove that the procedure Ψ returns the subgraph S_0. We compare edges and half-edges according to the $(\mathcal{G}, T_0)$-order denoted by $<$: we say that an edge or half-edge is *greater* or *less* than another. We also compare edges and half-edges according to their *order of visit* during the algorithm: we say that an edge or half-edge is *before* or *after* another. We denote by t the motion function associated with T_0. We denote by $h_i = t^i(h_0)$ the i^{th} half-edge for the $(\mathcal{G}, T_0)$-order. Also, for every half-edge h, we denote $F_h = \{e = \{h_1, h_2\}$ such that $\min(h_1, h_2) < h\}$, $T_h = T_0 \cap F_h$ and $S_h = S_0 \cap F_h$.
We want to prove that *at the beginning of the i^{th} core step, $h = h_i$, $F = F_h$, $T = T_h$, $S = S_h$,* where h is the current half-edge. We proceed by induction on the number of core steps. The property holds for the first ($i = 0$) core step since $h = h_0$ and $F_{h_0} = T_{h_0} = S_{h_0} = \emptyset$. Suppose the property holds for all $i \leq k$. By the induction hypothesis, the $(\mathcal{G}, T_0)$-order and the order of visit coincide on the edges and half-edges of F. In particular, if C is any set not contained in $\overline{F}$, then $h_{\min}(C) = h_{\text{first}}(C)$ and $e_{\min}(C) = e_{\text{first}}(C)$. Suppose the edge e containing the current half-edge h is not in $F = F_h$. In this case, the current half-edge h (resp. edge e) is less than any other half-edge (resp. edge) in $\overline{F}$. We consider the different cases $(a), (b), (c), (a'), (b'), (c')$. We will prove successively the following properties.

- *Condition (a) is equivalent to $e \in S_0 \cap \overline{T_0}$.*
  - Suppose Condition (a) holds: h is a tail and e is in a directed cycle $C \subseteq \overline{F}$. Since, $C \subseteq \overline{F}$, the current half-edge h is minimal in C. Since h is a tail, the directed cycle C is tail-min. Thus, e is in $S_0 \cap \overline{T_0}$ by Lemma 6.16.
  - Conversely, if e is in $S_0 \cap \overline{T_0}$, then e is minimal in a tail-min directed cycle C by Lemma 6.16. Therefore, h is a tail and $C \subseteq \overline{F}$.

- *Condition (a') is equivalent to $e \in \overline{S_0} \cap T_0$.*
  The proof is the similar to the proof of the preceding point.

- *Condition* (b) *is equivalent to* $e \in \overline{S_0} \cap \overline{T_0}$ *and* $\mathcal{O}_{S_0}(e)$ *is reverse.*
  - Suppose Condition (b) holds: h is a tail and e is in a head-first directed cocycle $D \nsubseteq \overline{F}$ such that for all directed cocycle D' with $e_{\text{first}}(D') = e_{\text{first}}(D)$ either $e \in D'$ or $D \vartriangle D' \nsubseteq \overline{F}$ and $e_{\text{first}}(D \vartriangle D') \in D'$. Since the $(\mathcal{G}, T_0)$-order and the order of visit coincide on F we have $h_{\min}(D) = h_{\text{first}}(D)$. Since the cocycle D is head-first, it is tail-min. The edge $e^* := e_{\min}(D)$ is minimal in a head-min directed cocycle, hence e^* is in $\overline{S_0} \cap T_0$ by Lemma 6.16. Let D^* be the fundamental cocycle of e^*. Recall that $e_{\min}(D^*) = e^* = e_{\min}(D)$ We want to prove that e is in D^*. Suppose e is not in D^*. By Condition (b), we have $D \vartriangle D^* \nsubseteq \overline{F}$ and $e_{\text{first}}(D \vartriangle D^*) \in D^*$. But this is impossible since $e_{\min}(D \vartriangle D^*) = e_{\text{first}}(D \vartriangle D^*)$ and D^* is tight by Lemma 6.17. Thus, e is indeed in the fundamental cocycle D^* of e^*. Since e^* is in $\overline{S_0} \cap T_0$, the edge e is in $\overline{T_0}$ and also in $\overline{S_0}$ by Lemma 6.1. Moreover the arc $\mathcal{O}_{S_0}(e)$ is reverse.
  - Conversely, suppose that e is in $\overline{S_0} \cap \overline{T_0}$ and that the arc $\mathcal{O}_{S_0}(e)$ is reverse. The current half-edge h is the least half-edge of e. Since e is external, h is the head of the arc $\mathcal{O}_{T_0}(e)$ and the tail of the reverse arc $\mathcal{O}_{S_0}(e)$. Since $\mathcal{O}_{S_0}(e)$ is reverse, the external edge e is in the fundamental cocycle D of an edge $e^* \in \overline{S_0} \cap T_0$. The cocycle D is head-min, directed and tight by Lemmas 6.13 and 6.17. Since $e^* = e_{\min}(D)$, the edge e^* is less than e. Therefore e^* is before e and $D \nsubseteq \overline{F}$. The cocycle D is head-first since $h_{\text{first}}(D) = h_{\min}(D)$. Consider any directed cocycle D' such that $e_{\text{first}}(D') = e_{\text{first}}(D) = e^*$ and $e \notin D'$. We want to prove that $D \vartriangle D' \nsubseteq \overline{F}$ and $e_{\text{first}}(D \vartriangle D') \in D'$. Since D is tight, the edge $e' = e_{\min}(D \vartriangle D')$ is in D'. Since e is in $D \vartriangle D'$, the edge e' is less than e, hence it is in F. Therefore, $D \vartriangle D' \nsubseteq \overline{F}$ and $e_{\text{first}}(D \vartriangle D') = e_{\min}(D \vartriangle D') = e'$ is in D'.

- *Condition* (b') *is equivalent to* $e \in S_0 \cap T_0$ *and* $\mathcal{O}_{S_0}(e)$ *is reverse.*
  The proof is the similar to the proof of the preceding point.

- *Condition* (c) *is equivalent to* $e \in S_0 \cap T_0$ *and is not reverse.*
  - Suppose Condition (c) holds. In this case, Conditions (a), (a'), (b), (b') do not hold. Hence (by the preceding points), the edge e is not in $S_0 \vartriangle T_0$ and the arc $\mathcal{O}_{S_0}(e)$ is not reverse. Since $\mathcal{O}_{S_0}(e)$ is not reverse and the half-edge h (which is the least half-edge of e) is a tail, the edge e is in T_0. Since e is not in $S_0 \vartriangle T_0$, it is in S_0.
  - Conversely, suppose that e is in $S_0 \cap T_0$ and that $\mathcal{O}_{S_0}(e)$ is not reverse. By the preceding points, none of the conditions (a), (a'), (b), (b') holds. Moreover, the half-edge h (which is the least half-edge of e) is a tail.

- *Condition* (c') *is equivalent to* $e \in \overline{S_0} \cap \overline{T_0}$ *and is not reverse.*
  The proof is the similar to the proof of the preceding point.

By the preceding points, e is added to S (resp. T) in the procedure Ψ if and only if e is in S_0 (resp. T_0). Hence, the next half-edge will be $t(h) = \sigma\alpha(h)$ if h is in T_0 and $\sigma(h)$ otherwise. Thus, all the properties are satisfied at the beginning of the $(k+1)^{th}$ core step. $\square$

This concludes the proof of Theorem 6.9. We have also proved the following property that will be useful in the next chapter.

Lemma 6.18 *During the execution of the procedure Ψ on an orientation $\mathcal{O}$, the half-edges are visited in $(\mathcal{G}, T)$-order, where T is the spanning tree $\Delta \circ \Psi(\mathcal{O})$.*

6.4 Concluding remarks

6.4.1 The planar case and duality

In this subsection we restrict our attention to planar graphs. Our goal is to highlight some nice properties of our bijections with respect to duality. Therefore we will handle simultaneously a planar embedding and its *dual*. In order to avoid confusion we shall indicate the implicit embedding $\mathcal{G}$ for the tree-intervals and the mapping Φ by writing $[T^-, T^+]_{\mathcal{G}}$ and $\Phi_{\mathcal{G}}$.

Let $G = (V, E)$ be a planar graph. The graph G can be drawn *properly* on the sphere, that is, in such a way the edges only intersect at their endpoints. If the graph G is properly drawn on the sphere, the dual graph G^* is obtained by putting a vertex in each face of G and an edge across each edge of G. A drawing of a graph on the oriented sphere defines an embedding $\mathcal{G} = (H, \sigma, \alpha)$ where the permutation σ corresponds to the counterclockwise order around each vertex. Proper drawings on the sphere correspond to planar embedding, that is, embeddings having *Euler characteristic* 0, where the Euler characteristic is the number of vertices (cycles of σ) plus the number of *faces* (cycles of $\sigma\alpha$) minus the number of *edges* (cycles of α) minus 2. If $\mathcal{G} = (H, \sigma, \alpha, h_0)$ is a planar embedding of G, then $\mathcal{G}^* = (H, \sigma\alpha, \alpha, h_0)$ is a planar embedding of G^* (observe that $\mathcal{G}^{**} = \mathcal{G}$).

Consider a planar embedding $\mathcal{G}$. Observe that the edges, subgraphs and orientations of $\mathcal{G}$ can also be considered as edges, subgraphs and orientations of $\mathcal{G}^*$. Given a subgraph S of $\mathcal{G}$ we denote by $\overline{S}^*$ the *co-subgraph*, that is, the complement of S considered as a subgraph of $\mathcal{G}^*$. Given an orientation $\mathcal{O}$ of $\mathcal{G}$ we denote by $\overline{\mathcal{O}}^*$ the *co-orientation*, that is, the orientation obtained from $\mathcal{O}$ by reversing all arcs considered as an orientation of $\mathcal{G}^*$. Observe that for any subgraph S and any orientation $\mathcal{O}$, we have $\overline{\overline{S}^*}^* = S$ and $\overline{\overline{\mathcal{O}}^*}^* = \mathcal{O}$. From the Jordan Lemma, a subgraph S is connected if and only if the co-subgraph $\overline{S}^*$ is acyclic. This implies the well known property (see [Mull 67]) that a subgraph T is a spanning tree of $\mathcal{G}$ if and only if the co-subgraph $\overline{T}^*$ is a spanning tree of $\mathcal{G}^*$. It follows that the fundamental cycle (resp. cocycle) of an internal (resp. external) edge e with respect to $\mathcal{G}$ and T is the fundamental cocycle (resp. cycle) of e with respect to $\mathcal{G}^*$ and $\overline{T}^*$. Moreover, it follows directly from the definitions that the motion function of the spanning tree T of $\mathcal{G}$ and the motion function of the spanning tree $\overline{T}^*$ of $\mathcal{G}^*$ are equal. In particular, the $(\mathcal{G}, T)$-order and the $(\mathcal{G}^*, \overline{T}^*)$-order are the same. Hence, an edge is $(\mathcal{G}, T)$-active if and only if it is $(\mathcal{G}^*, \overline{T}^*)$-active. Thus, the mapping $S \mapsto \overline{S}^*$ induces a bijection between the tree-intervals $[T^-, T^+]_{\mathcal{G}}$ and $[\overline{T}^{*-}, \overline{T}^{*+}]_{\mathcal{G}^*}$. It follows directly from this property and the definitions that the mappings $\Phi_{\mathcal{G}}$ and $\Phi_{\mathcal{G}^*}$ are related by the relation

$$\text{for any subgraph } S \text{ of } \mathcal{G}, \quad \overline{\Phi_{\mathcal{G}}(S)}^* = \Phi_{\mathcal{G}^*}(\overline{S}^*).$$

6.4.2 An alternative algorithmic description of the mappings Φ and Ψ in the planar case.

In this subsection, we give an alternative algorithmic description of the bijections Φ and Ψ between subgraphs and orientations in the case of planar embeddings. More precisely, we give two algorithms φ and ψ performing the bijection Φ and Ψ respectively in a linear time.

During the algorithms φ and ψ it is necessary to keep track of the number of times each half-edge has been visited. This is done by initializing a function f (an array) to be 0 on every half-edge h and increment the value $f(h)$ each time the half-edge h is visited.

Definition 6.19 *Let $\mathcal{G} = (H, \sigma, \alpha, h_0)$ be an embedded graph. Given a subgraph S the procedure φ returns the orientation $\mathcal{O}$ defined by the following procedure.*
Initialization: *Initialize the* current half-edge *h to be the root h_0. Initialize the function f to be 0 on every half-edge. Initialize the subgraph T to be S.*
Core: *Do:*
C1: *Let e be the edge containing the current half-edge h.*

- *Increment $f(h)$.*
- *If $f(h) = 1$ and $f(\alpha(h)) = 0$, then orient the edge e:*
  If e is in S, then set $\mathcal{O}(e) = (h, \alpha(h))$ else set $\mathcal{O}(e) = (\alpha(h), h)$.
- *If $f(h) = 2$ and $f(\alpha(h)) = 0$ then*
  If e is in T then remove it from T else add it to T.

C2: *Move to the next half-edge:*

- *If $f(h) = 1$ and e is in $S \triangle T$ then set h to be $\alpha(h)$.*
- *If e is in T, then set the current half-edge h to be $\sigma\alpha(h)$, else set it to be $\sigma(h)$.*

Repeat until $h = h_0$ and $f(h_0) = 2$.
End: *Return the orientation $\mathcal{O}$.*

Definition 6.20 *Let $\mathcal{G} = (H, \sigma, \alpha, h_0)$ be an embedded graph. Given an orientation $\mathcal{O}$ the procedure ψ returns the subgraph S defined by the following procedure.*
Initialization: *Initialize the* current half-edge *h to be the root h_0. Initialize the function f to be 0 on every half-edge. Initialize the subgraphs S and T to be empty.*

Core: *Do:*

C1: *Let e be the edge containing the current half-edge h.*

- *Increment $f(h)$.*
- *If $f(h) = 1$ and $f(\alpha(h)) = 0$, then*

  If h is a tail, then add it to S and T.
- *If $f(h) = 2$ and $f(\alpha(h)) = 0$ then*

  If e is in T then remove it from T else add it to T.

C2: *Move to the next half-edge:*

- *If $f(h) = 1$ and e is in $S \triangle T$ then set h to be $\alpha(h)$.*
- *If e is in T, then set the current half-edge h to be $\sigma\alpha(h)$, else set it to be $\sigma(h)$.*

Repeat until $h = h_0$ and $f(h_0) = 2$.

End: *Return the subgraph S.*

An execution of the procedure φ (resp. ψ) is represented in Figure 133 (resp. 134).

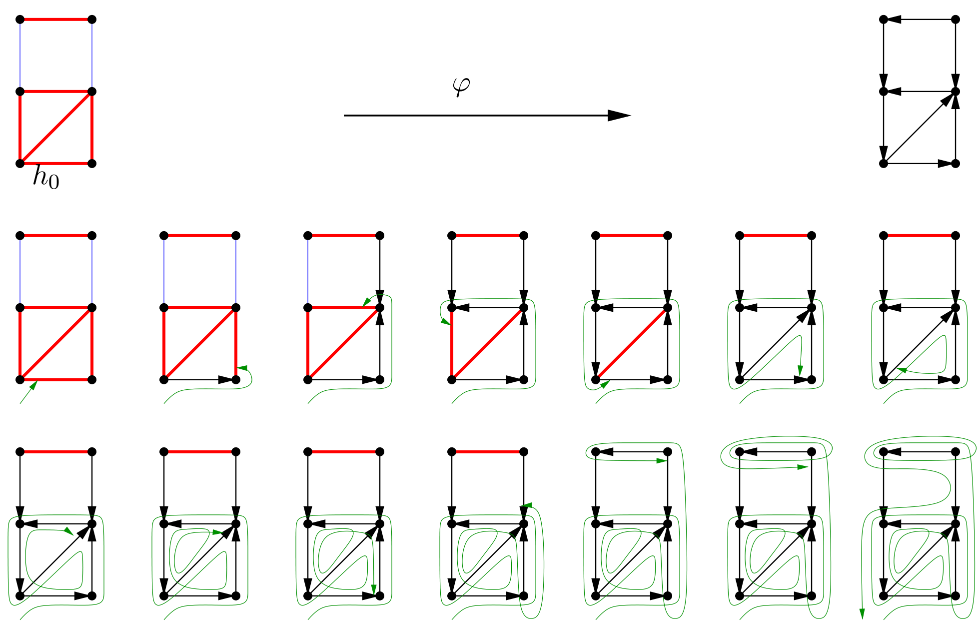


Figure 133: The mapping φ and some intermediate steps.

Theorem 6.21 *Let $\mathcal{G} = (H, \sigma, \alpha, h_0)$ be an embedded graph and S be a subgraph. If $\mathcal{G}$ is planar or S is a forest, then the procedure φ terminates and returns the orientation $\Phi(S)$. In these cases, the number of core steps of the procedure φ is $2|H|$, hence this procedure performs in a time linear in the number of edges of $\mathcal{G}$.*

Theorem 6.22 *Let $\mathcal{G} = (H, \sigma, \alpha, h_0)$ be an embedded graph and $\mathcal{O}$ be an orientation. If $\mathcal{G}$ is planar or $\Psi(\mathcal{O})$ is a forest, then the procedure ψ terminates and returns the subgraph $\Psi(\mathcal{O})$. In*

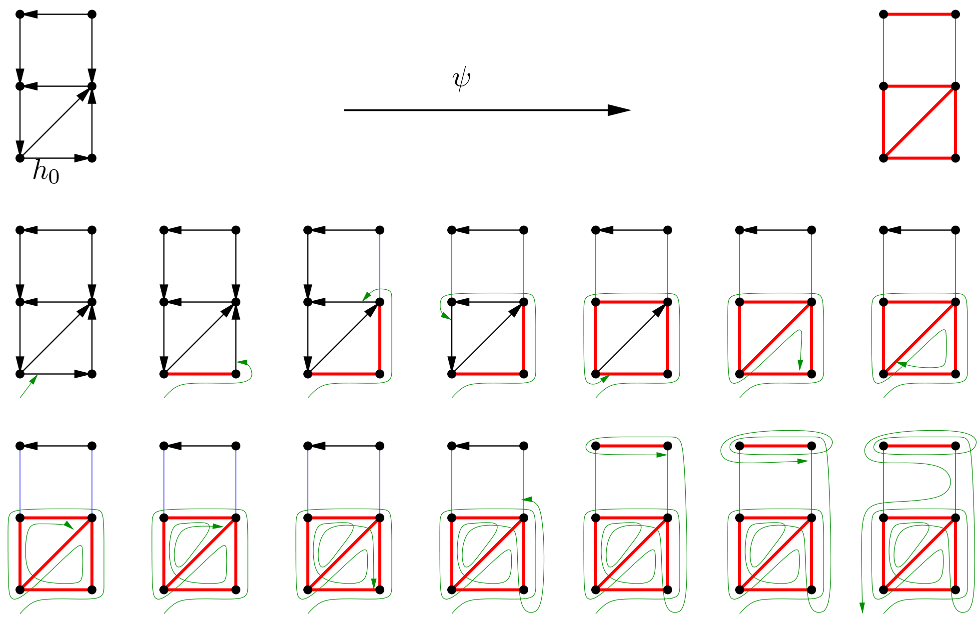


Figure 134: The mapping ψ and some intermediate steps.

these cases, the number of core steps of the procedure ψ is $2|H|$, hence this procedure performs in a time linear in the number of edges of $\mathcal{G}$.

