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Truncation of fractional derivative for online system identification

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Abstract: Fractional derivatives are non local operators that has compacity property in terms of parameter number for modeling diffusive phenomenon with very few parameters. One of its main properties is its non-local behavior, as it can be exploited to model long-memory phenomena such as heat transfers. However, such non-locality implies a constant knowledge of the full past of the function to be differentiated. In the context of real-time system identification, this may limit the experiences as calculations become slower as time progresses. This study deals with the relationship between frequency content of a signal and its truncation error in order to obtain real-time exploitable algorithms.

Keywords: fractional derivative, short memory principle, LMRPEM, system identification

1. INTRODUCTION

Fractional calculus has been a mathematical concept since its origin after knowing a huge interest in modeling diffusive phenomena since the 70s (Oustaloup (2014)). Diffusive phenomena have been shown to exhibit behavior that can be modelled through a half-order derivative. Thermal impedance at high frequency has also been proven to behave like a half-order integrator (Battaglia et al. (2001)). It has also been shown that finite thermal media can also be modeled through fractional order impedance models (Duhé et al. (2022)). Its properties have proven to be useful in medical scenarios, as is the case of lung modeling, cardiac tissue and muscle relaxation. Therefore, there is an increasing interest regarding fractional order models and their properties.

Fractional differentiation definitions is not unique (Kilbas et al. (2006); Garrappa et al. (2019)), but all definitions model long-memory phenomena. The past history of a given function $f(t)$ still has an influence on its fractional derivative after a long period of time. Fractional order dynamics can be interpreted as an infinite distribution of time constants (Oustaloup (1995)). Unfortunately, this implies the existence of infinitely slow time constants. A main drawback regarding this property is the non-local nature of fractional differentiation: the whole knowledge of the function past is required in order to estimate its fractional derivative at any given time. As time goes by, a signal "history" becomes progressively longer and this may lead to slower computing times if a strict definition for the operator is kept. Computing time may be critical in real-time applications and a continuous increase in computing time can lead to an infeasible algorithm. In consequence, a truncated version of the operator is sought.

The fractional derivative may be elegantly truncated in a really simple way through Podlubny's short memory principle (Podlubny (1999)). The principle allows to use a limited portion of a function past and still get an accurate estimation of its fractional derivative. Even though Podlubny's principle provides an accuracy guarantee, it gives a pessimistic limit that could still be too slow to compute for some real-time scenarios. Online system identification usually imply an excitation signal relying within a defined bandwidth (Ljung (1999)). Therefore, the spectrum of a signal could influence the required length of the past signal and the truncation error. One of the aims of this paper is to analyze the frequency and parameter influence on truncation error for fractional differentiation. This will be performed through a simple scenario.

The earliest proposed techniques for fractional order system identification come from the 90s (Le Lay (1998)). This first methods estimate coefficients for a fractional order transfer function. However, they are limited to the Grünwald-Letnikov discrete-time definition of the fractional derivative. By taking continuous-time identification methods, methods relying on state-variable filters, least-squares optimization and instrumental variables are introduced (Malti et al. (2008)).

Real-time system identification is based upon recursive estimation methods, as recursive least-squares or recursive prediction error method (Ljung (1999)). The analysis of recursive identification methods has also been extended to least squares with state-variable filters and instrumental variable techniques (Padilla (2017)). A first study on recursive prediction error identification for fractional order models was proposed in Djouambi et al. (2012) for only coefficient estimation. Long-memory Recursive Prediction

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Error Method (LMRPEM) has proven to be a useful tool for accurately estimating a fractional order model's coefficients (Victor et al. (2022)). In that paper, the signal full time length was considered without considering real-time implementation. An effective combination of a well chosen truncation method and LMRPEM algorithm could lead to an entirely real-time implementable identification technique for fractional order models.

This paper aims at finding a relationship between the frequency content of a signal and its truncation error in order to obtain real-time exploitable algorithms.

The paper is organized as follows: *Section 2* introduces the basics of fractional calculus and fractional order transfer functions, *Section 3* presents the Short Memory Principle and the frequency's study on truncation error, *Section 4* analyzes fractional order transient response for a simple scenario. *Section 5* presents LMRPEM algorithm and an application of a truncated estimation and conclusions and perspectives are presented in *Section 6*.