We do not give the proofs of Theorems 6.21 and 6.22.

Chapter 7

Specializations of the bijection between subgraphs and orientations

Abstract: We study several restrictions of the bijection Φ between subgraphs and orientations. For instance, we prove that the restriction of Φ to connected subgraphs induces a bijection between connected subgraphs and root-connected orientations. Since the connected subgraphs are counted by the evaluation $T_G(1,2)$ of the Tutte polynomial, we obtain an interpretation of this evaluation in terms of orientations. We will, in fact, give an interpretation for each of the evaluations $T_G(i,j), 0 \leq i,j \leq 2$ of the Tutte polynomial in terms of orientations. The strength of our approach is to derive all our results from the same bijection Φ specialized in various ways. Some of the results are expressed in term of outdegree sequences. For instance, we obtain a bijection between forests (counted by $T_G(2,1)$) and outdegree sequences, and also a bijection between spanning trees (counted by $T_G(1,1)$) and root-connected outdegree sequences.

Résumé : Nous étudions plusieurs restrictions de la bijection Φ entre les sous-graphes et les orientations. Par exemple, nous montrons que la restriction de Φ aux sous-graphes connexes induit une bijection entre les sous-graphes connexes et les orientations racine-accessibles. Puisque les sous-graphes connexes sont comptés par l'évaluation $T_G(1,2)$ du polynôme de Tutte, nous obtenons une interprétation de cette évaluation en termes d'orientations. Nous allons, en fait, donner une interprétation pour chacune des évaluations $T_G(i,j), 0 \leq i,j \leq 2$ du polynôme de Tutte en termes d'orientations. La force de notre approche est de déduire tous nos résultats d'une unique bijection que nous spécialisons de diverses manières. Certains résultats sont exprimés en termes de suites de degrés. Par exemple, nous obtenons une bijection entre les forets (comptées par $T_G(2,1)$) et les suites de degrés et aussi une bijection entre les arbres couvrants (comptés par $T_G(1,1)$) et les suites de degrés racine-accessibles.

7.1 Introduction

The Tutte polynomial of a connected graph $G = (V, E)$ can be defined by its subgraph expansion

$$T_G(x, y) = \sum_{S \text{ spanning subgraph}} (x-1)^{c(S)-1}(y-1)^{c(S)+|S|-|V|},$$

where the sum is over all subgraphs S (equivalently, subsets of edges), $c(S)$ denotes the number of connected components of S and $|.|$ denotes cardinality. From this definition, it is easy to see that $T_G(1, 1)$ (resp. $T_G(2, 1)$, $T_G(1, 2)$) counts the spanning trees (resp. forests, connected subgraphs) of G. A somewhat less interesting specialization is $T_G(2, 2) = 2^{|E|}$ counting the subgraphs of G. Note that this is also the number of orientations of G. As a matter of fact, all the specializations $T_G(i, j), 0 \leq i, j \leq 2$ as well as some of their refinements have nice interpretations in terms of orientations [Bryl 91, Gess 96, Gioa 06, Gree 83, Lass 01, Stan 73, Wind 66, Las 84b]. This makes it appealing to look for bijections between subgraphs and orientations that would allow us to prove these interpretations bijectively.

There is number of papers devoted to combinatorial proofs of the interpretations of the Tutte polynomial [Gebh 00, Gess 96, Gioa 06, Gioa 05, Lass 01]. In this chapter, we give new purely bijective proofs of the interpretations of $T_G(i, j), 0 \leq i, j \leq 2$ in terms of orientations. The strength of our approach is to derive all these interpretations from a single bijection Φ (defined in Chapter 6) between subgraphs and orientations that we specialize in various ways. For instance, we derive a bijection between connected subgraphs (counted by $T_G(1, 2)$) and root-connected orientations. We also derive a bijection between forests (counted by $T_G(2, 1)$) and outdegree sequences (this answers a question of Stanley [Stan 80a]). In particular, we derive a bijection between spanning trees (counted by $T_G(1, 1)$) and root-connected outdegree sequences. These sequences first enumerated in [Gioa 06] are in bijection with equivalence classes of orientations up to cycle and cocycle flips.

The outline of this chapter is as follows. In Section 7.2, we define several classes of subgraphs and prove that they are counted by evaluations of the Tutte polynomial. This counting properties are easily proved thanks to the characterization of the Tutte polynomial by embedding-activities established in Chapter 5. In Section 7.3, we study the restriction of Φ to connected and to *external* subgraphs. In Section 7.4, we study the restriction of Φ to forest and to *internal* subgraphs. The forest are seen to be in bijection with a class of orientation called *minimal*. In Section 7.5, the minimal orientations are proved to be in bijection with outdegree sequences. In Section 7.6, we summarize our results and explore some refinements. We conclude in Section 7.7 by some remarks about the cycle/cocycle reversing systems and some algorithmic applications.

7.2 Enumerative results for several classes of subgraphs

In this section we define several classes of subgraphs and obtain counting results for these classes by using the characterization (84) of the Tutte polynomial established in Chapter 5.

Let $\mathcal{G}$ be an embedded graph and let T be a spanning tree. Recall that the spanning tree T of $\mathcal{G}$ is said to be *internal* (resp. *external*) if it has no external (resp. internal) $(\mathcal{G}, T)$-active edge. In Figure 122, the first and second spanning trees (from left to right) are internal while the fourth and fifth are external. We say that a subgraph S in $[T^-, T^+]$ is *internal* or *external* if the spanning tree T is. The notion of *internal subgraph* is close to Whitney's notion of subgraphs *without broken circuit* [Whit 32b]. Observe that by Lemma 6.2 any internal subgraph is a forest and any external subgraph is connected (the converse is, of course, false). In Figure 135 we represented the subgraphs of figure 122 in each of the categories defined by the four criteria *forest*, *internal*, *connected*, *external*.

Proposition 7.1 *Let $\mathcal{G}$ be an embedded graph. The number of subgraphs in each category defined by the criteria* forest, internal, connected, external *is given by the following specialization of the Tutte polynomial:*

	General	*Connected*	*External*
General	$T_G(2,2) = 2^{\lvert E\rvert}$	$T_G(1,2)$	$T_G(0,2)$
Forest	$T_G(2,1)$	$T_G(1,1)$	$T_G(0,1)$
Internal	$T_G(2,0)$	$T_G(1,0)$	$T_G(0,0) = 0$

Proof: Let T be a spanning tree with $\mathcal{I}(T)$ internal and $\mathcal{E}(T)$ external $(\mathcal{G}, T)$-active edges. By Lemma 6.2, the connected subgraphs in $[T^-, T^+]$ are obtained by adding some external $(\mathcal{G}, T)$-active edges to T. Hence, there are $1^{\mathcal{I}(T)}2^{\mathcal{E}(T)}$ connected subgraphs in $[T^-, T^+]$. Thus, given the partition of the set of subgraphs into tree-intervals given by Theorem 6.3, the graph $\mathcal{G}$ has

$$\sum_{T \text{ spanning tree}} 1^{\mathcal{I}(T)}2^{\mathcal{E}(T)}$$

connected subgraphs. This sum is equal to $T_G(1,2)$ by the characterization (84) of the Tutte polynomial (that we established in Chapter 5). Observe that there are $0^{\mathcal{I}(T)}2^{\mathcal{E}(T)}$ external (connected) subgraphs in the interval $[T^-, T^+]$[1]. Hence there are $T_G(0,2)$ external subgraphs of G. Every other category admits a similar treatment.

$\square$

We will now study the restriction of the bijection Φ to each category of subgraphs.

[1]Here, as everywhere in this chapter, the convention is that $0^0 = 1$.

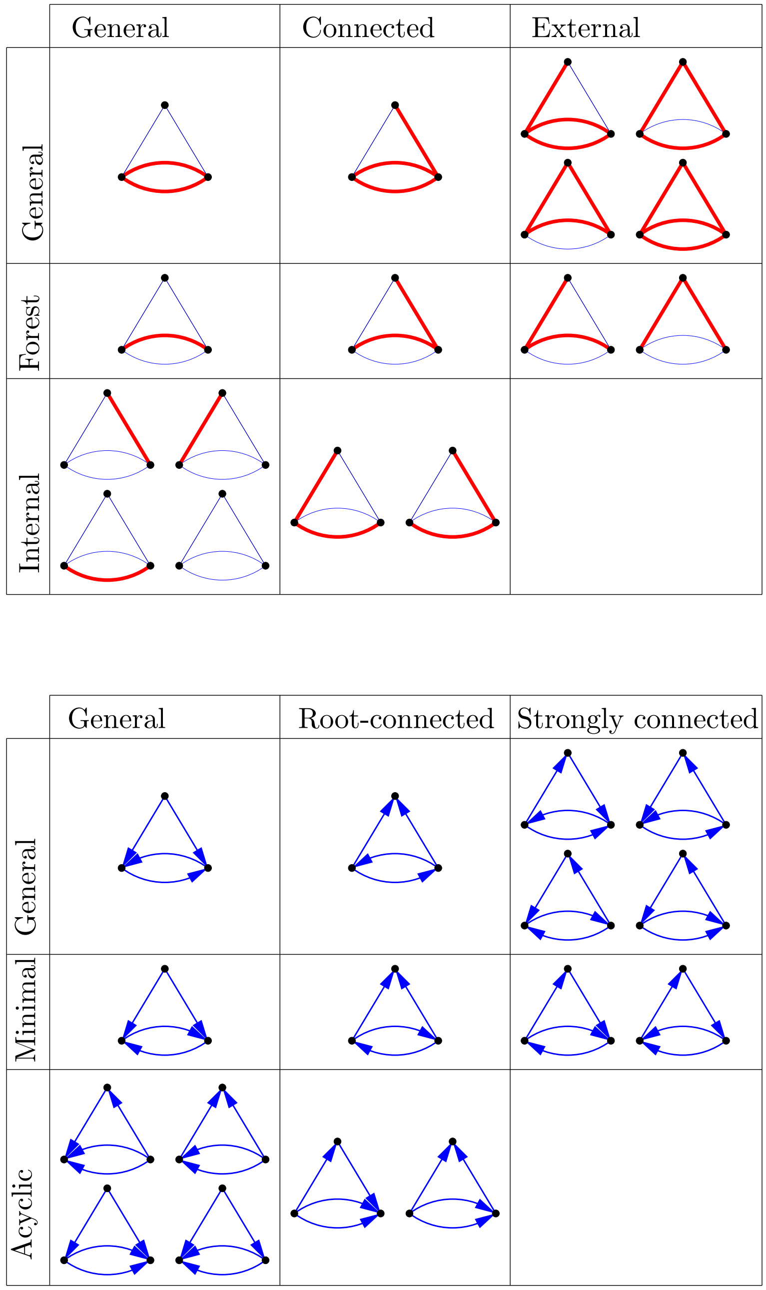


Figure 135: Subgraphs in each category defined by the four criteria *forest*, *internal*, *connected*, *external* and the corresponding orientations. The categories goes from the most general to the most constrained from left to right and from up to down. The non-connected subgraphs (resp. non-external connected subgraphs, external subgraphs) are in column 1 (resp. 2, 3). The subgraphs that are not forests (resp. the forests that are not internal, the internal forests) are in line 1 (resp. 2, 3).

7.3 Connected subgraphs and external subgraphs

In this section we study the restriction of Φ to connected and to external subgraphs.

Proposition 7.2 *Let $\mathcal{G}$ be an embedded graph and let v_0 be the root-vertex. The orientation $\mathcal{O}_S$ is v_0-connected if and only if the subgraph S is connected.*

Lemma 7.3 *Let $\mathcal{G}$ be an embedded graph and let T be a spanning tree. Let D be a cut and let $\mathcal{G}_0$ be the connected component of $\mathcal{G}$ containing the root-vertex v_0 after D is removed. Then, the half-edge $h_{\min}(D)$ is incident to $\mathcal{G}_0$. Moreover, every half-edge not in $\mathcal{G}_0$ is greater than or equal to $h_{\min}(D)$.*

Proof: Let t be the motion function of T. If a half-edge h is incident to $\mathcal{G}_0$ and is not in D then $t(h)$ is incident to $\mathcal{G}_0$. Since the root h_0 is incident to $\mathcal{G}_0$, the half-edge $h_{\min}(D)$ is also incident to $\mathcal{G}_0$ and is less than any half-edge not in $\mathcal{G}_0$. $\square$

Lemma 7.4 *An orientation is v_0-connected if and only if it has no head-min directed cocycle.*

Proof:
• If there is a head-min directed cocycle, this cocycle is directed toward the component containing v_0 by Lemma 7.3. Therefore, the vertices in the other components are not reachable from v_0 and the orientation is not v_0-connected.
• If the orientation is not v_0-connected we consider the cut D defined by the set V_0 of vertices reachable from v_0. The cut D is directed toward V_0, hence is head-min by Lemma 7.3. Let v_1 be the endpoint of the edge $e = e_{\min}(D)$ that is not in V_0. Let V_1 be the set of vertices in the connected component containing v_1 after the cut D is deleted. The set of edges D_1 with one endpoint in V_0 and one endpoint in V_1 is a cocycle contained in D. Since every edge in D_1 is directed away from V_0 the cocycle D_1 directed. Since $h_{\min}(D_1) = h_{\min}(D)$ is a head, the cocycle D_1 is head-min.

$\square$

Proof of Proposition 7.2. Let S be a subgraph in $[T^-, T^+]$. The orientation $\mathcal{O}_S$ is v_0-connected if and only if there is no head-min directed cocycle by Lemma 7.4. An edge is in $\overline{S} \cap T$ if and only if it is minimal in a head-min directed cocycle by Lemma 6.16. Thus, $\mathcal{O}_S$ is v_0-connected if and only if $\overline{S} \cap T = \emptyset$. And $\overline{S} \cap T = \emptyset$ if and only if S is connected by Lemma 6.2.

$\square$

We now study the restriction of the bijection Φ to external subgraphs.

Proposition 7.5 *Let $\mathcal{G}$ be an embedded graph and let S be a subgraph. The orientation $\mathcal{O}_S$ is strongly connected if and only if S is external.*

Lemma 7.6 *Let T be a spanning tree and let e be an edge of T. Let u and v be the endpoints of e with the convention that u is the father of v. For any* connected *subgraph S in $[T^-, T^+]$, the vertex v is $\mathcal{O}_s$-reachable from its father u.*

Proof: For any connected subgraph S in $[T^-, T^+]$, the set $\overline{S} \cap T$ is empty by Lemma 6.2. If the fundamental cocycle of the edge e contains no edge of $S \cap \overline{T}$, then the arc $\mathcal{O}_S(e)$ is not reverse. In this case, the arc $\mathcal{O}_S(e) = \mathcal{O}_T(e)$ is directed from u to v by Lemma 5.6. Suppose now that the fundamental cocycle of e contains an edge e^* of $S \cap \overline{T}$. In this case, e is in the fundamental cycle C^* of e^* which is $\mathcal{O}_S$-directed by Lemma 6.13. Therefore, the vertex v is $\mathcal{O}_s$-reachable from u (and vice-versa).

□

Lemma 7.7 *Let $\mathcal{G}$ be an embedded graph. Let T be a spanning tree and let S be a* connected *subgraph in $[T^-, T^+]$. An edge e is minimal in an $\mathcal{O}_S$-directed cocycle if and only if e is an internal $(\mathcal{G}, T)$-active edge.*

Proof: Since the subgraph S is connected, the subset $\overline{S} \cap T$ is empty by Lemma 6.2 and the orientation $\mathcal{O}_S$ is v_0-connected by Lemma 7.2.

• Suppose that the edge e is an internal $(\mathcal{G}, T)$-active edge. The edge e is minimal in its fundamental cocycle D. We want to prove that D is $\mathcal{O}_S$-directed. Note first that e is not in $S \bigtriangleup T$ (since e is in T and $\overline{S} \cap T = \emptyset$). No other edge of D is in $S \bigtriangleup T$ since none is $(\mathcal{G}, T)$-active. Hence, $\mathcal{O}_S(e) = \mathcal{O}_T(e)$. Let $e' \neq e$ be an edge in the fundamental cocycle D of e. The fundamental cycle of e' does not contain any edge of $\overline{S} \cap T$ since this edge is empty. Hence, $\mathcal{O}_S(e') = \mathcal{O}_T(e')$. Thus, the orientations $\mathcal{O}_S$ and $\mathcal{O}_T$ coincide on the cocycle D. By Lemma 5.12, the cocycle D is $\mathcal{O}_T$-directed, hence it is $\mathcal{O}_S$-directed.

• Suppose that $e = \{h_1, h_2\}$ with $h_1 < h_2$ is minimal in an $\mathcal{O}_S$-directed cocycle D. We want to prove that e is an internal $(\mathcal{G}, T)$-active edge. We prove successively the following properties:

- *The half-edge h_1 is a tail.* Otherwise, the cocycle D is head-min. (This is impossible by Lemma 7.4 since $\mathcal{O}_S$ is is v_0-connected.) - *The edge e is in T.* If e is not in T, then the arc $\mathcal{O}_S(e) = (h_1, h_2)$ is reverse. Thus, the fundamental cycle C of e contains an edge of $S \bigtriangleup T$. Since $C \subseteq T + e$ and $\overline{S} \cap T = \emptyset$, the edge e is in $S \cap \overline{T}$. Thus, the cycle C is $\mathcal{O}_S$-directed by Lemma 6.13. This is impossible since e cannot be both is a directed cycle and a directed cocycle.