2. MATHEMATICAL BACKGROUND

As previously stated, many different definitions for fractional order differentiation exist. One of the most well-known definitions is that of Riemann and Liouville (Samko et al. (1993)):

$$ {}_{0}^{RL}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-\tau)^{n-\alpha-1}f(\tau)d\tau \quad (1)$$

where $n-1 \leq \alpha \leq n$ with n being an integer and Euler's gamma function is defined as:

$$\Gamma(x) = \int_0^{\infty} e^{-t}t^{x-1}dt \quad (2)$$

for $x \in \mathbb{R} - \mathbb{Z}^-$.

If the initial conditions are set to zero, Riemann-Liouville's definition may be proven to be equal to the series definition given by Grünwald-Letnikov (Samko et al. (1993)):

$$p^{\alpha}f(t) = \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{j=0}^{\lfloor \frac{t}{h}\rfloor} c_j f(t-jh) \quad (3)$$

where $\lfloor . \rfloor$ stands for the floor operator and c_j stands for Grünwald's coefficients:

$$c_j = (-1)^j\binom{\alpha}{j} = (-1)^j\frac{\Gamma(\alpha+1)}{\Gamma(j+1)\Gamma(\alpha-j+1)}. \quad (4)$$

This coefficients exhibit a slow tendency towards zero and are what provides long-memory behavior for this definition.

If h parameter is replaced for a sampling time T_s, this last definition can be seen as a weighted sum of the functions past:

$$p^{\alpha}f(t) \approx \frac{1}{T_s^{\alpha}}\sum_{j=0}^{\lfloor \frac{t}{T_s}\rfloor} c_j f(t-jT_s) \quad (5)$$

This definition allows an easy implementation for computers and has led to a significant popularity. It is the definition that will be taken and used for the simulation scenarios presented in this study.

For null initial conditions, the Laplace transform of the the fractional derivative leads to a simple expression:

$$\mathscr{L}\{p^{\alpha}f(t)\} = s^{\alpha}F(s). \quad (6)$$

A fractional single-input single-output (SISO) model may relate its output $y(t)$ to input $u(t)$ through a fractional order differential equation:

$$\begin{aligned} y(t) + a_1p^{\alpha_1}y(t) + \ldots + a_{m_A}p^{\alpha_{m_A}} &= \\ b_0p^{\beta_0}u(t) + b_1p^{\beta_1}u(t) + \ldots + b_{m_B}p^{\beta_{m_B}}u(t) \end{aligned} \quad (7)$$

where (a_i, b_j) are real numbers and $\alpha_1 < \alpha_2 < \ldots < \alpha_{m_A}$ and $\beta_0 < \beta_1 < \ldots < \beta_{m_B}$ are allowed to be non-integer positive numbers.

Laplace's transform allows to relate input and output through a fractional order transfer function model:

$$G(s) = \frac{B(s)}{A(s)} = \frac{\sum_{i=0}^{m_B} b_i s^{\beta_i}}{1 + \sum_{j=1}^{m_A} a_i s^{\alpha_i}}. \quad (8)$$

A fractional order model is commensurate if all of its derivation orders are an integer multiple of a basic order ν (which is called the commensurate order):

$$G(s) = \frac{B(s)}{A(s)} = \frac{\sum_{i=0}^{m_B} b_i s^{i\nu}}{1 + \sum_{j=1}^{m_A} a_i s^{j\nu}}. \quad (9)$$

Stability of fractional order systems has been analyzed in different contexts. The most well-known stability criterion was established by Matignon (Matignon and d'Andréa-Novel (1996)) and allows to check the stability of a commensurate order system through the location of its s^{ν} poles. The original theorem was established for commensurate orders $0 < \nu < 1$, but this was extended for orders between 1 and 2 (Moze and Sabatier (2005)). Orders with commensurate orders beyond 2 can be proven to be unstable (Malti et al. (2011)). Further extensions have been developed in order to check stability of non-commensurate systems (Rivero et al. (2013)). More recently, stability of multi-variable and non-linear fractional systems has also been studied (Lenka (2019)).