- *The edge e is $(\mathcal{G}, T)$-active.* Since the edge e is in T, the arc $\mathcal{O}_S(e) = (h_1, h_2) = \mathcal{O}_T(e)$ is not reverse. Let v_1 and v_2 be the endpoints of h_1 and h_2 respectively. Let $\mathcal{G}_2$ be the connected component of $\mathcal{G}$ containing v_2 once the cocycle D is removed. The arc $\mathcal{O}_S(e)$ is directed toward v_2, thus the cocycle D is directed toward $\mathcal{G}_2$. By Lemma 7.6, all the descendants of v_2 are reachable from v_2, hence they are all in $\mathcal{G}_2$. Let e' be an edge in the fundamental cocycle D' of e. Since one of the endpoints of e' is a descendant of v_2, the edge e' is either in D or in $\mathcal{G}_2$. Since the minimal half-edge h_1 of D is not incident to $\mathcal{G}_2$, every edge in $D \cup \mathcal{G}_2$ is greater than or equal to e by Lemma 7.3. Thus, e' is greater than e. The edge e is minimal in its fundamental cocycle D, that is, e is $(\mathcal{G}, T)$-active.

□

Proof of Proposition 7.5. Let S be a subgraph in $[T^-, T^+]$.
• Suppose that the subgraph S is external. The subgraph S is connected and there is no $(\mathcal{G}, T)$-active edge, hence there is no $\mathcal{O}_S$-directed cocycle by Lemma 7.7. Thus, the orientation $\mathcal{O}_S$ is strongly connected.
• Suppose that the orientation $\mathcal{O}_S$ is strongly connected. The subgraph S is connected (since $\mathcal{O}_S$ is v_0-connected) and there is no $\mathcal{O}_S$-directed cocycle, hence there is no $(\mathcal{G}, T)$-active edge by Lemma 7.7. Thus, the subgraph S is external.

□

7.4 Forests and internal forests

In this section we study the restriction of the bijection Φ to forests and to internal subgraphs.

Let $\mathcal{G}$ be an embedded graph and let $\mathcal{O}$ be an orientation. We compare half-edges according to the $(\mathcal{G}, T)$-order, where $T = \Delta \circ \Psi(\mathcal{O})$. We say that the orientation $\mathcal{O}$ is *minimal* if there is no tail-min $\mathcal{O}$-directed cycle. We shall see (Lemma 7.11) that for any out degree sequence δ there is a unique minimal δ-orientation.

Proposition 7.8 *The orientation $\mathcal{O}_S$ is minimal if and only if the subgraph S is a forest.*

Proof: Let $T = \Delta(S)$. By Lemma 6.16, an edge is in $S \cap \overline{T}$ if and only if it is minimal in a tail-min directed cycle. Thus, the orientation $\mathcal{O}_S$ is minimal if and only if $S \cap \overline{T} = \emptyset$. And $S \cap \overline{T} = \emptyset$ if and only if S is a forest by Lemma 6.2.

□

Proposition 7.9 *The orientation $\mathcal{O}_S$ is acyclic if and only if the subgraph S is internal.*

In order to prove Proposition 7.9 we need to define a linear order, the *postfix order*, on the vertex set. For any vertex $v \neq v_0$ we denote by h_v the half-edge incident to v and contained in the edge linking v to its father in T. The *postfix order*, denoted by $<_{\text{post}}$, is defined by $v <_{\text{post}} v_0$ for $v \neq v_0$ and $v <_{\text{post}} v'$ if $h_v < h_{v'}$ for $v, v' \neq v_0$. The postfix order is illustrated in Figure 136.

Lemma 7.10 *Let T be a spanning tree and let e be an edge. The arc $\mathcal{O}_T(e)$ is directed toward its greatest endpoint (for the postfix order) if and only if the edge e is external $(\mathcal{G}, T)$-active.*

Lemma 7.10 is illustrated by Figure 136.

Proof: Recall from Lemma 5.7 that a half-edge h is incident to a descendant of v if and only if $h'_v < h \leq h_v$, where $h'_v = \alpha(h_v)$ is the other half of the edge containing h_v.

• Consider an internal edge e. Let u and v be the endpoints of e with u father of v. By Lemma 5.6, the arc $\mathcal{O}_T(e)$ is directed toward v. We want to prove that $v <_{\text{post}} u$. If $u = v_0$, the inequality holds. Else, the half-edges h_u and h_v exist. Moreover, the half-edge h_v is incident to a descendant of u, hence $h_v < h_u$ and $v <_{\text{post}} u$.

• Consider an external edge e. We write $e = \{h_1, h_2\}$ with $h_1 < h_2$ and denote by u and v the endpoints of h_1 and h_2 respectively. By definition, the arc $\mathcal{O}_T(e)$ is directed toward u. We want to prove that $v \leq_{\text{post}} u$ if and only if e is $(\mathcal{G}, T)$-active.

- Suppose the edge e is $(\mathcal{G}, T)$-active. Then, the vertex v is a descendant of u by Lemma 5.8. The half-edge h_v is incident to a descendant of u, hence $h_v \leq h_u$ and $v \leq_{\text{post}} u$.

- Suppose that $v \leq_{\text{post}} u$. If $u = v_0$, the vertex v is a descendant of u and the edge e is $(\mathcal{G}, T)$-active by Lemma 5.8. Else, the half-edges h_u and h_v exist and $h_v \leq h_u$. In this case, $\alpha(h_u) < h_1 < h_2 < h_v \leq h_u$ (indeed, $h_2 < h_v$ since h_2 is incident to v and $\alpha(h_u) < h_1$ since h_1 is incident to u), hence v is a descendant of u by Lemma 5.7. Thus, the edge e is $(\mathcal{G}, T)$-active by Lemma 5.8.

$\square$

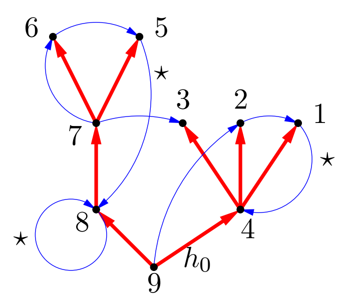


Figure 136: A spanning tree T, the postfix order, the orientation $\mathcal{O}_T$ and the external active edges (indicated by a $\star$).

Proof of Proposition 7.9. Let S be a subgraph in the tree-interval $[T^-, T^+]$. We compare half-edges according to the $(\mathcal{G}, T)$-order.

• Suppose that the subgraph S is internal (i.e. the tree T is internal). Recall that $S \cap \overline{T} = \emptyset$. We want to prove that the orientation $\mathcal{O}_S$ is acyclic. Observe first that the orientation $\mathcal{O}_T$ is acyclic since the vertices are strictly decreasing (for the postfix order) along any $\mathcal{O}_T$-directed path by Lemma 7.10. Suppose now that there is an $\mathcal{O}_S$-directed cycle C. The $\mathcal{O}_S$-directed cycle C contains a reverse arc $\mathcal{O}(e)$ or C would be $\mathcal{O}_T$-directed. Since $S \cap \overline{T} = \emptyset$, the reverse edges are in the fundamental cocycle of an edge of $\overline{S} \cap T$. Thus, the edge e is in the fundamental cocycle D of an edge of $\overline{S} \cap T$. The cocycle D is directed by Lemma 6.13. This is impossible since e cannot be both in a directed cycle and in a directed cocycle.

• Suppose that the orientation $\mathcal{O}_S$ is acyclic. We want to prove that the subgraph S is internal (i.e. the tree T is internal). Suppose there is an external $(\mathcal{G}, T)$-active edge e. Let C be the fundamental cycle of e. Since $\mathcal{O}_S$ is minimal, we know (by Proposition 7.8) that $S \cap \overline{T}$ is empty. Therefore, the reverse edges are in the fundamental cocycle of an edge of $\overline{S} \cap T$. Since e is active, it is not in the fundamental cocycle of an edge of $\overline{S} \cap T$. Since the other edges of C are not active (they are less than e) they are not in $\overline{S} \cap T$. Moreover, since they are in T, they are not in the fundamental cocycle of an edge of $\overline{S} \cap T$. Thus, the orientations $\mathcal{O}_S$ and $\mathcal{O}_T$ coincide on the cycle C. By Lemma 5.12, the cycle C is $\mathcal{O}_T$-directed, hence it is $\mathcal{O}_S$-directed. This is impossible since $\mathcal{O}_S$ is acyclic.

□

7.5 Minimal orientations and out-degree sequences

In the previous section we proved that the bijection Φ induces a bijection between forests and minimal orientations (Proposition 7.8). We are now going to link minimal orientations and out-degree sequences.

Proposition 7.11 *Let $\mathcal{G}$ be an embedded graph. For any outdegree sequence δ there exists a unique minimal δ-orientation.*

The rest of this section is devoted to the proof of Proposition 7.11. We first recall the link between outdegree sequences and the *cycle-flips*.

Consider an orientation $\mathcal{O}$ and an $\mathcal{O}$-directed cycle (resp. cocycle) C. *Flipping* the $\mathcal{O}$-directed cycle (resp. cocycle) C means reversing every arc in C. We shall talk about *cycle-flips* and *cocycle-flips*. Observe that flipping a directed cycle does not change the outdegree sequence. Therefore, any orientation $\mathcal{O}'$ obtained from $\mathcal{O}$ by a sequence of cycle-flips has the same outdegree sequence as $\mathcal{O}$. It was proved in [Fels 04] that the converse is also true.

Lemma 7.12 *[Fels 04] Two orientations $\mathcal{O}$ and $\mathcal{O}'$ have the same outdegree sequence if and only if they can be obtained from one another by a sequence of cycle-flips. Moreover, the flipped cycles can be chosen to be contained in the set $\{e/\mathcal{O}(e) \neq \mathcal{O}'(e)\}$.*

Lemma 7.12 is a direct consequence of the following result proved in [Fels 04].

Lemma 7.13 *[Fels 04] Let G be a graph and let $\mathcal{O}$ and $\mathcal{O}'$ be two orientations having the same outdegree sequence. For any edge e in the set $K = \{e'/\mathcal{O}(e) \neq \mathcal{O}'(e)\}$, there is an $\mathcal{O}$-directed cycle $C \subseteq K$ containing e.*

Proof: (Hint) Start from the end v of $\mathcal{O}(e)$ and look for an edge e_1 in K directed away from v. This edge exists except if v is also the origin of e (since the number of edges directed away from v is the same in $\mathcal{O}$ and $\mathcal{O}'$). Repeat the process until arriving to the origin of e.

□

Recall that any very arc of an oriented graph is either in a directed cycle or a directed cocycle but not both (Lemma 6.11). We say that an arc a is *cyclic* or *acyclic* depending on a being in a directed cycle or in a directed cocycle. We call *cyclic part* (resp. *acyclic part*) of an orientation the set of cyclic (resp. acyclic) edges.

It is well known that the cyclic and acyclic parts are unchanged by a cycle-flip or a cocycle flip [Fels 04, Gioa 06, Prop 93]. Indeed, it is easily seen that the cyclic part of an orientation can only grow when a directed cocycle D is flipped (since no directed cycle intersects with D). Since we return to the original orientation by flipping D twice, we conclude that the cyclic and acyclic parts are unchanged by a cocycle-flip. Similarly, the cyclic and acyclic parts are unchanged by a cycle-flip.

We will also need the following classical result (closely related to an axioms of oriented matroids theory [Björ 93]).

Lemma 7.14 (Elimination) *Let $\mathcal{O}$ be an orientation and let C and C' be two $\mathcal{O}$-directed cycles (resp. cocycles). Let $\mathcal{O}'$ be the orientation obtained from $\mathcal{O}$ by flipping C'. Then, the symmetric difference of C and C' is a union of $\mathcal{O}'$-directed cycles (resp. cocycles). In particular, any edge in the $\mathcal{O}$-directed cycle (resp. cocycle) C is in an $\mathcal{O}'$-directed cycle (resp. cocycle) $C'' \subseteq C \cup C'$.*

Lemma 7.14 is illustrated by Figure 137.

Figure 137: The $\mathcal{O}$-directed cycles (resp. cocycles) C and C' (thin and thick lines) and their intersection (dashed lines).

We are now ready to prove Proposition 7.11. A *false* proof of the uniqueness of the minimal δ-orientation in this proposition is as follows. If there are two different δ-orientations $\mathcal{O}$ and $\mathcal{O}'$, then these orientations differ on a directed cycle C. Hence, the cycle C is tail-min in either $\mathcal{O}$ or $\mathcal{O}'$. A *false* proof of the existence (of a minimal δ-orientation) is as follows. Take any δ-orientation and starts flipping cycles until no more tail-min directed cycle remains. Of course, both the uniqueness and existence proofs are false in this version since flipping a cycle changes the associated subgraph, hence the spanning tree and the order on the half-edges. However being a bit careful, one can make both proofs correct.

We consider the procedure Ψ on orientations (see Definition 6.10). For an orientation $\mathcal{O}$ we denote by $\Psi[\mathcal{O}]$ the execution of Ψ on $\mathcal{O}$. Recall (from Lemma 6.18) that the half-edges are visited in $(\mathcal{G}, T)$-order during $\Psi[\mathcal{O}]$, where T is the spanning tree $\Delta \circ \Psi(\mathcal{O})$. Therefore, *the orientation $\mathcal{O}$ is minimal if and only if Condition (a) never holds during the execution $\Psi[\mathcal{O}]$.*

Lemma 7.15 *Let $\mathcal{O}$ be an orientation. Consider the current half-edge h, the edge e and the sets F, S and T at the beginning of a given core step of the execution $\Psi[\mathcal{O}]$. Let $C_f \subseteq \overline{F+e}$ be an $\mathcal{O}$-directed cycle and let $\mathcal{O}'$ be the orientation obtained from $\mathcal{O}$ by flipping C_f. We want to prove that* Condition (a) (resp. (b), (c), (a'), (b'), (c')) holds for the orientation $\mathcal{O}$ if and only if it holds for the orientation $\mathcal{O}'$. *(Let us insist that when evaluating the Conditions $(a), \cdots, (c')$ for the orientation $\mathcal{O}'$, the symbols F, S, T, h_{first} and e_{first} continue to refer to the execution of $\Psi[\mathcal{O}]$.)*

Proof: Note first that the orientations $\mathcal{O}$ and $\mathcal{O}'$ coincide on the current half-edge h since $e \notin C_f$. We now study separately the different conditions.

• Recall that $\mathcal{O}$ and $\mathcal{O}'$ coincide on their acyclic part: the directed cocycles of $\mathcal{O}$ and $\mathcal{O}'$ are the same. Therefore, Condition (b) (resp. (a')) holds for $\mathcal{O}$ if and only if it holds for $\mathcal{O}'$.

• Suppose now that Condition (a) holds for $\mathcal{O}$: the current half-edge h is a tail and the edge e is in an $\mathcal{O}$-directed cycle $C \subseteq \overline{F}$. By Lemma 7.14, the edge e is also in an $\mathcal{O}'$-directed cycle $C' \subseteq C \cup C_f \subseteq \overline{F}$. Thus, Condition (a) holds for $\mathcal{O}'$. The same argument proves that if Condition (a) holds for $\mathcal{O}'$, then it holds for $\mathcal{O}$ ($\mathcal{O}$ is obtained from $\mathcal{O}'$ by flipping the $\mathcal{O}'$-directed cycle C_f).

• Suppose now that Condition (b') holds for $\mathcal{O}$: the current half-edge h is a head and the edge e is in a tail-first $\mathcal{O}$-directed cycle $C \nsubseteq \overline{F}$ such that for all $\mathcal{O}$-directed cycle C' with $e_{\text{first}}(C') = e_{\text{first}}(C)$ either $e \in C'$ or ($C \vartriangle C' \nsubseteq \overline{F}$ and $e_{\text{first}}(C \vartriangle C') \in C'$). By Lemma 7.14, the edge $e^* = e_{\text{first}}(C)$ is in an $\mathcal{O}'$-directed cycle $C_1 \subseteq C \cup C_f$. Note that $e_{\text{first}}(C_1) = e^*$. We want to prove that Condition (b') holds for $\mathcal{O}'$ by considering the $\mathcal{O}'$-directed cycle C_1. We prove successively the following properties.

- *The edge e is in C_1.*
  The edge e^* is in the $\mathcal{O}'$-directed cycle C_1 and not in C_f. By Lemma 7.14, there is an $\mathcal{O}$-directed cycle $C_2 \subseteq C_1 \cup C_f$ containing e^* (since $\mathcal{O}$ is obtained from $\mathcal{O}'$ by flipping C_f). Note that $e_{\text{first}}(C_2) = e^*$. Suppose that e is not in C_2. By Condition (b') on C, we have $C \vartriangle C_2 \nsubseteq \overline{F}$ and $e_{\text{first}}(C \vartriangle C_2) \in C_2$. This is impossible since $\overline{C} \cap C_2 \subseteq C_f$ (since $C_2 \subseteq C_1 \cup C_f \subseteq C \cup C_f$) and the edge e in $C \cap \overline{C_2}$ is visited before any edge in C_f. Thus $e \in C_2$. Since $e \in C_2 \subseteq C_1 \cup C_f$ and e is not in C_f, it is in C_1.

- *For all $\mathcal{O}'$-directed cycle C_1' with $e_{first}(C_1') = e_{first}(C_1)$ either $e \in C_1'$ or ($C_1 \vartriangle C_1' \nsubseteq \overline{F}$ and $e_{first}(C_1 \vartriangle C_1') \in C_1'$).* (This proves that Condition (b') is satisfied for $\mathcal{O}'$).
  Let C_1' be an $\mathcal{O}'$-directed cycle not containing e and such that $e_{\text{first}}(C_1') = e_{\text{first}}(C_1) = e^*$. We want to prove that $C_1 \vartriangle C_1' \subseteq \overline{F}$ and $e_{\text{first}}(C_1 \vartriangle C_1') \in C_1'$. The edge e^* is in the $\mathcal{O}'$-directed cycle C_1' but not in C_f. By Lemma 7.14, there exists an $\mathcal{O}$-directed cycle $C' \subseteq C_1' \cup C_f$ containing e^*. Note that $e_{\text{first}}(C') = e^*$ and that $e \notin C'$ (since e is not in C_f nor in C_1' by hypothesis). By Condition (b') on C, we have $C \vartriangle C' \nsubseteq \overline{F}$ and $e^{\vartriangle} = e_{\text{first}}(C \vartriangle C') \in C'$. We now prove the following properties.
  - *The edge $e^{\vartriangle}$ is in $\overline{C_1} \cap C_1'$.*
  The edge $e^{\vartriangle}$ is in C_1' since $e^{\vartriangle} \notin C_f$ and $e^{\vartriangle} \in C' \subseteq C_1' \cup C_f$. Moreover, $e^{\vartriangle}$ is not in C_1 since $e^{\vartriangle} \notin C$, $e^{\vartriangle} \notin C_f$ and $C_1 \subseteq C \cup C_f$. Thus, $e^{\vartriangle}$ is in $\overline{C_1} \cap C_1'$.
  - *Any edge in $C_1 \cap \overline{C_1'}$ is visited after $e^{\vartriangle}$ during the execution $\Psi[\mathcal{O}]$.*
  Let e' be an edge in $C_1 \cap \overline{C_1'}$. If e' is in C_f, it is visited after $e^{\vartriangle}$. Else, e' is in C since $e' \in C_1$, $e' \notin C_f$ and $C_1 \subseteq C \cup C_f$. Moreover, e' is not in C' since $e' \notin C_1'$, $e' \notin C_f$ and $C' \subseteq C_1 \cup C_f$. Since $e' \in C \vartriangle C'$, the edge e' is visited after $e^{\vartriangle} = e_{\text{first}}(C \vartriangle C')$ during the execution $\Psi[\mathcal{O}]$.
  Since $e^{\vartriangle}$ is in $\overline{C_1} \cap C_1'$ and any edge in $C_1 \cap \overline{C_1'}$ is visited after $e^{\vartriangle}$, the edge $e_{\text{first}}(C_1 \vartriangle C_1')$ is in C_1'. Thus, Condition (b') holds for $\mathcal{O}'$.

We have proved that if Condition (b') holds for $\mathcal{O}$, then it holds for $\mathcal{O}'$. The same argument proves that if Condition (b') holds for $\mathcal{O}'$, then it holds for $\mathcal{O}$.
• Condition (c) holds for $\mathcal{O}$ if h is a tail and Conditions (a) and (b) do not hold for $\mathcal{O}$ By the preceding points this is true if and only if h is a tail and Conditions (a) and (b) do not hold for $\mathcal{O}'$. Therefore, Condition (c) holds for $\mathcal{O}$ if and only if it holds for $\mathcal{O}'$. Similarly, Condition (c') holds for $\mathcal{O}$ if and only if it holds for $\mathcal{O}'$.

□

Lemma 7.16 *Consider two orientations $\mathcal{O}$ and $\mathcal{O}'$ having the same outdegree sequence. We consider the executions $\Psi[\mathcal{O}]$ and $\Psi[\mathcal{O}']$. For all $0 \leq i < |H|$, we denote by h_i, F_i, T_i and S_i the current half-edge and the sets F, T and S at the beginning of the i^{th} core step of the execution $\Psi[\mathcal{O}]$ (see Definition 6.10). We define h_i', F_i', T_i' and S_i' similarly for the orientation $\mathcal{O}'$. We want to prove that* if the orientations $\mathcal{O}$ and $\mathcal{O}'$ coincide on h_i for all $i < k$ (that is, $\mathcal{O}(e_i) = \mathcal{O}'(e_i)$ where e_i is the edge containing h_i), then the k first core steps of the executions $\Psi[\mathcal{O}]$ and $\Psi[\mathcal{O}']$ are the same. In particular, $h_i = h_i'$, $F_i = F_i'$, $S_i = S_i'$, and $T_i = T_i'$ for all $i \leq k$.

Proof: We proceed by induction on k. Recall from Lemma 7.12 that the orientation $\mathcal{O}'$ can be obtained from $\mathcal{O}$ by a sequence of cycle-flips such that the flipped cycles are contained in the set $K = \{e/\mathcal{O}(e) \neq \mathcal{O}'(e)\}$. For $k = 0$ the property obviously holds. Now suppose that the property holds for k and suppose that $\mathcal{O}$ and $\mathcal{O}'$ coincide on $h_i, i < k + 1$. By the induction hypothesis the current half-edge $h_k = h'_k$ and the sets $F = F_k = F'_k$, $S = S_k = S'_k$, and $T = T_k = T'_k$ are the same at the beginning of the $(k+1)^{th}$ core step of the procedures $\Psi[\mathcal{O}]$ and $\Psi[\mathcal{O}']$. Moreover, the set $K = \{e'/\mathcal{O}(e') \neq \mathcal{O}'(e')\}$ of reverse edges is contained in $\overline{F+e}$. Since $\mathcal{O}'$ is obtained from $\mathcal{O}$ by a sequence of flips of cycles contained in $\overline{F+e}$, we know by induction on Lemma 7.15 that Condition (a) (resp. (b), (c), (a'), (b'), (c')) holds for the orientation $\mathcal{O}$ if and only if it holds for the orientation $\mathcal{O}'$. Therefore, the $(k+1)^{th}$ core step is the same for the two executions $\Psi[\mathcal{O}]$ and $\Psi[\mathcal{O}']$. In particular, the sets F, S, and T are modified in the same way in both executions and $h_{k+1} = h'_{k+1}$. Thus, the property holds by induction.

□

Proof of Proposition 7.11. Recall that an orientation $\mathcal{O}$ is minimal if and only if Condition (a) never holds during the execution $\Psi[\mathcal{O}]$. Thus, we need to prove that for any outdegree sequence δ there exists a unique δ-orientation $\mathcal{O}$ such that Condition (a) never holds during the execution $\Psi[\mathcal{O}]$.

• **Uniqueness:** Let $\mathcal{O}$ and $\mathcal{O}'$ be two (distinct) orientations having the same outdegree sequence. We take the same notations h_i, F_i, T_i, S_i, h'_i, F'_i, T'_i, S'_i as in Lemma 7.16. Let k be the first index such that $\mathcal{O}$ and $\mathcal{O}'$ differ on h_k. By Lemma 7.16, we have $h_k = h'_k$ and $F_k = F'_k$, $T_k = T'_k$, $S_k = S'_k$. We can suppose without loss of generality that h_k is a tail in $\mathcal{O}$ and a head in $\mathcal{O}'$. We now prove that Condition (a) holds for $\mathcal{O}$. By hypothesis, the edge e containing h is such that $\mathcal{O}(e) \neq \mathcal{O}'(e)$. Hence, by Lemma 7.13, the edge e is contained in an $\mathcal{O}$-directed cycle $C \subseteq K = \{e/\mathcal{O}(e) \neq \mathcal{O}'(e)\}$. Since $\mathcal{O}$ and $\mathcal{O}'$ coincide on h_i for $i < k$, the set K is contained in $\overline{F_i}$. Since $C \subseteq \overline{F_i}$ is $\mathcal{O}$-directed, Condition (a) holds for $\mathcal{O}$.

• **Existence:** Let δ be an outdegree sequence. We want to find a δ-orientation $\mathcal{O}$ such that Condition (a) never holds during the execution $\Psi[\mathcal{O}]$. Let $\mathcal{O}_0$ be any δ-orientation. We are going to define a set of δ-orientations $\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_{|H|}$ such that Condition (a) is not satisfied during the i first core steps of the execution $\Psi[\mathcal{O}_i]$. We prove that $\mathcal{O}_k$ exists by induction on k. Suppose the δ-orientation $\mathcal{O}_{k-1}$ exists. We consider the current half-edge h, the edge e and the sets F, S and T at the beginning of the k^{th} core step of $\Psi[\mathcal{O}_{k-1}]$. If either $e \in F$ or Condition (a) does not hold, we define $\mathcal{O}_k = \mathcal{O}_{k-1}$. Else, the current half-edge h_k is a tail (for the orientation $\mathcal{O}_{k-1}$) and there is an $\mathcal{O}_k$-directed cycle $C \subseteq \overline{F}$ containing e. In this case, we define $\mathcal{O}_k$ to be the orientation obtained from $\mathcal{O}_{k-1}$ by flipping the cycle C. Observe that $\mathcal{O}_k$ is a δ-orientation in which h_k is a head. Moreover, since $C \subseteq \overline{F}$ the two orientations $\mathcal{O}_{k-1}$ and $\mathcal{O}_k$ coincide on the half-edges h_i for $i < k$, where h_i is the current half-edge at the beginning of the i^{th} core step of the execution $\Psi[\mathcal{O}_{k-1}]$. Thus, by Lemma 7.16, the k first core steps of the executions $\Psi[\mathcal{O}_{k-1}]$ and $\Psi[\mathcal{O}_k]$ are the same. Moreover, the current half-edge $h = h_k$ at the beginning of the k^{th} core step of $\Psi[\mathcal{O}_k]$ is a head (for the orientation $\mathcal{O}_k$). Hence, Condition (a) does not hold at this core step. Thus, $\mathcal{O}_k$ is a δ-orientation such that Condition (a) does not hold during the k^{th} first core steps of the execution $\Psi[\mathcal{O}_k]$. The orientations $\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_{|H|}$ exist by induction. In particular, the δ-orientation $\mathcal{O}_{|H|}$ is such that Condition (a) never holds during the execution $\Psi[\mathcal{O}_{|H|}]$. $\square$

From Proposition 7.8 and 7.11 one obtains the following bijection between outdegree sequences and forests. This answers a question raised by Stanley in [Stan 80a].

Proposition 7.17 *Let $\mathcal{G}$ be an embedded graph. The mapping Γ which associates with any subgraph S the outdegree sequence of the orientation $\mathcal{O}_S$ establishes a bijection between the forests and the outdegree sequences of $\mathcal{G}$.*

7.6 Summary of the specializations and further refinements

From Propositions 7.2, 7.5, 7.8 and 7.9 we can characterize the orientations associated with each class of subgraphs defined by the criteria *forest*, *internal*, *connected*, *external*. Each class of subgraphs is counted by a specialization of the Tutte polynomial given in Proposition 7.1. Our results are summarized in the following theorem.

Theorem 7.18 *Let $\mathcal{G}$ be an embedded graph and let v_0 be the root-vertex.*

1. *The v_0-connected orientations are in bijection with the connected subgraphs counted by $T_G(1,2)$.*

2. *The strongly connected orientations are in bijection with the external subgraphs counted by $T_G(0,2)$.*

3. *The outdegree sequences are in bijection with minimal orientations, which are in bijection with forests, counted by $T_G(2,1)$.*

4. *The acyclic orientations are in bijection with internal forests counted by $T_G(2,0)$.*

5. *The v_0-connected outdegree sequences are in bijection with v_0-connected minimal orientations which are in bijection with spanning trees counted by $T_G(1,1)$.*

6. *The strongly connected outdegree sequences are in bijection with strongly connected minimal orientations which are in bijection with external spanning trees counted by $T_G(0,1)$.*

7. *The v_0-connected acyclic orientations are in bijection with internal spanning trees counted by $T_G(1,0)$.*

Theorem 7.18 is illustrated by Figure 135. The enumeration of acyclic orientations by $T_G(2,0)$ was first established by Winder in 1966 [Wind 66] and rediscovered by Stanley 1973 [Stan 73]. The result of Winder was stated as an enumeration formula for the number of faces of hyperplanes arrangements and was independently extended to reel arrangements by Zaslavsky [Zasl 75] and to orientable matroids by Las Vergnas [Las 75]. The enumeration of v_0-connected acyclic orientations by $T_G(1,0)$ was found by Greene and Zaslavsky [Gree 83].

In [Gess 96], Gessel and Sagan gave a bijective proof of both results. In [Gebh 00], Gebhard and Sagan gave three other proofs of Greene and Zaslavsky's result. The enumeration of strongly connected orientations by $T_G(0,2)$ is a direct consequence of Las Vergnas' characterization of the Tutte poynomial [Las 84b]. The enumeration of outdegree sequences by $T_G(2,1)$ was discovered by Stanley [Bryl 91, Stan 80a]. The enumeration of v_0-connected orientations by $T_G(1,2)$, the enumeration of v_0-connected outdegree sequences by $T_G(1,1)$ and the enumeration of strongly connected outdegree sequences by $T_G(0,1)$ were proved by Gioan [Gioa 06].

Refinements. It is possible to refine the results of Theorem 7.18. For instance, we have proved that the acyclic orientations of a graph G are counted by $T_G(2,0)$. This is the sum of the coefficients of the polynomial $T_G(1+x,0)$ (which is closely related to the chromatic polynomial of G). Let us generically denote by $[x^i]P(x)$ the coefficient of x^i in a polynomial $P(x)$. The identities

$$\sum_{i\in\mathbb{N}}[x^i]T_G(1+x,0) = T_G(2,0) = |\{\text{acyclic orientations}\}|,$$

and

$$\sum_{i\in\mathbb{N}}[x^i]T_G(x,0) = T_G(1,0) = |\{v_0\text{-connected acyclic orientations}\}|,$$

make it appealing to look for a partition of the acyclic orientations (resp. root-connected acyclic orientations) in parts of size $[x^i]T_G(1+x,0)$, $i \geq 0$ (resp. $[x^i]T_G(x,0)$). Such partitions were defined by Lass in [Lass 01] using set functions algebra. More generally, one can try to interpret the coefficients of $T_G(x,1)$, $T_G(1+x,1)$, $T_G(x,2)$, $T_G(1+x,2)$ *etc.* in terms of orientations in order to interpolate between the different specializations $T_G(i,j), 0 \leq i,j \leq 2$. Observe that the coefficients of each of these polynomials can be given an interpretation in terms of subgraphs. For instance, $[x^i]T_G(1+x,0)$ counts internal forests with $i+1$ trees (by Theorem 6.3 and Lemma 6.2) and $[x^i]T_G(x,0)$ counts internal spanning trees with i internal embedding-active edges (by Theorem 5.5).

We will give an interpretation of the coefficients $[x^i]T_G(1+x,j)$ for $i \geq 0$ and $j = 0,1,2$ in terms of orientations. Let $\mathcal{O}$ be an orientation. We define the partition of the vertex set V into *root-components* $V = \biguplus_{0\leq i\leq k} V_i$ as follows. The first root-component V_0 is the set of vertices reachable from the root-vertex v_0. If $W_k = \cup_{0\leq i\leq k} V_i \subsetneq V$, we consider the minimal edge e_k with one vertex in W_k and one vertex v_k in $\overline{W_k}$ (the edges are compared according to the $(\mathcal{G},T)$-order, where $T = \Delta(\Psi(\mathcal{O}))$). Then, the $(k+1)^{th}$ root-component is the set of vertices in $\overline{W_k}$ that are reachable from v_k. For instance, the root-components have been indicated for the orientation in Figure 138 (left). It is clear that v_0-connected orientations have only one root-component. Given a v_0-connected orientation $\mathcal{O}$, we define the partition of the vertex set V into *root-strong-components* $V = \biguplus_{0\leq i\leq k} U_i$ as follows. The first root-strong-component

U_0 is the set of vertices that can reach the root-vertex v_0. If $W_k = \cup_{0\leq i\leq k} U_i \subsetneq V$, we consider the minimal edge e_k with one vertex in W_k and one vertex v_k in $\overline{W_k}$. Then, the $(k+1)^{th}$ root-strong-component is the set of vertices in $\overline{W_k}$ that can reach v_k. For instance, the root-strong-components have been indicated for the v_0-connected orientation in Figure 138 (right).

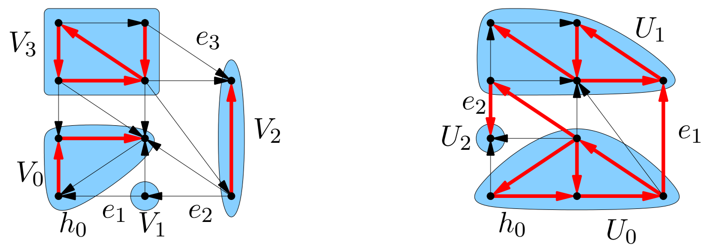


Figure 138: Left: root-components of an orientation. Right: root-strong-components of a v_0-connected orientation. The thick edges correspond to the subgraph associated with the orientation by the bijection Ψ.

Theorem 7.19 *Let $\mathcal{G}$ be an embedded graph and let v_0 be the root-vertex. The coefficient $[x^i]T_G(1+x,2)$ (resp. $[x^i]T_G(1+x,1)$, $[x^i]T_G(1+x,0)$) counts orientations (resp. minimal orientations, acyclic orientations) with $i+1$ (non-empty) root-components. The coefficient $[x^i]T_G(x,2)$ (resp. $[x^i]T_G(x,1)$, $[x^i]T_G(x,0)$) counts v_0-connected orientations (resp. minimal v_0-connected orientations, acyclic v_0-connected orientations) with $i+1$ (non-empty) root-strong-components.*

As mentioned above, the coefficients $[x^i]T_G(1+x,0)$ and $[x^i]T_G(x,0)$ had already been interpreted by Lass in [Lass 01]. We now prove Theorem 7.19.

Lemma 7.20 *Let $\mathcal{G}$ be an embedded graph and let $\mathcal{O}$ be an orientation. We consider the spanning tree $T = \Delta(\Psi(\mathcal{O}))$ and compare the half-edges and edges according to the $(\mathcal{G},T)$-order. Let $V_0,\ldots,V_k$ be the root-components and let $W_i = \cup_{0\leq j\leq i} V_j$. Let D_i for $i = 1\ldots k$ be the cut defined by W_{i-1} and let e_i be the minimal edge in D_i. Then, an edge is minimal in a head-min directed cocycle if and only if it is in the set $\{e_1,\ldots,e_k\}$.*

Proof:

• We first prove that for all $1\leq i\leq k$ *the edge e_k is minimal in a head-min directed cocycle.* Clearly, every edge in the set D_i is directed toward the vertices in W_{i-1}. Let v_i be the endpoint of $e_i = e_{\min}(D)$ which is not in W_{i-1}. Let X_i be the set of vertices contained in the connected component containing v_i once the cut D is removed. The set D of edges with one endpoint in W_{i-1} and one endpoint in X_i is a directed cocycle contained in D_i. Thus, the edge e_i is minimal in the directed cocycle D directed toward W_{i-1}. Since the cocycle D is directed toward the component containing the root-vertex, it is head-min by Lemma 7.3.

• Consider an edge e minimal in a head-min directed cocycle D. We want to prove that e is in $\{e_1, \ldots, e_k\}$. Let $\mathcal{G}_0$ and $\mathcal{G}_1$ be the connected components after the cocycle D is removed with the convention that $\mathcal{G}_0$ contains the root-vertex v_0. The directed cocycle D is head-min, hence it is directed toward $\mathcal{G}_0$ by Lemma 7.3. Let i be the first index such that the root-component V_i contains a vertex v of $\mathcal{G}_1$. The cocycle D is directed toward $\mathcal{G}_0$, hence no edge of $\mathcal{G}_1$ is reachable from v_0 and the index i is positive. Let u_i and v_i be the endpoints of e_i in W_{i-1} and $\overline{W_{i-1}}$ respectively. By definition, the endpoint u_i is in $\mathcal{G}_0$. Moreover, the vertex $v \in \mathcal{G}_1$ is reachable from v_i, hence the endpoint v_i is in $\mathcal{G}_1$. Thus, the edge e_i is in D and $e_i \geq e = e_{\min}(D)$. We will now prove that $e_i \leq e$. The subset of vertices W_{i-1} contains the root-vertex and the subset of edges D_i separate W_{i-1} and $\overline{W_{i-1}}$, hence every edge with one endpoint in $\overline{W_{i-1}}$ is greater than $e_i = e_{\min}(D_i)$ by Lemma 7.3. The edge e has one endpoint in $\mathcal{G}_1 \subseteq \overline{W_{i-1}}$, hence $e_i \leq e$. Thus, $e = e_i$.