Matignon's Stability Theorem:

Let S be a commensurate transfer function and ν its commensurate order. $G(s) = \frac{Q_{\nu}(s)}{P_{\nu}(s)}$ is BIBO-stable if and only if:

$$0 < \nu < 2 \quad (10)$$

and, for every pole s_k ($P_{\nu}(s_k) = 0$):

$$|arg(s_k)| > \nu\frac{\pi}{2} \quad (11)$$

3. TRUNCATED FRACTIONAL DERIVATIVE

As Grünwald-Letnikov's definition is the most commonly used for numerical simulation, its definition for a truncation of length L should be provided:

$${}_{L}p^{\alpha}f(t) \approx \frac{1}{T_s^{\alpha}}\sum_{j=0}^{\lfloor \frac{L}{T_s}\rfloor} c_j f(t-jT_s) \quad (12)$$

The maximal error is then defined as follows:

$$\epsilon_{max} = \max|p^{\alpha}f(t) -_L p^{\alpha}f(t)| \quad (13)$$

A simple cosine function will be used for simulations. The sampling period is taken as $T_s = 0.01$s for the whole subsection.

Figure 1 shows maximum error for varying frequencies within the band $[1-100]$rad/s. Maximal error has been plotted for different memory lengths and keeping $\alpha = 0.5$ constant. As can be observed, an increasing length of memory leads to naturally lower errors. However, frequency of signal also plays an important role in error. As frequency increases, there is a significant error reduction for all memory lengths tested. It could be considered that a higher frequency implies a higher number of periods of the cosine function available for fractional differentiation estimation. Therefore, a reduction of error with frequency may be considered as a natural consequence of an increased “knowledge” of the function’s past.

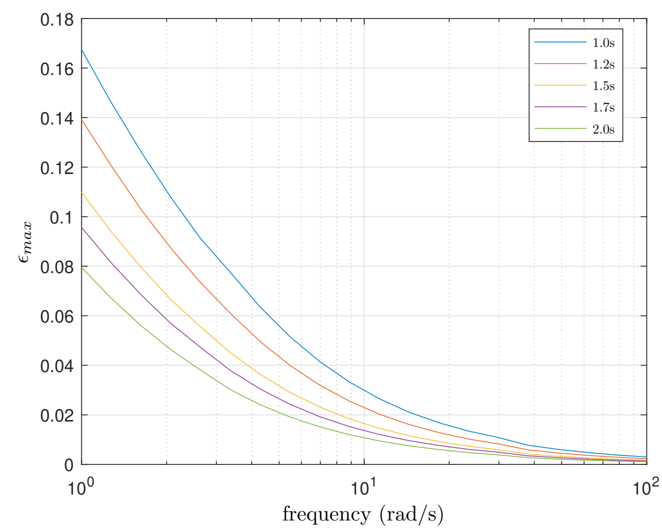

Fig. 1. Maximal error vs frequency at different truncation lengths

This result may suggest that the error stays relatively the same if the same number of periods are taken to estimate its fractional derivative. Therefore, truncation length L is now taken as:

$$L = \frac{X}{\omega} \tag{14}$$

where X is an arbitrary factor and ω is the signal’s frequency. This choice of length leads to an estimation based upon a fixed number of the signal’s periods rather than a time length. Results from this simulations are presented in figure 2.

Even though a higher X factor leads to lower error, maximal error is not preserved by keeping a constant number of periods. As observed, as frequency increases, keeping the same number of periods for fractional differentiation estimation actually leads to an important increase in error. However, it should be noted that a high enough number of previous periods may severely limit frequency effect, as is the case of $X = 50$.