□

Here is a counterpart of Lemma 7.20 for root-strong-components.

Lemma 7.21 *Let $\mathcal{G}$ be an embedded graph and let $\mathcal{O}$ be a v_0-connected orientation. We consider the spanning tree $T = \Delta(\Psi(\mathcal{O}))$ and compare the half-edges and edges according to the $(\mathcal{G}, T)$-order. Let $U_0, \ldots, U_k$ be the root-strong-components and let $W_i = \cup_{0 \leq j \leq i} U_j$. Let D_i for $i = 1 \ldots k$ be the cut defined by W_{i-1} and let e_i be the minimal edge in D_i. Then, an edge is minimal in a directed cocycle if and only if it is in the set $\{e_1, \ldots, e_k\}$.*

Proof: The proof of Lemma 7.21 very similar to the proof of Lemma 7.20 and is left to the reader.

□

Proof of Theorem 7.19.

• We first prove that *the coefficient* $[x^i]T_G(1+x,2)$ *(resp.* $[x^i]T_G(1+x,1)$, $[x^i]T_G(1+x,0)$*) counts orientations (resp. minimal orientations, acyclic orientations) with* $i+1$ *root-components.* Let T be a spanning tree with $\mathcal{I}(T)$ internal and $\mathcal{E}(T)$ external $(\mathcal{G}, T)$-active edges. By Lemma 6.2, the coefficient $[x^i](1+x)^{\mathcal{I}(T)}2^{\mathcal{E}(T)}$ counts the subgraphs S in the tree-interval $[T^-, T^+]$ having i edges in $\overline{S} \cap T$. Given that the tree-intervals form a partition of the set of subgraphs, the coefficient $[x^i]\sum_{T \text{ spanning tree}} (1+x)^{\mathcal{I}(T)}2^{\mathcal{E}(T)}$ counts the subgraphs S having i edges in $\overline{S} \cap \Delta(S)$. Moreover, by the characterization (84) of the Tutte polynomial, the sum $\sum_T (1+x)^{\mathcal{I}(T)}2^{\mathcal{E}(T)}$ is equal to $T_G(1+x,2)$. Similarly, the coefficient $[x^i]T_G(1+x,1)$ (resp. $[x^i]T_G(1+x,0)$) counts the forests (resp. internal forests) S having i edges in $\overline{S} \cap \Delta(S)$. By Theorem 7.18 and Lemma 6.16, the coefficient $[x^i]T_G(1+x,2)$ (resp. $[x^i]T_G(1+x,1)$, $[x^i]T_G(1+x,0)$) counts the orientations (resp. minimal orientations, acyclic orientations) having exactly i edges which are minimal in some head-min directed cocycle. Moreover, by

Lemma 7.20, an orientation has i edges which are minimal in some head-min directed cocycle if and only if it has $i+1$ root-components.

• We now prove that *the coefficient* $[x^i]T_G(x,2)$ *(resp.* $[x^i]T_G(x,1)$, $[x^i]T_G(x,0)$*) counts* v_0*-connected orientations (resp. minimal* v_0*-connected orientations, acyclic* v_0*-connected orientations) with* $i+1$ *root-strong-components.* Let T be a spanning tree with $\mathcal{I}(T)$ internal $(\mathcal{G},T)$-active edges and $\mathcal{E}(T)$ external $(\mathcal{G},T)$-active edges. By Lemma 6.2, the coefficient $[x^i]x^{\mathcal{I}(T)}2^{\mathcal{E}(T)}$ is the number of connected subgraphs in the tree-interval $[T^-,T^+]$ if $\mathcal{I}(T)=i$ and 0 otherwise. Given that the tree-intervals form a partition of the set of subgraphs, the coefficient $[x^i]\sum_{T \text{ spanning tree}} x^{\mathcal{I}(T)}2^{\mathcal{E}(T)}$ counts the connected subgraphs S such that the tree $T=\Delta(S)$ has i internal $(\mathcal{G},T)$-active edges. Moreover, by the characterization (84) of the Tutte polynomial, the sum $\sum_T x^{\mathcal{I}(T)}2^{\mathcal{E}(T)}$ is equal to $T_G(x,2)$. Similarly, the coefficient $[x^i]T_G(x,1)$ (resp. $[x^i]T_G(x,0)$) counts the spanning trees (resp. internal spanning trees) T having i internal $(\mathcal{G},T)$-active edges. By Theorem 7.18 and Lemma 7.7, the coefficient $[x^i]T_G(x,2)$ (resp. $[x^i]T_G(x,1)$, $[x^i]T_G(x,0)$) counts the v_0-connected orientations (resp. minimal v_0-connected orientations, acyclic v_0-connected orientations) having exactly i edges which are minimal in some directed cocycle. Moreover, by Lemma 7.20, an orientation has i edges which are minimal in some directed cocycle if and only if it has $i+1$ root-strong-components.

□

One specialization of this result is of special interest: the coefficient $[x^1]T_G(x,0)$ counts *bipolar orientations*. Given two vertices u and v, a (u,v)*-bipolar* orientation is an acyclic orientation such that u is the unique source and v is the unique sink. The bipolar orientations are of crucial importance for many graph algorithms [Mend 94]. Moreover, a bijection between spanning trees of ordering-activities $(1,0)$ and bipolar orientation is the building block used in [Gioa 05] in order to define a general *activity preserving* correspondence between spanning trees and orientations.

Proposition 7.22 *Let* $\mathcal{G}$ *be an embedded graph, let* v_0 *be the root-vertex and let* v_1 *be the other endpoint of the root-edge. The mapping* Φ *establishes a bijection between the spanning trees having embedding-activities* $(\mathcal{I}(T),\mathcal{E}(T))=(1,0)$ *(counted by* $[x^1]T_G(x,0)$*) and the* (v_0,v_1)*-bipolar orientations.*

Proposition 7.22 is illustrated by Figure 139.

Proof: Observe first that an acyclic orientation $\mathcal{O}$ is (v_0,v_1)-bipolar if and only if any vertex is reachable from v_0 and can reach v_1. By Theorem 7.19 the coefficient $[x^1]T_G(x,0)$ counts acyclic v_0-connected orientation having 2 root-strong-components. No vertex $v\neq v_0$ can reach v_0 in an acyclic v_0-connected orientation (there would be a directed path from v_0 to v and back). Hence the first root-component U_0 of an acyclic v_0-connected orientation is reduced

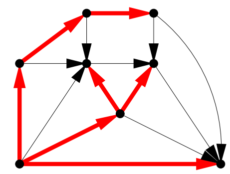


Figure 139: A bipolar orientation and the corresponding spanning tree (indicated by thick lines).

to $\{v_0\}$. The minimal edge with one endpoint in $U_0 = \{v_0\}$ and one endpoint outside U_0 is the root-edge. Hence an acyclic v_0-connected orientation has 2 root-strong-components if and only if every vertex can reach v_1. Thus, the coefficient $[x^1]T_G(x, 0)$ counts (v_0, v_1)-bipolar orientations.

□

7.7 Concluding remarks

7.7.1 The cycle and cocycle reversing systems

Let us consider the *cycle reversing system* and the *cocycle reversing system*. A transition in the cycle (resp. cocycle) reversing system consists in flipping a directed cycle (resp. cocycle). The cycle and cocycle reversing systems appear implicitly in many works (e.g. [Fels 04, Fray 01, Prop 93, Boni 05]). The *cycle-cocycle reversing system* in which a transition consists in flipping either a directed cycle or a directed cocycle was introduced in [Gioa 06]. It was observed in this paper that the cycle and cocycle flips are really independant since they act on the cyclic part and acyclic part respectively and do not modify the other part. It is known from [Prop 93] that there is a unique v_0-connected orientation (equivalently, orientation without head-min directed cocycle by Lemma 7.4) in each equivalence class of the cocycle reversing system. The counterpart of this property for the cycle reversing system is given by Proposition 7.11. Indeed, it is clear from Lemma 7.12 that the equivalence classes of the cycle reversing system are in one-to-one correspondence with outdegree sequences. Thus, Proposition 7.11 proves that there is a unique minimal orientation (that is, orientation without tail-min directed cycle) in each equivalence class of the cycle reversing system. Since the cycle and cocycle flips are really independant, there is a unique v_0-connected minimal orientation in each equivalence class of the cycle-cocycle reversing system.

As observed in [Gioa 06], the enumerative results of Theorem 7.18 can be expressed in terms of cycle/cocycle reversing systems. For instance, the equivalence classes of the cocycle reversing system (in bijection with minimal orientations) are counted by $T_G(1, 2)$, the equivalence classes of the cocycle reversing system reduced to one element (equivalently, the stongly connected orientations) are counted by $T_G(0,2)$ *etc.*

7.7.2 Algorithmic applications

The bijections exhibited in this chapter have interesting applications in the framework of random sampling. Indeed, the bijection Φ allows us to turn a random sampling algorithm for a class of subgraphs (e.g. spanning trees, forests, internal subgraphs) into a random sampling algorithm for the corresponding class of orientations (see Theorem 7.18). The bijection Ψ allows us to perform the converse operation.

We have seen in the previous chapter (Subsection 6.4.2) that in certain cases the mappings Φ and Ψ could be performed in linear time. Hence, in these cases there is no increase of complexity in the transfer between subgraphs and orientations. In particular, in the planar case any random sampling algorithm for a class of subgraphs gives a random sampling algorithm with the same complexity for the corresponding class of orientations and conversely. Moreover, the mapping Φ restricted to forest can always be performed in linear time. Hence, any algorithm for the random generation of forests (resp. spanning tree [Wils 96]) gives an algorithm for the random generation of outdegree sequences (resp. v_0-connected outdegree sequences) with the same complexity.

Chapter 8

A bijection between spanning trees and recurrent sandpile configurations

Abstract: We define a bijection between spanning trees and recurrent configurations of the sandpile model. The image of any spanning tree having k external embedding-active edges is a recurrent configuration at *level* k. This gives a new bijective proof that the coefficient of y^k in the specialization $T_G(1, y)$ of the Tutte polynomial counts the recurrent configurations at level k. (This result of Merino [Meri 97] was already proved bijectively by Cori and Le Borgne [Cori 03].) In the previous chapter, we established a bijection between spanning trees and root-connected outdegree sequences. Combining our results, we obtain a bijection between recurrent configurations and root-connected outdegree sequences which leaves the configurations at level 0 unchanged. This answers a question raised by Gioan [Gioa 06].

Résumé : Nous définissons une bijection entre les arbres couvrants et les configurations récurrentes du modèle du tas de sable. L'image d'un arbre couvrant ayant k arêtes externes actives par plongement est une configuration récurrente au *niveau* k. Cela constitue une nouvelle preuve bijective du fait que le coefficient de y^k dans la spécialisation $T_G(1, y)$ du polynôme de Tutte compte les configurations récurrentes au niveau k. (Ce résultat dû à Merino [Meri 97] avait déjà été prouvé bijectivement par Cori et Le Borgne [Cori 03]). Dans le chapitre précédent, nous avons établi une bijection entre les arbres couvrants et les suites de degrés racine-accessibles. En combinant nos résultats nous obtenons une bijection entre les configurations récurrentes et les suites de degrés racine-accessibles qui laisse inchangées les configurations au niveau 0. Nous répondons ainsi à une question posée par Gioan [Gioa 06].

8.1 Introduction

In this chapter we deal with the sandpile model (a short introduction to this model is given in Subsection 4.1.4). The sandpile model was originally defined in statistical physics [Bak 87, Dhar 90]. It appeared independently in combinatorics as the *chip firing game* [Björ 91]. In this model, a *configuration* is an attribution of a non-negative integer to each vertex of the graph: the number of *sand grains* on this vertex. A vertex having a number of sand grains not less than its degree can *topple* and send a grain through each of the incident edges. The sandpile model is appreciated in physics because it provides an analytically tractable model of self-organized criticality. The sandpile model is also studied in combinatorics for its algebraic properties. Indeed, the formal sum of configurations (that is, the vertex by vertex sum of the number of sand grains) induces a group structure displaying interesting properties [Cori 00]. *Recurrent configurations* play an important role in the sandpile model. From the physical point of view, the recurrent configurations are the one that can be observed after a long period of time. Moreover, from the algebraic perspective, each element of the group defined by the sandpile model is associated with a recurrent configuration.

It is known that the recurrent configurations of the sandpile model on G are counted by $T_G(1,1)$ [Dhar 92]. Observe that this is the number of spanning trees. The following refinement is also true: the coefficient of y^k in $T_G(1,y)$ is the number of recurrent configurations at *level* k [Meri 97]. A bijective proof of this result was given in [Cori 03]. We give an alternative bijective proof based on the characterization of the Tutte polynomial via embedding-activities. We also answer a question of Gioan [Gioa 06] by establishing a bijection between recurrent configurations of the sandpile model and root-connected outdegree sequences that leaves the configurations at level 0 unchanged.

In Section 8.2, we define a mapping Λ from spanning trees to configurations of the sandpile model. We prove that the image of any spanning tree is a recurrent configuration. In Section 8.3, we define a mapping Υ from recurrent configurations to spanning trees. The mapping Υ is reminiscent of the *burning algorithm* introduced by Dhar in order to distinguish between recurrent and non-recurrent configurations [Dhar 90]. We proceed to prove that Λ and Υ are inverse bijections between spanning trees and recurrent sandpile configurations.

8.2 A bijection between spanning trees and recurrent configurations

In chapter 5 (Section 5.5), we defined a mapping $\Lambda : T \mapsto \mathcal{S}_T$ from spanning trees to configurations of the sandpile model. Recall from Definition 5.11 that the number of grains $\mathcal{S}_T(v)$ on the vertex v in the configuration $\mathcal{S}_T = \lambda(T)$ is the number of tails plus the number of external $(\mathcal{G},T)$-active heads incident to v in the orientation $\mathcal{O}_T = \Phi(T)$. In this chapter, we prove that the mapping Λ is a bijection between spanning trees and recurrent configurations of the sandpile model. From now on, the *recurrent configurations of the sandpile model* are simply called *recurrent configurations* .

Theorem 8.1 *Let $\mathcal{G}$ be an embedded graph. The mapping $\Lambda : T \mapsto \mathcal{S}_T$ is a bijection between the spanning trees and the recurrent configurations of $\mathcal{G}$.*

Let $G = (V, E)$ be the graph underlying the embedding $\mathcal{G}$. Observe that the *level* of the configuration $\mathcal{S}_T$, that is, $\sum_{v\in V} \mathcal{S}_T(v) - |E|$, is the number of external $(\mathcal{G},T)$-active edges. Indeed, every edge of G has contribution 1 to the sum $\sum_v \mathcal{S}_T(v)$ except the external $(\mathcal{G},T)$-active edges which have contribution 2.

Corollary 8.2 *Let $\mathcal{G}$ be an embedded graph. The number of recurrent configurations at level i is the number $[y^i]T_G(1, y)$ of spanning trees having i external $(\mathcal{G},T)$-active edges.*

As mentioned above, Corollary 8.2 is not new. It was first proved recursively in [Meri 97] and then bijectively in [Cori 03] (by using the *order-activities* of Tutte [Tutt 54]). The Theorem 8.1 and Corollary 8.2 are illustrated by Figure 140.

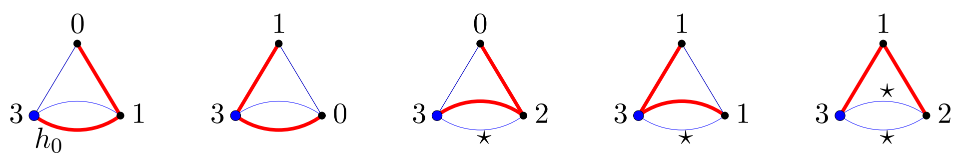


Figure 140: The spanning trees (thick lines) and the corresponding recurrent configurations. The external active edges are indicated by a $\star$.

We first prove that the image of any spanning tree is a recurrent configuration.

Proposition 8.3 *Let $\mathcal{G}$ be an embedded graph. For any spanning tree T, the configuration $\mathcal{S}_T = \Lambda(T)$ is a recurrent configuration.*

Proof: Let v_0 be the root-vertex. We consider the orientation $\mathcal{O}_T$ and prove successively the following properties.

• *The configuration $\mathcal{S}_T$ is stable.* Let v be any vertex distinct from v_0. We want to prove that $\mathcal{S}_T(v) < \deg(v)$. Observe that any half-edge incident to v has contribution at most one to $\mathcal{S}_T(v)$. Moreover, the half-edge h_v incident to v and contained in the edge of T linking v to its father is a head by Lemma 5.6. Thus, h_v has no contribution to $\mathcal{S}_T(v)$, and $\mathcal{S}_T(v) \leq \deg(v) - 1$.

• *$\mathcal{S}_T(v_0) = \deg(v_0)$.* We must prove that every half-edge incident to v_0 has contribution 1 to $\mathcal{S}_T(v_0)$. By Lemma 5.6, the internal edges are oriented from father to son in $\mathcal{O}_T$. Therefore any internal half-edge incident to v_0 is a tail, hence has contribution 1 to $\mathcal{S}_T(v_0)$. Let h be an external half-edge incident to v_0. By definition, if the half-edge h is greater than the half-edge

$h' = \alpha(h)$, then h is a tail. Else, the edge $e = \{h, h'\}$ is $(\mathcal{G}, T)$-active by Lemma 5.8 (since the endpoint v_0 of h is an ancestor of the endpoint of h'). Thus, any external half-edge incident to v_0 has contribution 1 to $\mathcal{S}_T(v_0)$.

• *The configuration $\mathcal{S}_T$ is recurrent.* We want to prove that there is a labeling of the vertices $v_0, v_1, \ldots, v_{|V|-1}$ such that the sequence of topplings $\mathcal{S}_T \overset{v_0}{\dashrightarrow} \mathcal{S}_T^1 \overset{v_1}{\dashrightarrow} \cdots \overset{v_{|V|-1}}{\dashrightarrow} \mathcal{S}_T^{|V|}$ is valid. Observe that in this case the configuration $\mathcal{S}_T$ is recurrent. Indeed, the final configuration $\mathcal{S}_T^{|V|}$ is equal to $\mathcal{S}_T$ since every vertex v has been toppled once, hence has sent and received exactly $\deg(v, *)$ grains during the sequence of topplings (recall that $\deg(v, *)$ is the number of non-loop edges incident to v). In Chapter 7, we defined a linear order, the *postfix order*, on the vertex set V (see Lemma 7.10). The root-vertex v_0 is the maximal element for this order. We want to prove that taking the unique labeling such that $v_0 > v_1 > \cdots > v_{|V|-1}$ for the postfix order, the sequence of topplings $\mathcal{S}_T \overset{v_0}{\dashrightarrow} \mathcal{S}_T^1 \overset{v_1}{\dashrightarrow} \cdots \overset{v_{|V|-1}}{\dashrightarrow} \mathcal{S}_T^{|V|}$ is valid. From the preceding point, the toppling of v_0 is valid. Suppose that the sequence $\mathcal{S}_T \overset{v_0}{\dashrightarrow} \mathcal{S}_T^1 \overset{v_1}{\dashrightarrow} \cdots \overset{v_{i-1}}{\dashrightarrow} \mathcal{S}_T^i$ is valid. After these topplings, the number of grains on the vertex v_i is $\mathcal{S}_T^i(v_i) = \mathcal{S}_T(v_i) + \sum_{j<i} \deg(v_i, v_j)$ (recall that $\deg(v_i, v_j)$ is the number of edges linking v_i and v_j). We want to prove that v_i can be toppled, that is, $\mathcal{S}_T^i(v_i) \geq \deg(v_i)$. By Lemma 7.10, any arc $\mathcal{O}_T(e)$ is directed toward its least endpoint (for the postfix order) unless e is external $(\mathcal{G}, T)$-active. Let h be an half-edge in an edge linking v_i to a vertex v_j, $j \geq i$. The vertex v_j is less than or equal to v_i for the postfix order, hence h is either a tail or an external $(\mathcal{G}, T)$-active half-edge. In both cases, the half-edge h has contribution 1 to $\mathcal{S}_T(v_i)$. Hence,

$$\mathcal{S}_T(v_i) \geq \sum_{j \geq i} \deg(v_i, v_j).$$

Thus,

$$\mathcal{S}_T^i(v_i) = \mathcal{S}_T(v_i) + \sum_{j \geq i} \deg(v_i, v_j) \geq \sum_{j \geq 0} \deg(v_i, v_j) = \deg(v_i)$$

and v_i can be toppled. By induction, the sequence of topplings $\mathcal{S}_T \overset{v_0}{\dashrightarrow} \mathcal{S}_T^1 \overset{v_1}{\dashrightarrow} \cdots \overset{v_{|V|-1}}{\dashrightarrow} \mathcal{S}_T^{|V|}$ is valid.

□

It remains to prove that $\Lambda : T \mapsto \mathcal{S}_T$ is a bijection between the spanning trees and the recurrent configurations. This will be done in the next section by defining the inverse mapping Υ.

8.3 The inverse bijection

In this section we define a mapping Υ that we shall prove to be the inverse of Λ. This mapping Υ is a variant of the *burning algorithm* introduced by Dhar in order to distinguish between recurrent and non-recurrent configurations [Dhar 90]. The spanning tree returned by the algorithm can be seen as the path through which the *fire* (the sequence of topplings) propagates. The intuitive principle of the algorithm is to decompose each toppling and consider its effect grain after grain. When a grain makes another vertex topple, we add the edge by which the grain has traveled into the tree. Different variants of this algorithm have been proposed [Cori 03, Cheb 05]. These variants differ by the rule used for choosing the next grain to be sent, and also differ from the procedure Υ given below. Let us insist that these variants are really unequivalent.

If v is a vertex and $F \subseteq E$ be a subgraph, we denote by $\deg_F(v)$ the degree of v in the subgraph F.

Definition 8.4 *Let $\mathcal{G} = (H, \sigma, \alpha, h_0)$ be an embedded graph. The mapping Υ associates with a recurrent configuration $\mathcal{S}$ of the sandpile model the spanning tree defined by the following procedure.*

Initialization: *Initialize the* current half-edge h *to be* $h'_0 = \sigma^{-1}(h_0)$. *Initialize the tree T and the set of visited edges F to be empty.*

Core: *Do:*

C1: *Let e be the edge containing h, let u be the vertex incident to h and let v be the other endpoint of e.*

If e is not in F, then

- *Add e to F.*
- *If u is not connected to v by T and $\mathcal{S}(v) + \deg_F(v) \geq \deg(v)$ then*

  Add e to T.

C2: *Move to the next half-edge clockwise around T:*

If e is in T, then set the current half-edge h to be $\sigma^{-1}\alpha(h)$, else set it to be $\sigma^{-1}(h)$.

Repeat until the current half-edge h is h'_0.

End: *Return the tree T.*

Observe that during the procedure Υ our motion (step **C2**) around the spanning tree is reverse (compared to our previous algorithms). This way of visiting the half-edges would be the usual tour of the spanning tree in the embedded graph $\mathcal{G}' = (H, \sigma^{-1}, \alpha, h'_0)$.

We represented the intermediate steps of the procedure Υ in Figure 141.

We will now prove that Υ and Λ are inverse bijections. We first prove that the mapping Υ is well defined on recurrent configurations and returns a spanning tree (Proposition 8.5). Then we prove that Υ and Λ are inverse mappings (Propositions 8.12 and 8.13).

Proposition 8.5 *The procedure Υ is well defined on recurrent configurations and returns a spanning tree.*

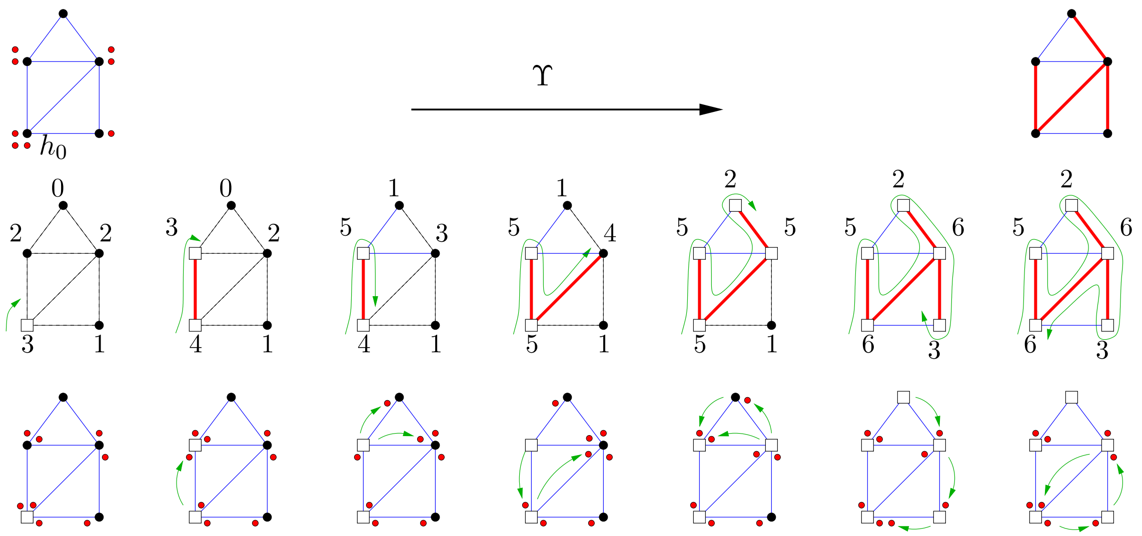


Figure 141: The mapping Υ. Some intermediate steps of the procedure are represented in the middle line. The set $\overline{F}$ of unvisited edges is indicated by dashed lines. The number associated to each vertex v is equal to $\mathcal{S}(v) + deg_F(v)$. In the bottom line are represented the *burning algorithm* representation of each of the intermediate steps.

Lemma 8.6 *Let $\mathcal{S}$ be a recurrent configuration. Then, at any time of the execution of the procedure Υ on $\mathcal{S}$, the endpoint u of the current half-edge h is connected to v_0 by T.*

Proof: The property holds at the beginning of the execution. Clearly, it remains true each time a step **C2** is performed. □

Proof of Proposition 8.5. Let $\mathcal{S}$ be a recurrent configuration. We denote by $\Upsilon[\mathcal{S}]$ the execution of the procedure Υ on $\mathcal{S}$. We prove successively the following properties on the execution $\Upsilon[\mathcal{S}]$.

• *At any time of the execution, the subgraph T is a tree incident to v_0.* The property holds at the beginning of the execution. Suppose that it holds at the beginning of a given core step and consider the edge e with endpoints u and v containing the current half-edge. If the edge e is added to T, the subgraph T remains acyclic since u is not connected to v by T. Moreover the subgraph T remains connected and incident to v_0 since (by Lemma 8.6) the vertex u is connected to v_0 by T.

• *No half-edge is visited twice, hence the execution terminates.* Suppose that a half-edge h is visited twice during the execution. We consider the first time this situation happens. First note that $h \neq h_0'$ or the execution would have stopped just before the second visit to h. Let h_1 and h_2 be respectively the current half-edge just before the first and second visit to h.. Let T_1 and T_2 be the trees constructed by the procedure Υ at the time of the first and second visit to h. Let e be the edge containing $\sigma^{-1}(h)$. For $i = 1, 2$ we have $h = \sigma^{-1}\alpha(h_i)$ if e is in

T_i and $h = \sigma^{-1}(h_i)$ otherwise. Since $h_1 \neq h_2$ and $T_1 \subseteq T_2$, the edge e is in T_2 but not in T_1. This is impossible since after the visit of h_1 the edge e is in F and cannot be added to the tree T anymore.

We denote by T_0 the tree returned by the execution $\Upsilon[\mathcal{S}]$ and by F_0 the set of visited edges at the end of this execution.

• *If $e = \{h_1, h_2\}$ is an edge in $T_0 = \Upsilon(\mathcal{S})$ and the endpoint of h_1 is the father of the endpoint of h_2, then h_1 is visited during the execution $\Upsilon[\mathcal{S}]$.* Consider the core step at which the edge e is added to the tree T. Let h be the current half-edge, let u be the vertex incident to h and let v be the other endpoint of e. By Lemma 8.6, the vertex u is connected to v_0 by $T \subseteq T_0 - e$, hence u is the father of v. Hence $h_1 = h$ is visited during the execution Υ.

• *At the end of the execution, any edge adjacent to T_0 is in F_0.* We want to show that any half-edge incident to T_0 is visited during the execution $\Upsilon[\mathcal{S}]$. First observe that no edge can be added to T after its first visit. Therefore, when a step **C2** is performed, the edge e containing the current half-edge is in T if and only if it is in T_0. Let h be a half-edge incident to T_0 which has not been visited during the execution Υ. If the half-edge $\sigma^{-1}(h)$ is not in T_0 then it has not been visited (or h would have been the next half-edge visited during the execution). Thus by applying σ^{-1} repeatedly we find an unvisited half-edge h such that $\sigma^{-1}(h)$ is in T_0. Then, the half-edge $\alpha\sigma^{-1}(h)$ has not been visited during the execution Υ (or h would have been the next half-edge visited during the execution). Thus (by the preceding point) the endpoint of $\alpha\sigma^{-1}(h)$ is the son of the endpoint of $\sigma^{-1}(h)$. We have proved that if there is an unvisited half-edge h incident to T_0, then there is an unvisited half-edge incident to one of its sons in T_0. We reach an impossibility.

• *The tree $T_0 = \Upsilon(\mathcal{S})$ is spanning.* Let $v_0, v_1, \ldots, v_{|V|-1}$ be a labeling of the vertices such that the sequence $\mathcal{S} \overset{v_0}{\dashrightarrow} \mathcal{S}^1 \overset{v_1}{\dashrightarrow} \cdots \overset{v_{|V|-1}}{\dashrightarrow} \mathcal{S}^{|V|}$ is valid. In the configuration $\mathcal{S}_i$, the number of sand grains on the vertex v_i is $\mathcal{S}^i(v_i) = \mathcal{S}(v_i) + \sum_{j<i} \deg(v_j, v_i)$ and is more than the degree of v_i. Suppose now that the tree T_0 is not spanning and consider the least index i such that v_i is not connected to v_0 by T. Each vertex v_j for $j < i$ is incident to T, hence (by the preceding point) every edge joining v_j and v_i is in F_0. Moreover v_i is adjacent to at least one of the vertices $v_j, j < i$ since $\mathcal{S}(v_i)$ is less than its degree and $\mathcal{S}^i(v_i)$ is not. Consider the last edge e (in order of visit) joining v_i to a vertex $v_j, j < i$. When the edge e is visited, we have $\deg_F(v_i) \geq \sum_{j<i} \deg(v_i, v_j)$. Therefore, the condition $\mathcal{S}(v_i) + \deg_F(v_i) \geq \deg(v_i)$ holds and the edge e should have been added to the tree T. We reach a contradiction. □

We proceed to prove that Λ and Υ are inverse mappings.

Lemma 8.7 *Consider a given core step of the procedure Υ. Let e be the edge containing the current half-edge h and let v be the endpoint of $\alpha(h)$. If the edge e is added to T, then the inequality $\mathcal{S}(v) + \deg_F(v) \geq \deg(v)$ (tested in the procedure Υ) is an equality.*

Proof: Observe first that the vertex v is distinct from v_0, otherwise adding e to the tree T

would create a cycle by Lemma 8.6. While v is not connected to v_0 by T, it is not the endpoint of the current half-edge h (Lemma 8.6). Thus, each time the quantity $\deg_F(v)$ increases, that is, each time an edge incident to v is added to F, the condition $\mathcal{S}(v) + \deg_F(v) \geq \deg(v)$ is tested and the edge is added to T if the condition holds.

□

Lemma 8.8 *Let $\mathcal{G} = (H, \sigma, \alpha, h_0)$ be an embedded graph and let T be a spanning tree. We consider the $(\mathcal{G}, T)$-order on half-edges. Let v be a vertex distinct from v_0 and let h_v be the half-edge incident to v in the edge of T linking v to its father. Any half-edge h incident to v and such that $\alpha(h) > h_v$ is external. Moreover, there are* $\deg(v) - \mathcal{S}_T(v) - 1$ *such half-edges.*

Proof: We consider the orientation $\mathcal{O}_T$. Recall from Lemma 5.6 that $\alpha(h_v) < h_v$ and that the half-edges h incident to a descendant of v are characterized by $\alpha(h_v) < h \leq h_v$. In particular, the inequalities $\alpha(h_v) < h \leq h_v$ hold for the half-edges incident to v. We now prove successively the following properties.

• *Any half-edge h incident to v and such that $\alpha(h) > h_v$ is external.* Suppose that the half-edge h is internal and consider the edge e containing h. If e links v to its father, then $h = h_v$ and $\alpha(h) = \alpha(h_v) < h_v$. If e links v to one of its sons, then $\alpha(h)$ is incident to a descendant of v and $\alpha(h) \leq h_v$. In either cases, the hypothesis $\alpha(h) > h_v$ does not hold.

• *An external half-edge h incident to v is a non-active head if and only if $\alpha(h) > h_v$.* The three following properties are sufficient to prove the equivalence:

- *If h is a tail then $\alpha(h) < h_v$.* Indeed, we have $\alpha(h) < h$ since h is a tail and $h \leq h_v$ since h is incident to v.

- *If h is a head and $\alpha(h) < h_v$ then h is $(\mathcal{G}, T)$-active.* Since h is a head, we have $h < \alpha(h)$ hence, $\alpha(h_v) < h < \alpha(h) < h_v$. Thus, $\alpha(h)$ is incident to a descendant of v and the edge $e = \{h, \alpha(h)\}$ is $(\mathcal{G}, T)$-active by Lemma 5.8.

- *If h is a head and $\alpha(h) > h_v$ then h is not $(\mathcal{G}, T)$-active.* Since h is a head we have $h < \alpha(h)$. Since $\alpha(h) > h_v$, the half-edge $\alpha(h)$ is not incident to a descendant of v and the edge $e = \{h, \alpha(h)\}$ is not $(\mathcal{G}, T)$-active by Lemma 5.8.

• *There are* $\deg(v) - \mathcal{S}_T(v) - 1$ *half-edges h incident to v and such that $\alpha(h) > h_v$.* By definition, $\mathcal{S}_T(v)$ is the number of tails plus the number of external $(\mathcal{G}, T)$-active heads incident to v. Hence, $\deg(v) - \mathcal{S}_T(v)$ is the number of heads incident to v which are not external $(\mathcal{G}, T)$-active. By Lemma 5.6, internal edges are oriented from father to son. Hence, the vertex v is incident to exactly one internal head. Thus $\deg(v) - \mathcal{S}_T(v) - 1$ is the number of external non-active heads. By the preceding point, these half-edges are characterized by the condition $\alpha(h) > h_v$.

□

We now define the *clockwise-tour* of a tree. Let $\mathcal{G} = (H, \sigma, \alpha, h_0)$ be an embedded graph.

Given a spanning tree T, we define the *clockwise-motion function* τ on half-edges by

$$\tau(h) = \sigma^{-1}\alpha(h) \text{ if } h \text{ is internal and } \tau(h) = \sigma^{-1}(h) \text{ otherwise.}$$

As observed above, the clockwise-motion function τ is the usual motion function for the embedded graph $\mathcal{G}^{-1} = (H, \sigma^{-1}, \alpha, \sigma^{-1}(h_0))$. This defines the $(\mathcal{G}^{-1}, T)$-*order* on the half-edge set H for which $h'_0 = \sigma^{-1}(h_0)$ is the least element. The $(\mathcal{G}, T)$-order denoted by $<$ and the $(\mathcal{G}^{-1}, T)$-order denoted by $<^{-1}$ are closely related.