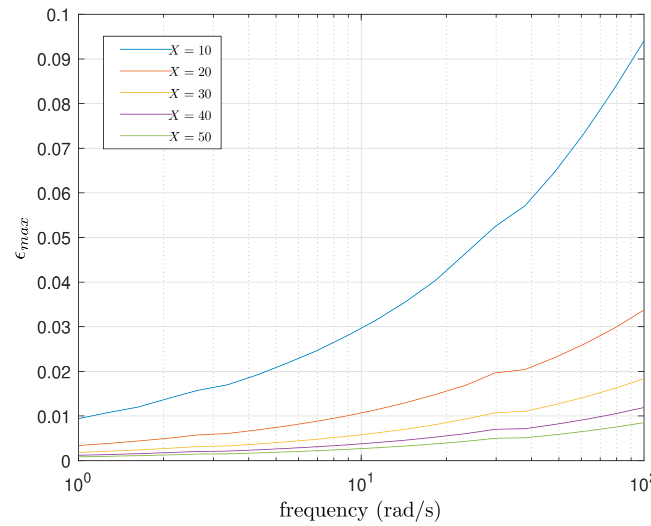

Fig. 2. Maximal error vs frequency at different X factors

4. ONLINE SYSTEM IDENTIFICATION

4.1 Long-Memory recursive prediction error method

In this section, a recursive algorithm for fractional order system identification will be used in order to estimate the system parameters. The Long-Memory Recursive Prediction Error Method (LMRPEM) has proven to have advantages as variance reduction and unbiased results. It will be the chosen method for the example presented in this paper.

If one starts with classical prediction error method, error function is originally defined as:

$$\epsilon(t) = y(t) - \hat{y}(t), \tag{15}$$

where the estimated output $\hat{y}(t)$ is computed as:

$$\hat{y}(t) = G(p, \hat{\theta})u(t). \tag{16}$$

However, nature of fractional order derivatives lead to the choice of an extended error in order to take into account fractional system’s natural long-memory.

$$\tilde{\epsilon}(kT_s) = [\epsilon(0)\, \epsilon(T_s)\, \epsilon(2T_s) \ldots \epsilon(kT_s)]^T \tag{17}$$

This error signal $\tilde{\epsilon}$ will include errors at all instants from $t = 0$ to the current time $t = kT_s$ and will be increased by one data-point per iteration. Additionally, the gradient $\tilde{\psi}_\rho(kT_s, \hat{\theta})$ used will be a matrix:

$$\tilde{\psi}_\rho(kT_s, \theta) = - \begin{bmatrix} \frac{\partial\epsilon(0)}{\partial b_0} & \frac{\partial\epsilon(T_s)}{\partial b_0} & \frac{\partial\epsilon(2T_s)}{\partial b_0} & \ldots & \frac{\partial\epsilon(kT_s)}{\partial b_0} \\ \frac{\partial\epsilon(0)}{\partial b_1} & \frac{\partial\epsilon(T_s)}{\partial b_1} & \frac{\partial\epsilon(2T_s)}{\partial b_1} & \ldots & \frac{\partial\epsilon(kT_s)}{\partial b_1} \\ & & \vdots & & \\ \frac{\partial\epsilon(0)}{\partial a_{m_A}} & \frac{\partial\epsilon(T_s)}{\partial a_{m_A}} & \frac{\partial\epsilon(2T_s)}{\partial a_{m_A}} & \ldots & \frac{\partial\epsilon(kT_s)}{\partial a_{m_A}} \end{bmatrix} \tag{18}$$

where k indicates the present iteration.

By introducing the extended measured output $\tilde{Y}^*(kT_s)$ and the extended estimated output $\tilde{Y}(kT_s)$ as:

$$\tilde{Y}^*(kT_s) = [y^*(0)\, y^*(T_s)\, y^*(2T_s) \ldots y^*(kT_s)]^T \tag{19}$$

$$\tilde{Y}(kT_s) = [y(0)\, y(T_s)\, y(2T_s) \ldots y(kT_s)]^T, \tag{20}$$

the long memory recursive prediction-error method (LMRPEM) is proposed for fractional order systems:

$$\begin{cases} \tilde{\epsilon}(kT_s) = \tilde{Y}^*(kT_s) - \hat{\tilde{Y}}^*(kT_s) \\ \hat{\theta}(kT_s) = \hat{\theta}(k-1) + \gamma_\rho R^{-1}(k)\tilde{\psi}_\rho(kT_s,\theta)\tilde{\epsilon}(kT_s) \\ R(k) = R(k-1) + \gamma_\rho \left[\tilde{\psi}_\rho(kT_s,\theta)\tilde{\psi}_\rho^T(kT_s,\theta) - R(k-1)\right] \end{cases} \tag{21}$$

where γ_ρ is a refining gain analogous to the step in a gradient descent.