Lemma 8.9 *Let $\mathcal{G}$ be an embedded graph and let T be a spanning tree. The $(\mathcal{G}, T)$-order and $(\mathcal{G}^{-1}, T)$-order are related by $h < h'$ if and only if $\beta(h') <^{-1} \beta(h)$, where β is the involution defined by $\beta(h) = h$ if h is external and $\beta(h) = \alpha(h)$ otherwise.*

Proof: Let t be the usual motion function and let τ be the clockwise-motion function. Observe that $t\beta = \sigma$ and $\tau\beta = \sigma^{-1}$. Thus, $\tau = \beta t^{-1}\beta$. Let us write $t = (h_0, h_1, \ldots, h_{|H|-1})$ in cyclic notation. Then $t^{-1} = (h_{|H|-1}, \ldots, h_1, h_0)$ and $\tau = \beta t^{-1}\beta = (\beta(h_{|H|-1}), \ldots, \beta(h_1), \beta(h_0))$. Moreover, $\sigma\beta(h_{|H|-1}) = t(h_{|H|-1}) = h_0$, hence $\beta(h_{|H|-1}) = h'_0 = \sigma^{-1}(h_0)$. Therefore, $h_i < h_j$ if and only if $i < j$ if and only if $\beta(h_j) <^{-1} \beta(h_i)$. $\square$

Lemma 8.10 *Let $\mathcal{S}$ be a recurrent configuration and let $T_0 = \Upsilon(\mathcal{S})$ be the spanning tree returned by the procedure Υ. The half-edges of $\mathcal{G}$ are visited in $(\mathcal{G}^{-1}, T_0)$-order during the procedure Υ.*

Proof: During the procedure Υ, no edge can be added to the tree T after its first visit. Therefore, when a step **C2** is applied, the edge e containing the current half-edge is in T if and only if it is in T_0. Hence, a step **C2** corresponds to an application of the clockwise-motion function τ of the spanning tree T_0. Since the first visited half-edge is $h'_0 = \sigma^{-1}(h_0)$, the half-edges are visited in $(\mathcal{G}^{-1}, T_0)$-order. $\square$

Lemma 8.11 *Let $\mathcal{G}$ be an embedded graph and let T be a spanning tree. Let v be a vertex distinct from v_0 and let e_v be the edge of T linking v to its father. There are $\deg(v) - \mathcal{S}_T(v) - 1$ edges incident to v and less than e_v for the $(\mathcal{G}^{-1}, T)$-order.*

Proof: Let h_v be the half-edge of e_v incident to v. Let $h \neq h_v$ be a half-edge incident to v and let e be the edge containing h. We prove successively the following properties.

• *The edge e is less than e_v if and only if $\alpha(h) <^{-1} \alpha(h_v)$. Moreover, in this case e is not a loop.* By Lemma 5.6 applied to the embedded graph $\mathcal{G}^{-1}$, the half-edges h incident to v are such that $\alpha(h_v) <^{-1} h \leq^{-1} h_v$. Hence, the edge containing h is less than e_v for the $(\mathcal{G}^{-1}, T)$-order if and only if $\alpha(h) <^{-1} \alpha(h_v)$. In this case, $\alpha(h)$ is not incident to v by Lemma 5.6, that is, e is not a loop.

• *The conditions* $\alpha(h) <^{-1} \alpha(h_v)$ *and* $\alpha(h) > h_v$ *are equivalent. Moreover, there are* $\deg(v) - \mathcal{S}_T(v) - 1$ *half-edges satisfying this condition.* Suppose $\alpha(h) <^{-1} \alpha(h_v)$. In this case, h external. Indeed, h is not in e_v and is not incident to a son of v by Lemma 5.6 applied to the embedded graph $\mathcal{G}^{-1}$. Hence, by Lemma 8.9, we get $\alpha(h) > h_v$. Conversely, if $\alpha(h) > h_v$, the edge e is external by Lemma 8.8, hence $\alpha(h) <^{-1} \alpha(h_v)$ by Lemma 8.9. Moreover, there are $\deg(v) - \mathcal{S}_T(v) - 1$ half-edges satisfying this condition by Lemma 8.8. □

Proposition 8.12 *The mapping* $\Lambda \circ \Upsilon$ *is the identity on recurrent configurations.*

Proof: Let $\mathcal{S}$ be a recurrent configuration and let $T = \Upsilon(\mathcal{S})$. We want to prove that the recurrent configuration $\mathcal{S}_T = \Lambda(T)$ is equal to $\mathcal{S}$. We already know that $\mathcal{S}_T(v_0) = \deg(v_0) = \mathcal{S}(v_0)$ since $\mathcal{S}_T$ and $\mathcal{S}$ are recurrent configurations. Let v be a vertex distinct from v_0 and let e_v be the edge of T linking v to its father. Let F be the set of visited edges when e_v is added to T during the execution $\Upsilon[\mathcal{S}]$. We know that $\mathcal{S}(v) = \deg(v) - \deg_F(v)$ by Lemma 8.7. It remains to prove that $\mathcal{S}_T(v) = \deg(v) - \deg_F(v)$. By Lemma 8.10, the half-edges are visited in $(\mathcal{G}^{-1}, T)$-order during the execution $\Upsilon[\mathcal{S}]$. Therefore, the value $\deg_F(v)$ is the number of edges incident to v which are less or equal to e_v for the $(\mathcal{G}^{-1}, T)$-order. There are $\deg(v) - \mathcal{S}_T(v)$ such edges by Lemma 8.11. We obtain $\deg_F(v) = \deg(v) - \mathcal{S}_T(v)$, or equivalently, $\mathcal{S}_T(v) = \deg(v) - \deg_F(v)$. Thus, $\mathcal{S}_T(v) = \mathcal{S}(v)$. □

Proposition 8.13 *The mapping* $\Upsilon \circ \Lambda$ *is the identity on spanning trees.*

Proof: Let T_0 be a spanning tree. We denote by $T_1 = \Upsilon(\mathcal{S}_{T_0})$ the image of T_0 by $\Upsilon \circ \Lambda$ and want to prove that $T_1 = T_0$. Recall that every edge of $\mathcal{G}$ is visited during the execution $\Upsilon[\mathcal{S}_{T_0}]$. Hence, it is sufficient to prove that at the beginning of any core step of the execution $\Upsilon[\mathcal{S}_{T_0}]$, *the tree* T *constructed by the procedure* Υ *is* $T_0 \cap F$*, where* F *denotes the set of visited edges.* We proceed by induction on the number of core steps. The property holds at the beginning of the first core step. Suppose that it holds at the beginning of the k^{th} core step. If the edge e containing the current half-edge is already in the set F of visited edges, then the set F and the tree T are unchanged during this core step and the property holds at the beginning of the $k+1^{th}$ core step. Suppose now that the edge e is not in F at the beginning of the k^{th} core step. By the induction hypothesis, the tree T constructed by the procedure Υ is $T_0 \cap F$. Moreover, no edge is added to the tree T after its first visit, hence $T = T_1 \cap F$. In other words, the spanning trees T_0 and T_1 coincide on F. By Lemma 8.10, the half-edges are visited in $(\mathcal{G}^{-1}, T_1)$-order during the execution $\Upsilon[\mathcal{S}_{T_0}]$, hence the edges visited before e during the execution $\Upsilon[\mathcal{S}_{T_0}]$ have been visited in $(\mathcal{G}^{-1}, T_0)$-order. Thus, the edges visited before e during the execution $\Upsilon[\mathcal{S}_{T_0}]$ are the edges which are less than e for the $(\mathcal{G}^{-1}, T_0)$-order. Suppose now that the edge e is in the tree T_0. In this case the endpoints u and v of e are not connected by $T \subseteq T_0 - e$. Moreover, the value $\deg_{F+e}(v)$ which corresponds to the number of edges incident to v and visited before e during the execution $\Upsilon[\mathcal{S}_{T_0}]$, that is, the edge which are less or equal to e for the $(\mathcal{G}^{-1}, T_0)$-order, is $\deg(v) - \mathcal{S}_{T_0}(v)$ by Lemma 8.11. Thus, the condition $\mathcal{S}_{T_0}(v) + \deg_{F+e}(v) \geq \deg(v)$ (tested by the procedure Υ) holds and the edge e is added to the tree T. Suppose now that e is not in T_0. In this case, the edge e_v linking v to its father in T_0 is greater than e for the $(\mathcal{G}^{-1}, T_0)$-order. Hence, the value $\deg_{F+e}(v)$ is less or equal to the number of edges incident to v which are less than e_v for the $(\mathcal{G}^{-1}, T_0)$-order. Thus, $\deg_{F+e}(v) < \deg(v) - \mathcal{S}_{T_0}(v) - 1$ by Lemma 8.11. The condition $\mathcal{S}_{T_0}(v) + \deg_{F+e}(v) \geq \deg(v)$ (tested by the procedure Υ) does not hold, hence the edge e is not added to the tree T. In any case, the property holds at the beginning of the $k + 1^{th}$ core step. $\square$

This concludes our proof of Theorem 8.1. $\square$

Chapter 9

Perspectives

La combinatoire des cartes est un sujet riche et en plein renouvellement. Nous avons établi des résultats énumératifs pour plusieurs familles de triangulations et jeté un pont entre le polynôme de Tutte et les cartes. Nous avons aussi exhibé des bijections qui résolvent de vénérables énigmes combinatoires concernant les chemins de Kreweras, les cartes boisées et l'interprétation en termes d'orientations de nombreuses évaluations du polynôme de Tutte.

Certains résultats établis durant cette thèse appellent manifestement des développements ultérieurs. En premier lieu, le comptage récursif des triangulations effectué au chapitre 1 a permis de montrer l'algébricité des séries génératrices de plusieurs familles de triangulations. Les familles en considération sont *doublements contraintes*, c'est-à-dire sont définies par des contraintes de degrés portant à la fois sur les sommets et sur les faces. Les familles de cartes doublement contraintes sont souvent difficiles à énumérer et l'on dispose d'assez peu d'information sur ces familles. Il serait pourtant intéressant de préciser la frontière de l'algébricité des cartes, c'est-à-dire de déterminer quelles contraintes portant sur le degré des sommets et des faces donnent lieu à des séries génératrices algébriques.

Les bijections présentées au chapitre 2 permettent le comptage des triangulations et celui des chemins de Kreweras arrivant en $(0, 0)$. Une évidente perspective consiste à chercher une généralisation qui permette le comptage des triangulations d'un polygone à $i + 2$ côtés ou des chemins de Kreweras arrivant en $(i, 0)$.

Au chapitre 3, nous avons défini une bijection entre les *cartes boisées* (cartes dont un arbre couvrant est distingué) et les couples formés d'un arbre et d'une partition non-croisée. Cette bijection pourrait apporter un cadre unifié pour le comptage bijectif des cartes planaires. Rappelons que, bien souvent, la première étape pour le comptage bijectif d'une famille de cartes consiste à définir un arbre couvrant canonique pour chaque carte de cette famille. Nous avons établi une bijection générale entre les arbres couvrants et les orientations minimales des cartes. Cette bijection a ensuite été étendue aux cartes de genre quelconque au chapitre 7. Il devient donc possible de définir des arbres couvrants en termes d'orientations minimales ou, de manière équivalente, de suites de degrés sortants. Considérons, par exemple, la classe des cartes planaires *eulériennes* (i.e. dont les sommets sont de degré pair). On peut *définir* l'arbre *eulérien* d'une carte eulérienne comme l'arbre couvrant en bijection avec l'unique orientation minimale telle que le degré sortant de chaque sommet soit égal à la moitié de son degré total. Il est montré dans [Fusy 03] que l'arbre eulérien est précisément l'arbre couvrant utilisé dans [Scha 97] pour réaliser le comptage bijectif des cartes eulériennes par conjugaison d'arbres. D'autre part, nous avons établi une bijection entre les cartes planaires munies d'une orientation minimale et les couples formés d'un arbre et d'une partition non-croisée. On peut espérer appliquer cette bijection à des familles de cartes particulières en vue de leur comptage bijectif. Par exemple, le comptage bijectif des cartes eulériennes pourrait être réalisé en caractérisant puis en comptant les couples (formés d'un arbre et d'une partition non-croisée) associés aux cartes eulériennes munies de l'arbre couvrant eulérien.

Dans la troisième partie de cette thèse (chapitres 5 à 8) nous avons plongé le polynôme de Tutte au coeur de la combinatoire des cartes. Le projet naturel qui en découle est l'énumération du polynôme de Tutte des cartes, c'est-à-dire l'évaluation de la somme

$$S_n(\mu,\nu) = \sum_{C \in \mathbf{C}_n} T_C(\mu,\nu),$$

où $\mathbf{C}_n$ est l'ensemble des cartes de taille n et $T_C(\mu,\nu)$ est le polynôme de Tutte de la carte C. Nous avons mentionné dans l'introduction de cette thèse (section 0.3) que le polynôme de Tutte est équivalent (à changement de variables près) à la fonction de partition du modèle de Potts. La somme $S_n(\mu,\nu)$ est donc équivalente à la fonction de partition du modèle de Potts sur le réseau aléatoire de taille n.

Au chapitre 5, nous avons montré que le polynôme de Tutte est égal à la série génératrice des arbres couvrants comptés selon leurs activités de plongement. Ainsi la somme S_n peut aussi s'écrire :

$$S_n(\mu,\nu) = \sum_{C_A \text{ carte boisée}} x^{\mathcal{I}(C_A)} y^{\mathcal{E}(C_A)},$$

où la somme porte sur l'ensemble des cartes boisées de taille n et $\mathcal{I}(C_A)$ (resp. $\mathcal{E}(C_A)$) est l'activité de plongement interne (resp. externe) de la carte boisée C_A. Nous savons que les cartes boisées sont en bijection avec les mélanges de deux mots de parenthèses [Mull 67, Lehm 72]. Nous avons aussi prouvé au chapitre 3 que les cartes boisées sont en bijection avec les couples formés d'un arbre et d'une partition non-croisée. Il est tentant d'essayer d'évaluer la somme $S_n(\mu,\nu)$ en se basant sur l'une ou l'autre de ces bijections.

Nous venons de proposer une approche bijective pour l'énumération du polynôme de Tutte des cartes. Alternativement, on peut aborder ce problème avec une approche récursive. On considère alors la série génératrice des cartes pondérées par leur polynôme de Tutte

$$F(x,y) \equiv F(x,y,z,\mu,\nu) = \sum_{C\in\mathbf{C}} x^{f(C)} y^{s(C)} z^{|C|} T_C(\mu,\nu),$$

où $|C|$, $f(C)$ et $s(C)$ sont respectivement la taille, le degré de la face externe et le degré du sommet racine de la carte C. Les propriétés de récurrence de polynôme de Tutte permettent [Tutt 71] de caractériser la série génératrice $F(x,y)$ par l'équation fonctionnelle

$$\begin{aligned} F(x,y) \;&=\; 1 + xyz(x\mu-1)F(x,y)F(x,1) + xyz\left(\frac{xF(x,y)-F(1,y)}{x-1}\right) \\ &\quad + xyz(y\nu-1)F(x,y)F(1,y) + xyz\left(\frac{yF(x,y)-F(x,1)}{y-1}\right). \end{aligned}$$

Cette équation fait intervenir deux variables catalytiques x et y et est quadratique en les séries inconnues $F(x,y)$, $F(x,1)$ et $F(1,y)$. À ce jour, il n'existe pas de méthode pour la résolution des équations non-linéaires à deux variables catalytiques. La littérature ne fournit qu'un seul exemple d'équation non-linéaire à deux variables catalytiques ayant été résolue. Cet exploit technique revient à Tutte (encore lui !) qui y consacra une série de 8 articles entre 1973 et 1982 [Tutt 73a, Tutt 73b, Tutt 73c, Tutt 73d, Tutt 74, Tutt 78, Tutt 82a, Tutt 82b]. L'équation résolue par Tutte concerne la série génératrice

$$G(x,y) \equiv G(x,y,z,\lambda) = \sum_{C\in\mathbf{Q}} x^{f(C)} y^{s(C)} z^{|C|} \frac{P_C(\lambda)}{\lambda},$$

de la classe $\mathbf{Q}$ des quasi-triangulations *coloriées* (pondérées par leur polynôme chromatique). La résolution de Tutte qui est retracée dans l'article de synthèse [Tutt 95] permet de passer de l'équation fonctionnelle

$$\begin{aligned} G(x,y) \;&=\; yz(\lambda-1) + xyzG(x,y)G(x,1) \\ &\quad + yz\left(\frac{G(x,y)-G(0,y)}{x}\right) - xy^2z^2\left(\frac{G(x,y)-G(x,1)}{y-1}\right), \end{aligned} \tag{95}$$

à l'équation différentielle

$$H''(z)(\lambda z + 10H(z) - 6H'(z)) - 2\lambda^3 z + 2\lambda^2 z + \lambda(4-\lambda)(20H(z) - 18zH'(z) + 9z^2H''(z)) = 0,$$

caractérisant la série génératrice univariée $H(z) = \sum_{C\in\mathbf{T}} z^{|C|/3} \frac{P_C(\lambda)}{\lambda}$ de la classe $\mathbf{T}$ des triangulations coloriées.

L'obtention d'équations fonctionnelles du type de (95) pour une famille de cartes coloriées ne constitue pas un obstacle majeur. Nous avons établi des équations fonctionnelles pour de nombreuses familles de cartes coloriées. Ces équations sont présentées dans l'annexe 9.1. Par contre, la résolution des équations fonctionnelles est pour le moins problématique. La méthode de Tutte pour résoudre l'équation (95) des triangulations coloriées est assez alambiquée. Nous allons tenter de donner un aperçu de cette méthode sur d'autres objets qui nous sont chers : les chemins de Kreweras. Nous ne prétendons pas, par cet exemple, démonter tous les mécanismes de la méthode de Tutte mais simplement faire ressortir l'un des rouages clefs, à savoir, l'élimination d'une des deux variables catalytiques par la *méthode des invariants*. L'équation que nous nous proposons de résoudre s'écrit

$$K(x,y) = 1 + xyzK(x,y,z) + z\left(\frac{K(x,y)-K(0,y)}{x}\right) + z\left(\frac{K(x,y)-K(x,0)}{y}\right), \qquad (96)$$

et caractérise la série génératrice de la classe $\mathbf{K}$ de chemins de Kreweras

$$K(x,y) \equiv K(x,y,z) = \sum_{\kappa\in\mathbf{K}} x^{i(\kappa)}y^{j(\kappa)}z^{|\kappa|},$$

où $i(\kappa)$, $j(\kappa)$, $|\kappa|$ sont respectivement l'abcisse et l'ordonnée du point d'arrivée et la longueur du chemin κ. L'équation (96) (qui s'obtient par une décomposition récursive des chemins) est le point de départ du résultat énumératif de Kreweras [Krew 65] que nous démontrons bijectivement au chapitre 2. Cette équation est linéaire (par rapport aux séries $K(x,y)$, $K(0,y)$ et $K(x,0)$) et peut être résolue par la *méthode du noyau obstinée* [Bous 05a, Bous 02, Bous 03a]. Nous présentons ci-dessous la *méthode des invariants* inspirée de [Tutt 95] qui constitue une méthode de résolution alternative.