4.2 Simulation results

Suppose a first kind fractional order system:

$$G(s) = \frac{K}{1+\tau s^\alpha} \tag{22}$$

where true values are $K = 1$, $\tau = 1$ and $\alpha = 0.5$.

The input is a pseudo-random binary signal oscillating between -5 and 5 and containing 2000 points with a sampling time of $T_s = 1.00\,s$. Note that system (22) was simulated in (Duhé et al., 2022, Section 3.1.1). The system cut-off frequency at $-3dB$ is obtained at $0.2679 rad/s$ which provides a time constant of 23.45s. Moreover, as the system (22) is fractional, its time response is more than 3 times this time constant. Therefore, the sampling time T_s is sufficiently small.

In system identification, it should be noted that in order to correctly identify the system parameters, the input signal should be sufficiently persistent: namely, the input signal spectrum should have at least half the cosine components of the number of the system parameters. In the case of fractional system identification, the input signal persistency has been studied in Abrashov et al. (2017) for first kind fractional systems (or generalized first order fractional systems) and in Malti et al. (2022) for second kind fractional systems (or generalized second order fractional systems).

All simulations are initialized with $\hat{K}_0 = 2$, $\hat{\tau}_0 = 4$ and $\alpha = 0.5$ is kept constant. Gamma factor $\gamma_\rho = 0.01$ and $SNR = 15\,dB$. Figure 3 shows input-output data for this simulations.

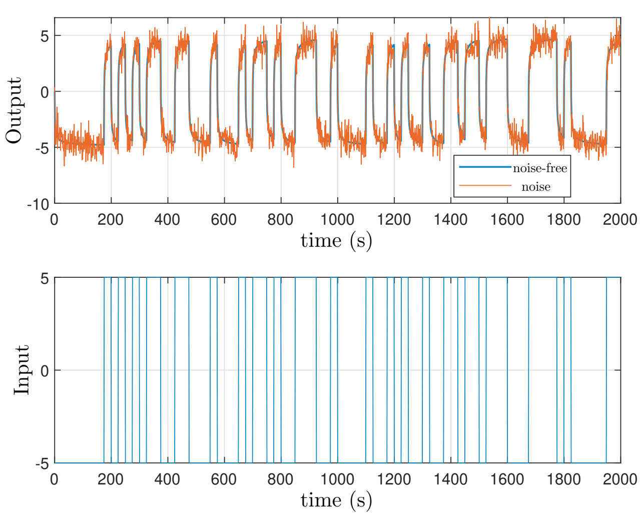

Fig. 3. Input-output data with $SNR = 15$dB

Case 1: no truncation

Parameter estimation as well as calculation time are presented in figures 4 and 5, respectively. An accurate estimation of the parameters is obtained. However, as expected, calculation time dramatically increases with each iteration.

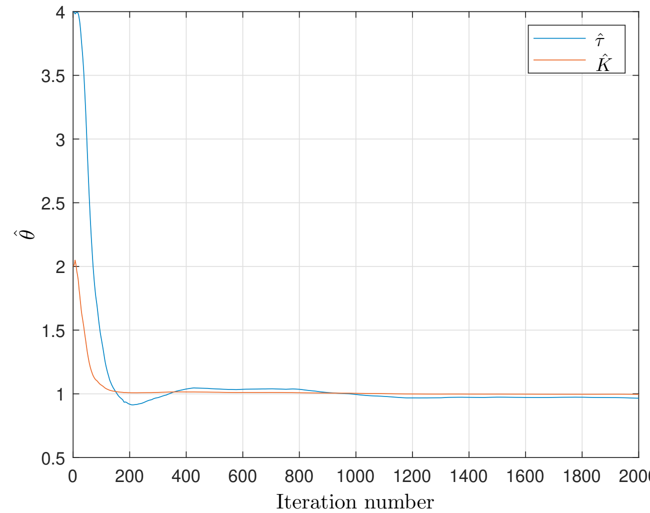

Fig. 4. LMRPEM system identification with $L = 128$s

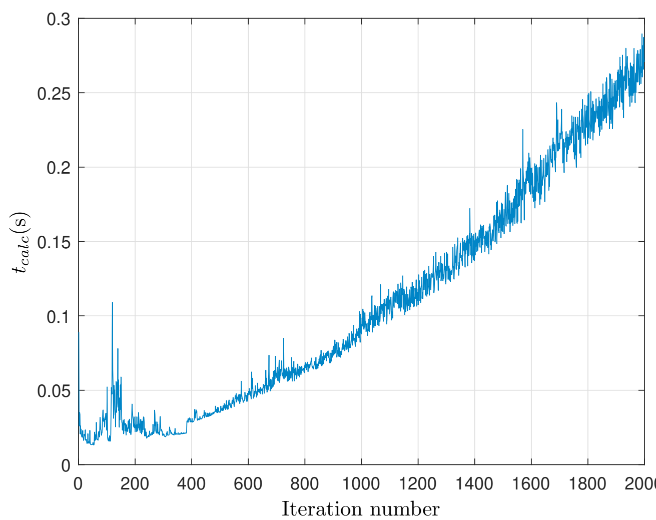

Fig. 5. Computation time vs iteration number

Case 2: truncation equal to relaxation time

Parameter estimation as well as calculation time are presented in figures 6 and 7, respectively. Truncation length is chosen to be $L = 128$s based on long-time approximation. For this case, truncation leads to oscillations in parameters. However, it can be seen that they are still correctly estimated. Calculation time has been dramatically decreased and is found to have a mean value of $t_{calc} = 14.7$ms per iteration. This is far beyond the sampling time limit, which enables real-time implementation. Parameter fluctuations may be due to rather short memory length L, as LMRPEM method uses an extended error and the whole relaxation error is non-negligible.

Case 3: truncation longer than relaxation time

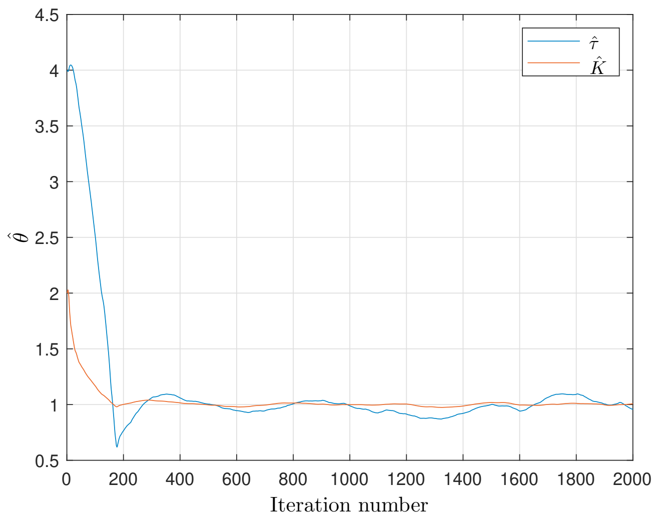

Fig. 6. LMRPEM system identification without truncation

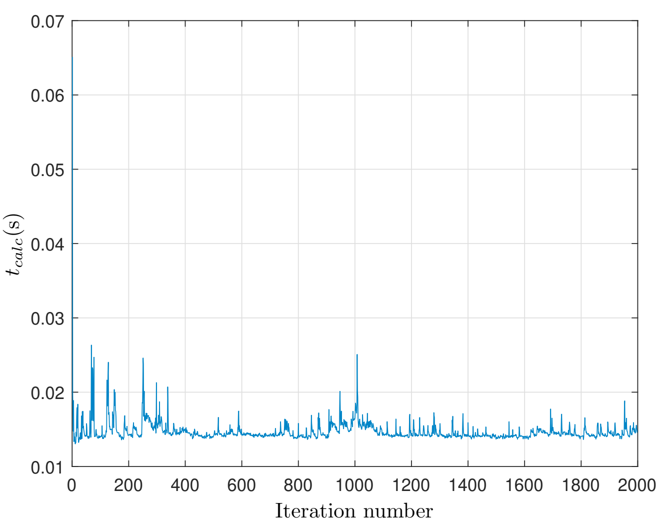

Fig. 7. Computation time vs iteration number

Parameter estimation as well as calculation time are presented in figures 8 and 9, respectively. In order to reduce parameter oscillations seen in previous case, truncation length is chosen to be $L = 256$s (twice the relaxation time). For this case, oscillations are significantly reduced. Even though the performance is not identical to that of the non-truncated case, it remains close. On the other hand, calculation cost is not significantly increased with respect to the previous case. Mean calculation time is $t_{calc} = 17.5$ms, which remains far from the sampling time. Calculation cost curve shows a slight increase in calculation time during the first iterations and then truncation imposes a limit to this increase.