La première étape de la méthode des invariants est similaire à celle de la méthode du noyau (applicable pour les équations linéaires à une variable catalytique). On met en facteur la série inconnue principale $K(x,y)$ et on obtient

$$N(x,y)\cdot K(x,y) = E(x,y), \qquad (97)$$

où

$$N(x,y) = z(x+y+x^2y^2) - xy \qquad \text{et} \qquad E(x,y) = xzG(x,0) + yzG(0,y) - xy.$$

On cherche ensuite les séries $S(x,z)$ dont la substitution à y annule le *noyau* $N(x,y)$. Le noyau étant quadratique, on trouve deux racines (exprimables par radicaux) qui sont des séries de Laurent en la variable z :

$$S_1(x,z) = z + o(z) \qquad \text{et} \qquad S_2(x,z) = \frac{1}{xz} + o(\frac{1}{z}).$$

La méthode des invariants nécessite l'obtention de deux racines du noyau qui soient substituables dans l'équation (96) (contrairement à la méthode du noyau pour laquelle une seule racine subtituable suffisait). Malheureusement, la série $S_2(x,z)$ n'est pas substituable à y dans la série $K(x,y)$. Cette difficulté est contournée en imposant le changement de variable (brutal) consistant à remplacer x par la série $S_1(u,z)$.[1] Après cette substitution, le noyau $N(S_1(u,z),y)$ est toujours quadratique en la variable y et admet deux racines

$$Y_1(u,z) = u \qquad \text{et} \qquad Y_2(u,z) = S_2(u,z) = \frac{1}{uz} + o(\frac{1}{z}),$$

substituables à y dans l'équation (97).

À ce stade, nous disposons de deux équations

$$N(S_1, Y_1) = 0 \qquad \text{et} \qquad N(S_1, Y_2) = 0, \tag{98}$$

liant les séries $S_1 \equiv S_1(u,z)$, $Y_1 \equiv Y_1(u,z)$ et $Y_2 \equiv Y_2(u,z)$. En reportant ces équations dans (97) on obtient deux équations supplémentaires

$$E(S_1, Y_1) = 0 \qquad \text{et} \qquad E(S_1, Y_2) = 0. \tag{99}$$

Notons que les équations (98) et (99) ne font pas intervenir la série trivariée principale $K(x,y,z)$ mais seulement ses spécialisations $K(0,y,z)$ et $K(x,0,z)$. On peut également se débarrasser de la série S_1 en remarquant que le produit Y_1Y_2 des racines du polynôme quadratique $N(S_1,y)$ est égal à $\dfrac{1}{S_1}$. En utilisant cette égalité, l'expression du noyau et du second membre deviennent

$$N(S_1, y) = \frac{zy^2}{Y_1^2Y_2^2} + y(z - \frac{1}{Y_1Y_2}) + \frac{z}{Y_1Y_2},$$

et

$$E(S_1, y) = zyK(0,y,z) + \frac{z}{Y_1Y_2}K(\frac{1}{Y_1Y_2}, 0, z) - \frac{y}{Y_1Y_2}.$$

Nous arrivons maintenant au coeur de la méthode des invariants de Tutte. Nous allons convertir les équations (98) et (99) en une unique équation caractérisant la série $K(0,y,z)$ et ne faisant intervenir qu'une seule variable catalytique y.

Considérons une série $I(u,y,z)$ en la variable z dont les coefficients sont des fractions rationnelles en u et y. La série $I(u,y,z)$ est un *invariant* si elle vérifie la relation $I(u,Y_1,z) = I(u,Y_2,z)$. Un invariant $I(u,y,z)$ est *pur* si il ne fait pas intervenir la variable u. Par exemple, la série $\frac{y}{Y_1Y_2} + \frac{1}{y}$ est un invariant impur de même que les séries $N(S_1,y)$ et $E(S_2,y)$. Notons que les séries ne faisant pas intervenir la variable y sont des invariants et que la somme et le produit d'invariants sont des invariants. Ces propriétés de clôture permettent de construire des invariants purs à partir d'invariants impurs. Ainsi, en ajoutant l'invariant

$$-\frac{zy2}{Y_1^2Y_2^2} - \frac{z}{y^2} + \frac{y}{Y_1Y_2} + \frac{1}{y} - \frac{z}{Y_1Y_2},$$

[1] Ce changement de variable nous est apparu en traçant le *diagramme des racines* qui sert de point de départ à la méthode du noyau obstinée [Bous 05a].

à $N(S_1, y)$ on obtient l'invariant pur

$$I(y,z) = \frac{z}{y^2} - zy - \frac{1}{y}.$$

De même, en ajoutant l'invariant

$$-\frac{z}{Y_1Y_2}K(\frac{1}{Y_1Y_2}, 0, z) + \frac{y}{Y_1Y_2} + \frac{1}{y}$$

à $E(S_1, y)$ on obtient l'invariant pur

$$J(y,z) = zyK(0,y,z) + \frac{1}{y}.$$

Une technique pour la construction d'invariants purs à partir d'invariants impurs a été étudié dans [Bern 03]. Précisons néanmoins que ce travail s'avère nettement plus complexe dans le cas des équations fonctionnelles concernant les cartes coloriées.

La méthode des invariants s'appuie sur une propriété structurelle forte de l'espace des invariants purs.

Lemme 9.1 (Invariants) *Soit $L(y,z) = \sum_n f_n(y)z^n$ un invariant pur et soit k un entier tel que pour tout $n \in \mathbb{N}$, $f_n(y) =_{y\to 0} O(\frac{1}{y^k})$. Alors, il existe des séries $c_0(z), \ldots, c_k(z)$ telles que $L(y,z) = \sum_{i\leq k} c_i(z)J(y,z)^i$.*

Nous ne donnerons pas la démonstration du lemme des invariants qui n'éclairerait guère notre propos. Ce lemme assure l'existence de trois séries formelles $a(z)$, $b(z)$, $c(z)$ telles que

$$I(y,z) = a(z)J(y,z)^2 + b(z)J(y,z) + c(z). \tag{100}$$

En comparant le développement des deux membres de cette équation pour y tendant vers 0, on obtient

$$a(z) = z, \qquad b(z) = -1 \qquad \text{et} \qquad c(z) = -2z^2G(0,y,z).$$

En reportant ces identités dans (100), on montre que la série $L(y) \equiv G(0,y,z)$ vérifie l'équation

$$L(y) = 1 + yz^2L(y)^2 + 2z\frac{L(y) - L(0)}{y}.$$

Cette équation à *une* variable catalytique peut être résolue par la méthode générale présentée dans l'introduction de cette thèse (sous-section 0.2.3). On obtient alors l'équation algébrique

$$64z^6K^3 + 16z^3K^2 + (1 - 72z^3)K + 54z^3 - 1 = 0,$$

caractérisant la série $K = K(0,0,z)$ des chemins de Kreweras retournant en $(0,0)$.

Le lemme des invariants constitue la clef de voûte de la méthode de Tutte pour la résolution des équations à deux variables catalytiques. La méthode consiste à produire deux invariants purs indépendants puis à les relier grâce au lemme des invariants. On obtient alors une équation fonctionnelle à *une* variable catalytique[2] que l'on sait résoudre [Bous 05b].

Nous avons présenté la méthode des invariants sur l'exemple des chemins de Kreweras. Notre projet est maintenant d'appliquer cette méthode aux équations fonctionnelles des cartes coloriées présentées dans l'annexe 9.1. Au delà des cartes coloriés, nous aimerions énumérer les cartes pondérées par leur polynome de Tutte. Ce projet a de multiples facettes et des applications importantes. Lorsque Tutte entreprit le comptage des cartes coloriées, il avait en tête de prouver le théorème de quatre couleurs par une approche quantitative. Depuis lors, les motivations se sont déplacées du côté de la physique, autour du modèle de Potts sur réseau aléatoire. Ce modèle suscite beaucoup d'intérêt en physique statistique. D'autre part, le type d'équation auquel conduit l'approche récursive des cartes coloriées constitue le point d'achoppement de nombreuses questions sur les cartes. Une meilleure compréhension des techniques pouvant s'appliquer à ces équations, en particulier de la méthode des invariants de Tutte, aurait des applications dans de nombreux domaines de la combinatoire énumérative.

[2]Contrairement au cas des chemins de Kreweras, l'équation fontionnelle obtenue par Tutte dans le cas des triangulations coloriées n'est pas totalement explicite.

9.1 Annexe : Équations fonctionnelles des cartes coloriées

Nous avons établi des équations fontionnelles pour douze familles de cartes coloriées. Les familles en considération sont définies par différents critères : non-séparables, sans arête double, sans *digones* (faces de degré 2) *etc.* Nous obtenons un panel de douze équations pour les douze familles de cartes définies par la figure 142. Certaines de ces équations apparaissaient déjà dans la littérature et nous avons indiqué la référence de l'article correspondant.

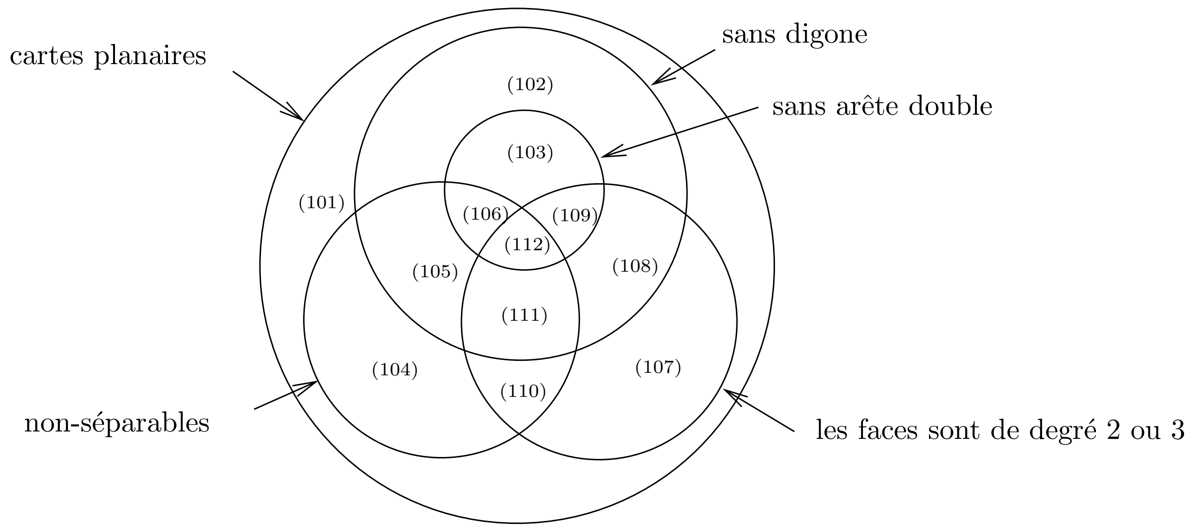


Figure 142: Quelques familles de cartes planaires. Nous avons indiqué pour chaque famille le numéro de l'équation fonctionnelle correspondante.

Cartes générales.

• Cartes générales [Tutt 71] :

$$\begin{aligned} G(x,y) &= 1 + yz(x^2(\lambda-1)+x)G(x,y)G(x,1) + xyz\left(\frac{xG(x,y)-G(1,y)}{x-1}\right) \\ &\quad -xyzG(x,y)G(1,y) - xyz\left(\frac{yG(x,y)-G(x,1)}{y-1}\right). \end{aligned} \tag{101}$$

• Cartes sans digone :

$$\begin{aligned} G(x,y) &= 1 + x^2yz(\lambda-1)G(x,y)G(x,1) \\ &\quad +xyz\left(\frac{xG(x,y)-G(1,y)}{x-1}\right) - xyzG(x,y)G(1,y) - yz(G(x,y)-1)) \\ &\quad -xyz\left(\frac{yG(x,y)-G(x,1)}{y-1}\right) + xyzG(x,y)G(x,1) - xy^2z^2\left(\frac{G(x,y)-G(x,1)}{y-1}\right). \end{aligned} \tag{102}$$

• Cartes sans arête double :

$$\begin{aligned} G(x,y) &= 1+x^2yz(\lambda-1)G(x,y)G(x,1) \qquad (103)\\ &+xyz\left(\frac{xG(x,y)-G(1,y)}{x-1}\right)-xyzG(x,y)G(1,y)-\frac{(G(x,y)-1)(G(1,y)-1)}{\lambda-1}\\ &-xyz\frac{\left(\frac{G(x,y)-G(x,1)}{y-1}\right)(1+\frac{G(1,y)-1}{l-1})}{1-\frac{z}{z(\lambda-1)}\left(\frac{G(1,y)-yG(1,1)}{y-1}+1\right)}+xyzG(x,y)(G(x,1)-1). \end{aligned}$$

Cartes *non-séparables* (sans boucle ni sommet séparateur).

• Cartes non-séparables [Liu 90] :

$$G(x,y) = x^2yz(\lambda-1)+xyz\frac{\left(\frac{G(x,y)-xG(1,y)}{x-1}\right)}{1-\left(\frac{G(x,1)-xG(1,1)}{x-1}\right)}-xyz\frac{\left(\frac{G(x,y)-yG(x,1)}{y-1}\right)}{1-\left(\frac{G(1,y)-yG(1,1)}{y-1}\right)}. \qquad (104)$$

• Cartes non-séparables sans digone :

$$\begin{aligned} G(x,y) &= x^2yz(\lambda-1)+xyz\frac{\left(\frac{G(x,y)-xG(1,y)}{x-1}\right)}{1-\left(\frac{G(x,1)-xG(1,1)}{x-1}\right)}-yzG(x,y) \qquad (105)\\ &-xyz\frac{\left(\frac{G(x,y)-yG(x,1)}{y-1}+yz\frac{G(x,y)-G(x,1)}{y-1}\right)}{1-\left(\frac{G(1,y)-yG(1,1)}{y-1}+yz\frac{G(1,y)-G(1,1)}{y-1}\right)}. \end{aligned}$$

• Cartes non-séparables sans arête double :

$$G(x,y)=x^2yz(\lambda-1)+xyz\frac{\left(\frac{G(x,y)-xG(1,y)}{x-1}\right)}{1-\left(\frac{G(x,1)-xG(1,1)}{x-1}\right)}-\frac{G(x,y)G(1,y)}{(\lambda-1)}-xyz\frac{H(x,y)}{1-H(1,y)}\ ,(106)$$

où

$$H(x,y)=\frac{\left(\frac{G(x,y)-G(x,1)}{y-1}\right)\left(1+\frac{z}{z(\lambda-1)}G(1,y)\right)}{1-\frac{G(1,y)-yG(1,1)}{(\lambda-1)(y-1)}}-G(x,1).$$

Triangulations.

• Cartes dont toute face est de degré 2 ou 3 :

$$\begin{aligned} G(x,y) &= 1+x^2z(\lambda-1)G(x,y)G(x,1)+yz(G(x,y)-1)\\ &+\frac{z}{x}(G(x,y)-1-x^2G(x,y)[x^2]G(x,y))-xz\left(\frac{G(x,y)-G(x,1)}{y-1}\right). \qquad (107) \end{aligned}$$

• *Triangulations* (toute face est de degré 3) :

$$\begin{aligned} G(x,y) &= 1+x^2yz(\lambda-1)G(x,y)G(x,1)+\frac{yz}{x}(G(x,y)-1-x^2G(x,y)[x^2]G(x,y))\\ &-xy^2z^2\left(\frac{G(x,y)-G(x,1)}{y-1}\right). \qquad (108) \end{aligned}$$

• Triangulations sans arête double :

$$\begin{aligned} G(x,y) &= 1 + x^2yz(\lambda-1)G(x,y)G(x,1) + \frac{yz}{x}(G(x,y) - 1 - x^2yz(\lambda-1)G(x,y)) \quad (109) \\ &-\frac{z}{z(\lambda-1)}(G(x,y)-1)[x^3]G(x,y) - xy^2z^2\frac{\left(\frac{G(x,y)-G(x,1)}{y-1}\right)}{1-[x^3]\left(\frac{G(x,y)-yG(x,1)}{(\lambda-1)(y-1)}\right)}. \end{aligned}$$

Triangulations non séparables.

• Cartes non-séparables dont toute face est de degré 2 ou 3 :

$$\begin{aligned} G(x,y) &= yz(\lambda-1) + +yzG(x,y) + xyzG(x,y)G(x,1) \quad (110) \\ &+yz\left(\frac{G(x,y)-G(0,y)}{x}\right) - xy^2z^2\left(\frac{G(x,y)-G(x,1)}{y-1}\right). \end{aligned}$$

• Triangulations non-séparables [Tutt 73a] :

$$\begin{aligned} G(x,y) &= yz(\lambda-1) + xyzG(x,y)G(x,1) \quad (111) \\ &+yz\left(\frac{G(x,y)-G(0,y)}{x}\right) - xy^2z^2\left(\frac{G(x,y)-G(x,1)}{y-1}\right). \end{aligned}$$

• Triangulations non-séparables sans arête double :

$$\begin{aligned} G(x,y) &= yz(\lambda-1) + xyzG(x,y)G(x,1) + yz\left(\frac{G(x,y)-yz(\lambda-1)}{x}\right) \quad (112) \\ &-\frac{z}{z(\lambda-1)}G(x,y)[x^1]G(x,y) - xy^2z^2\frac{\left(\frac{G(x,y)-G(x,1)}{y-1}\right)}{1-[x^1]\left(\frac{G(x,y)-yG(x,1)}{(\lambda-1)(y-1)}\right)}. \end{aligned}$$

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RÉSUMÉ :

Combinatoire des cartes et polynôme de Tutte

Les cartes sont les plongements, sans intersection d'arêtes, des graphes dans des surfaces. Les cartes constituent une discrétisation naturelle des surfaces et apparaissent aussi bien en informatique (codage d'informations visuelles) qu'en physique (surfaces aléatoires de la physique statistique et quantique). Nous établissons des résultats énumératifs pour de nouvelles familles de cartes. En outre, nous définissons des bijections entre les cartes et des classes combinatoires plus simples (chemins planaires, couples d'arbres). Ces bijections révèlent des propriétés structurelles importantes des cartes et permettent leur comptage, leur codage et leur génération aléatoire. Enfin, nous caractérisons un invariant fondamental de la théorie des graphes, le polynôme de Tutte, en nous appuyant sur les cartes. Cette caractérisation permet d'établir des bijections entre plusieurs structures (arbres couvrants, suites de degrés, configurations du tas de sable) comptées par le polynôme de Tutte.

Mots-clés : cartes planaire, graphe, triangulation, énumération, bijection, polynome de Tutte, arbres couvrant.

Combinatorics of maps and the Tutte polynomial

A map is a graph together with a particular (proper) embedding in a surface. Maps are a natural way of representing discrete surfaces and as such they appear both in computer science (encoding of visual data) and in physics (random lattices of statistical physics and quantum gravity). We establish enumerative results for new classes of maps. Moreover, we define several bijections between maps and simpler combinatorial classes (planar walks, pairs of trees). These bijections highlight some important structural properties and allows one to count, sample randomly and encode maps efficiently. Lastly, we give a new characterization of an important graph invariant, the Tutte polynomial, by making use of maps. This characterization allows us to establish bijections between several structures (spanning trees, sandpile configurations, outdegree sequences) counted by the Tutte polynomial.

Keywords : planar map, graph, triangulation, counting, bijection, Tutte polynomial, spanning tree.