5. CONCLUSIONS AND PERSPECTIVES

Non-locality of fractional derivatives constitutes both a main advantage and drawback to its in mathematical modeling. It allows to capture long-memory behavior, but may impose implementation limits in real-time scenarios. Podlubny's Short Memory Principle provides a mathematical proof of a truncation length L to be used in order to get an approximated fractional derivative without the whole past

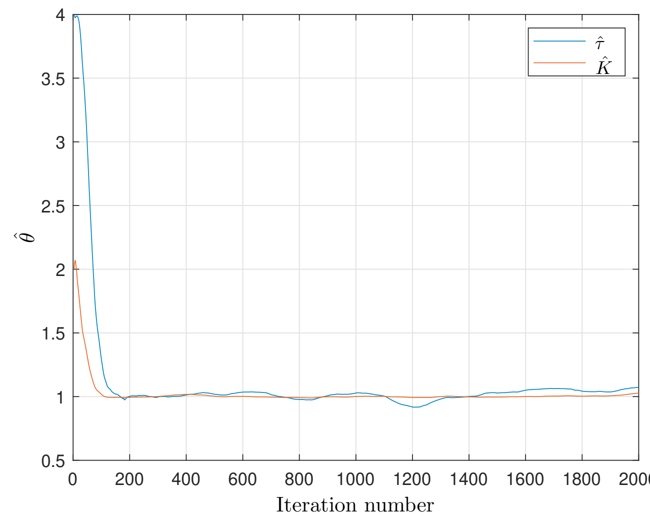

Fig. 8. LMRPEM system identification with truncation $L = 256$s

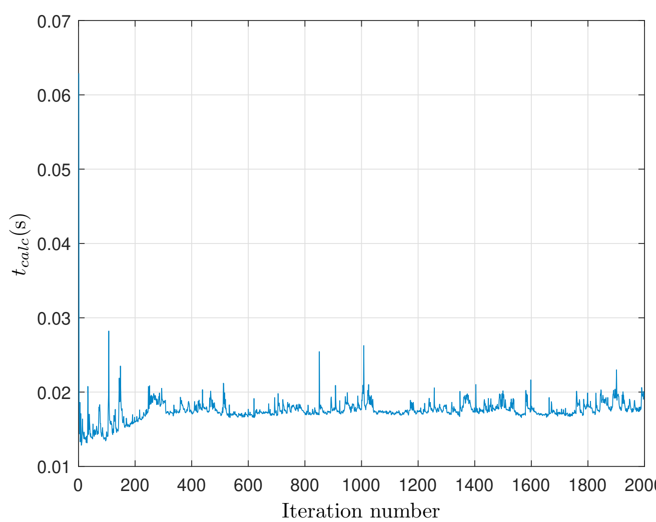

Fig. 9. Computation time vs iteration number

of the function. However, even though it effectively provides a worst-case scenario, Podlubny's limit doesn't take into account frequency of the signal. A simple cosine function was analyzed by starting with Riemann-Liouville's definition, leading to an unsolvable integral for the analytic error expression. As a consequence, numerical simulations using Grünwald-Letnikov definition are carried out. It is then seen that a fixed memory length L will lead to smaller error as the signal's frequency ω increases. Even though this may suggest that error reduction is due to an increased number of periods of the signal taken into account, this was shown to not be accurate. A fixed number of periods of a periodic function past increases truncation error as frequency increases.

There are two main perspectives for this study. The first one is to further analyze the link between parameter oscillation and calculation time. Depending on the scenario, optimizing truncation length L could lead to satisfying identification without exceeding sampling time. On the other hand, a more global fractional order identification could be analyzed. If fractional order of a system is also to be identified, it may have a severe influence on the choice of length. Influence of truncation over differentiation order estimation could be further analyzed. Adaptive methods could also be considered.

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